Complex Analysis ..

Unit - I

complex integration

Fundamental theorems - line integrals - Rectifiable arcs - line integrals as arcs cauchy's theorem for a rectangle and in a disk - cauchys integrals formula - Index of point [illegible] with respect to a closed curve. The integral formula - higher order derivatives - local properties of analytic functions - Taylor's Theorem - zeros and poles local mapping - maximum principle

chapter 4 [sec : 1 to 3]

Unit - II

complex integration [continued]

The general form of cauchy's theorem - chain rules and cycles simple connectivity - Homology - general statement of cauchy theorem - proof of cauchy theorem - locally exact Differentials - multiply connected regions - calculus of residues theorem - Argument principle - Evaluation of definite integrals

chapter : 4 [sec 4.[illegible].5]

unit - III

Harmonic functions and power series expansion →

Harmonic functions - Definition and basis Properties - mean value Propert

Poissons formula schwartz's theorem-
Reflexion principle weieris trassi-
Taylor's series. laurents series

chapter 4 [sec 6]
chapter 5 [sec 1]

unit - IV

Entire function : Jensons formula -Hadamards theorem.

Normal Families : Equicontinuity -Normality and compactness. Arzela's theorem. Families of analytic function the classical definition

chapter . 5 [sec 3 and 5]

unit - V

conformal mapping:

The Riemann mapping theorem conformal mapping of polygons A closure look at harmonic functions

[chapter -6 [sec 1.2 and 3]

Text Book:

complex Analysis
L. V AHLFors.

complex line integral.

complex line integral of $f(z)$ is extended over 'γ' be a piecewise differentiable arc is given by,

$$z = z(t) \; : a \le t \le b$$

If the function $f(z)$ is defined and continuous on 'γ' then '$f[z(t)]$' is also continuous and we define

$$\int_{\gamma} f(z)\, dz = \int_{a}^{b} f(z(t))\, z'(t)\, dt$$

Unit - I

complex integration.

line integrals:-

If $f(t) = u(t) + iv(t)$ is a continuous function defined in an interval $(a.b)$ Then we defined

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Note:-

i) $c = x+iy$ is a complex constant Then $\int_a^b c\,f(t)\,dt = c\int_a^b f(t)\,dt$

ii) when $a<b$ the fundamental inequality

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$$

proof:-

let $\int_a f(t)\,dt = re^{i\theta}$

$\Rightarrow \left|\int_a^b f(t)\,dt\right| = |re^{i\theta}|$

$\Rightarrow \left|\int_a^b f(t)\,dt\right| = re^{i\theta}$

$\Rightarrow r = e^{-i\theta}\int_a^b f(t)\,dt$

$\Rightarrow r = Re\left(e^{-i\theta}\int_a^b f(t)\,dt\right)$

$$= Re \int_a^b e^{-i\theta} f(t)\, dt$$

$$= \int_a^b Re\, (e^{-i\theta})\, f(t)\, dt$$

$$= \int_a^b | e^{-i\theta} f(t) |\, dt$$

$$\leq \int_a^b |e^{-i\theta}|\, |f(t)| \cdot dt$$

$$\therefore \left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)| dt$$

Property: 1

The line integral is invariant under a change of parameters.

A change of Parameter is determinded by an increasing function.

$t = f(\tau)$ which maps an interval

$\alpha \leq \tau < \beta$ on a $a \leq t \leq b$

we assume that.

$t(\tau)$ is piecewise differentiable.

$z = z(t)$

let.

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$= \int_\alpha^\beta f(z[t(\tau)])\, z'[t(\tau)]\, d[t(\tau)]$$

$$\int_\gamma f(z) dz = \int_\alpha^\beta f[z(t(\tau))]\, z'[t(\tau)]\, t'(\tau) dt$$

but.

$$\frac{d\,[z(t(\tau))]}{d\tau} = z'(t(\tau))\,\frac{d}{d\tau}\,t(\tau)$$

$$= z'(t(\tau))\,t'(\tau)$$

$$\Rightarrow d\,[z(t(\tau))] = z'(t(\tau))\,t'(\tau) \quad \text{— ②}$$

sub ② in ①

$$\int_\gamma f(z)dz = \int_\alpha^\beta f[z(t(\tau))]\,d\,[z(t(\tau))]\,dt$$

Property:2

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)dz$$

when $(-\gamma)$ is the opposite arc of (γ)

we know that

The arc (γ) is the

defined as.

$$\gamma = \{z(t) \,/\, t \in [a.b]\}$$

$$-\gamma = \{z(t) \,/\, t \in [-b,-a]\}$$

$$\Rightarrow \int_{-\gamma} f(z)dz = \int_{-b}^{-a} f[z(-t)]\,d\,[z(-t)]$$

$$= \int_{-b}^{-a} f\,[z(-t)]\,z'(-t)\,(-dt)$$

Put $t' = -t \Rightarrow dt' = -dt$

when,

$t = -b \Rightarrow t' = b$

$t = -a \Rightarrow t' = a$

$$① \Rightarrow \int_{-\gamma} f(z)\,dz = \int_{b}^{a} f[z(t')]\, z'(t')\, dt'$$

$$= -\int_{a}^{b} f(z(t))\, z'(t)\, dt$$

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Property: 3

we subdivide an arc 'γ' into a finite no. of sub arcs $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_n$ then the corresponding integrals satisfies the relation.

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz \ldots + \int_{\gamma_n} f(z)\,dz.$$

No proof

A line integral w.r.to $\bar{z}$

A complex line integral over the arc γ w.r.to $\bar{z}$ is defined by,

$$\int_{\gamma} f\, d\bar{z} = \int_{\gamma} \overline{\bar{f}\, dz}$$

note:-

i> $\int_{\gamma} f\, dx = \frac{1}{2}\left[\int_{\gamma} f\, dz + \int_{\gamma} f\, d\bar{z}\right]$

ii> $\int_{\gamma} f\, dy = \frac{1}{2i}\left[\int_{\gamma} f\, dz - \int_{\gamma} f\, d\bar{z}\right]$

Corollary:

$$\int_{-\gamma} f(z)\,|dz| = \int_{\gamma} f(z)\,|dz|$$

$$\int_{-\gamma} f(z)\,|dz| = \int_{-\gamma} f(z(-t))\,|dz(-t)|$$

$$= \int_{-b}^{-a} f(z(-t))\,|z'(t)(-dt)|$$

$$= \int_{-b}^{-a} f(z(-t))\,|z'(-t)|\,dt$$

Put, $t' = t$ $\qquad t = -b \Rightarrow t' = b$

$dt' = -dt$ $\qquad t = -a \Rightarrow t' = a$

$$= \int_{b}^{a} f(z(t'))\,|z'(t')|\,(-dt')$$

$$= -\int_{a}^{b} f(z(t'))\,|z'(t')|\,dt'$$

$$= \int_{a}^{b} f(z(t))\,|z'(t)|\,dt'$$

$$= \int_{a}^{b} f(z(t))\,|z'(t)|\,dt$$

$$= \int_{a}^{b} f(z(t))\,|dz(t')|$$

$$= \int_{a}^{b} f(z)\,|d(z)|$$

$$= \int_{\gamma} f(z)\,dz \quad \text{——} \quad ①$$

Note:

when, $f(z)=1$

$$① \Rightarrow \int_{\gamma} |dz| = \int_{\gamma} |dz| = \int_{\gamma} ds$$

where, s is the arc length

Eg:-

let γ be a circle

$$z = z(t) = a+\rho e^{it};\ 0 \le t \le 2\pi$$

$$\therefore \int_{\gamma} |dz| = \int_{0}^{2\pi} |d(z(t))| = \int_{0}^{2\pi} |z'(t)dt|$$

$$= \int_{0}^{2\pi} |i\rho e^{it}| dt$$

$$= \int_{0}^{2\pi} |i||e||e^{it}| dt$$

$$= \int_{0}^{2\pi} (1)(1)\rho\, dt$$

$$= \rho(t)_{0}^{2\pi} = e(2\pi-0)$$

$$= 2\pi e$$

$\int_{\gamma} |dz| = 2\pi e$ = circumference of a circle with radius 'ρ'

Defn: Rectifiable arcs

The length of an arc can also be defined has the least upper bound of all sums

$$|z(t_1)-z(t_0)| + |z(t_2-z(t_1)| + |z(t_3)-z(t_2)| + \dots$$

~~Wh~~ $$+|z(t_n)-z(t_{n-1})|$$

where; $a=t_0<t_1<t_2<t_3 \dots\dots <t_{n-1}<t_n=b$

If this least upper ~~a~~bound is finite ~~we~~ say that arc is rectifiable.

Note:-

Picesurse differentiable arcs are rectifiable.

Defn: Bounded variation:

If $z(t_k) = x(t_k) + iy(t_k)$ and

$z(t_{k-1}) = x(t_{k-1}) + iy(t_{k-1})$

Then

$$\sum_{k=1}^{n} z(t_k) - z(t_{k-1}) = \sum_{k=1}^{n} [x(t_k) - x(t_{k-1})] + i\sum_{k=1}^{n} [y(t_k) - y(t_{k-1})]$$

If $\sum_{i=1}^{n} [x(t_k) - x(t_{k-1})]$ and

$\sum_{k=1}^{n} [y(t_k) - y(t_{k-1})]$ are bounded

then we say that the function $x(t)$ and $y(t)$ are bounded variation.

Theorem:-1

An arc $z=z(t)$ is rectifiable if and only i real and imaginary parts of $z(t)$ are bounded variation

Necessary condition.

Assume that,

An arc $z(t)$ is rectifiable.

ie, $\sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$ is finite,

$\Rightarrow \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$ is bounded

We know that

$z(t) = x(t) + iy(t)$

$z(t_k) = x(t_k) + iy(t_k)$

$z(t_{k-1}) = x(t_{k-1}) + y(t_{k-1})$

$\Rightarrow z(t_k) - z(t_{k-1}) = x(t_k) + iy(t_k) - [x(t_{k-1}) + iy(t_{k-1})]$

$= x(t_k) + iy(t_k) - x(t_{k-1}) - iy(t_{k-1})$

$\therefore \sum_{k=1}^{n} [z(t_k) - z(t_{k-1})] = \sum_{k=1}^{n} [x(t_k) - x(t_{k-1})] + i \sum_{k=1}^{n} [y(t_k) - y(t_{k-1})$

$\Rightarrow \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})| \leq \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$

from ① $\Rightarrow \sum |x(t_k) - x(t_{k-1})|$ is bounded

|||ly

$\sum |y(t_k) - y(t_{k-1})|$ is bounded

i.e.) The real $[x(t)]$ and imagenary parts $[y(t)]$ of $z(t)$ are bounded variation.

sufficient condition.

Assume that

The real and imagenary parts of $z(t)$ are bounded variation.

i.e.) $\sum_{k=1}^{n} |x(t_k) - x(t_{k-1})|$ is bounded.

& $\sum_{k=1}^{n} |y(t_k) - y(t_{k-1})|$ is bounded $\Bigg\} \to ①$

$\Rightarrow z(t_k) - z(t_{k-1}) = [x(t_k) - x(t_{k-1})] + i[y(t_k) - y(t_{k-1})$

$\Rightarrow |z(t_k) - z(t_{k-1})| = |x(t_k) - x(t_{k-1}) + iy[(t_k) - y(t_{k-1})$

$\leq |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})|$

$\leq \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})| + \sum_{k=1}^{n} |y(t_k) - y(t_{k-1})|$

using ① $\sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$ is bounded

$\therefore$ $z(t)$ is rectifiable. arc.

Definition : line integral as functions of arcs.

The general line integral of the form

$$\int_\gamma p\,dx + q\,dy \quad \text{where, } p = \frac{\partial f}{\partial x} \; \& \; q = \frac{\partial f}{\partial y}$$

are often studied as functions of the arc γ its assume that p & q are defined and continuous in Ω

Note:

i) $\int p\,dx + q\,dy$ depends only on end points of 'γ' on a region Ω

ii) $\int_\gamma p\,dx + q\,dy = 0 \Leftrightarrow$ 'γ' is closed.

$\therefore$ Fundamental theorem of calculus

statement.

The line integral $\int p\,dx + q\,dy$ defined on depends only on the end points of 'γ' $\Leftrightarrow$ There exist a function $U(x,y)$ in 'Ω' with the partial derivatives $\frac{\partial U}{\partial x} = p$ & $\frac{\partial U}{\partial y} = q$

Assume that

There exist a function $U(x,y)$ in 'Ω' with the partial derivaties $\frac{\partial u}{\partial x} = p$ & $\frac{\partial U}{\partial y} = q$

$$\Rightarrow \int_{\gamma} p\,dx + q\,dy = \int_{\gamma} \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy$$

$$= \int_a^b \frac{\partial u}{\partial x}\, x'(t)\,dt + \left(\frac{\partial u}{\partial y}\right) y'(t)\,dt$$

[where $x = x(t)$ & $y = y(t)$]

$$= \int_a^b \left[\frac{\partial u}{\partial x}\, x'(t) + \frac{\partial u}{\partial y}\, y'(t)\right] dt$$

$$= \int_a^b \frac{d}{dt}\left[u(x(t)), y(t))\right] dt$$

$$= \left[u(x(t), y(t))\right]_a^b$$

$$\int_{\gamma} p\,dx + q\,dy = \left[u[x(a), y(b)] - u[x(a), y(a)]\right]$$

Hence the value of integrals depends only ~~ont~~ on the end points.

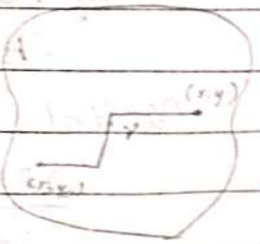
conversely

Let us assume that

the value of integral $\int_{\gamma} p\,dx + qy$ depends only on the end points.

To Prove. There exist a function $u(x,y)$ in 'Ω' with the partial derivaties $\frac{\partial u}{\partial x} = p$ & $\frac{\partial u}{\partial y} = q$

Let (x_0, y_0) be any fixed point in Ω and (x,y) be any point in Ω joint this two point by a polygons with arc γ in Ω



Parameter & obtain $U(x,y) = \int^{x} pdx + qdy$ (the l. limit of integral is irrevant.)

from this expression. $\frac{\partial u}{\partial x} = p$

whose sides are in Ω parallel to the co-ordinates axis in fig.

in same way by choosing last seg. is vertical we S.T. $\frac{\partial u}{\partial y} = q$

$$\text{let}, \; U(x,y) = \int_{\gamma} pdx + qdy$$

since, The value of $\int_{\gamma} pdx + qdy$ depends only on the end points if we choose the last segment of γ to horizonal

we can keep 'y' as constant and let 'x' varying with out changing the other segment

[The lower Limit of the integral is irrelavant]

by choosing the last segmant of 'x' vertically

$$\frac{\partial u}{\partial y} = q(x,y)$$

w.r.t y. $\int a(x,y)dy + const$

$\partial u/\partial x = p$

Note:-

(i) Exact differential.

If we write $du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right) dy$ and to say that the expression $(pdx + qdy)$ is exact differentiable

Thus, The integral ~~different~~ depends only on the end Points iff The integral is exact

ii) when, $f(z)\,dz = f(z)\,dx + if(z)\,dy$ on exact differential

sol:

by the exact differential

If a function $f(z) \in \Omega$ with Partial derivatives

$$\frac{\partial F(x)}{\partial x} = f(z) \; ; \; \frac{\partial F(z)}{\partial y} = if(z)$$

$$\Rightarrow \frac{\partial F(x)}{\partial x} = \frac{1}{i}\frac{\partial F(y)}{\partial y} = f(z)$$

$$\frac{\partial F(x)}{\partial x} = -i\frac{\partial F(y)}{\partial y}$$

which ⇒ that satisfy C-R equation.
and F(z) is analytic with derivatives f(z)

∴ f(z) dz is exact iff F(z) is analytic with derivative f(z)

Theorem: 3

The integral $\int_{\gamma} f\,dz$ with continuous function 'f' depends only on the end point of 'γ' iff 'f' is derivative of an analytic function in Ω [(ie) F'(z) = f(z)]

let $\int_{\gamma} f(z)\,dz = \int_{\gamma} f(z)\,[dx + i dy]$

$$= \int_{\gamma} (f(z)dx + if(z)dy)$$

By a known theorem:

'$\int_{\gamma} f(z)\,dz$ depends only on the end points of γ iff there exist a function F(z) defined on Ω

such that, $\frac{\partial F}{\partial x} = f(z)$; $\frac{\partial F}{\partial y} = if(z)$

$$\Leftrightarrow \frac{\partial F}{\partial z} = \frac{1}{i}\frac{\partial F}{\partial y} \quad (= f(z))$$

$$\Leftrightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

$$\Leftrightarrow \frac{\partial(u+iv)}{\partial x} = \frac{-i\partial(u+iv)}{\partial y}$$

$$\Leftrightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$\Leftrightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \; ; \; \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$\Leftrightarrow$ 'F' satisfies the C-R equation.

$\Leftrightarrow F(z)$ is analytic in Ω $f(z) = \frac{\partial F}{\partial x}$

$$= \frac{\partial(u+iv)}{\partial x}$$

$$f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= u_x + iv_x$$

$$\therefore f(z) = F'(z)$$

Note:

i) let $f(z)$ be a continuous complex valued function defined on a region 'Ω' Then $\int f(z)\,dz = 0$ for every closed curve 'γ' lying in 'Ω' then $\exists$ an analytic function such that $F'(z)$ & $f(z)$ in Ω

ii) $\int_\gamma (z-a)^n f(z) = 0$ for every closed curve 'γ' provided that integer 'n' is greater then '0'
Since, $(z-a)^n$ is the derivative of $(z-a)^{n+1}/n+1$ a function which is analytic in the whole plane is $\int_\gamma (z-a)^n dz = 0$

iii) consider a circle 'c' with centre 'a' and $z = a + \rho e^{it}$; $0 \le t \le 2\pi$

Then, $\int_c dz/_{z-a} = \int_0^{2\pi} \frac{\rho(ie^{it})}{\rho e^{it}} dt$

$= \int_0^{2\pi} (i) dt = i(t)_0^{2\pi}$

$\int_c \frac{dz}{z-a)} = 2\pi i$

1 Find $\int_\gamma x dz$ where 'γ' is the directed line segment from 0 to $(1+i)$

we know that

The eqn of line is

$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$

$\frac{x-0}{0-1} = \frac{y-0}{0-1}$

$\frac{x}{-1} = \frac{y}{-1}$

$x = y$

$\Rightarrow z = x + iy$

$\Rightarrow z = x + i(x)$

$\Rightarrow z = (1+i)x$

$dz = (1+i)dx$

on 'x' 'x' varies from 0 to 1

$\int x dz = \int_0^1 x(1+i)dx$

$$= (1+i)\left[x^2/2\right]_0^1$$

$$= (1+i)\left(\tfrac{1}{2}\right)$$

$$\int_\gamma x\,dx = \frac{(1+i)}{2}$$

Find $\int_{|z|=r} x\,dz$

let $z(t) = re^{it}$; $0 \leq t \leq 2\pi$

$$dz = re^{it}(i)dt$$

$$x = Re(z) = Re(re^{it}) = r\cos t$$

$$\int_{|z|=r} x\,dz = \int_0^{2\pi} (r\cos t)((ie^{it})(r))\,dt$$

$$= i(r^2)\int_0^{2\pi} (\cos t)(\cos t + i\sin t)\,dt$$

$$= ir^2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2} + \frac{i\sin 2t}{2}\right)dt$$

$$= ir^2\left[\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)_0^{2\pi} + \frac{i}{2}\left(\frac{-\cos 2t}{2}\right)_0^{2\pi}\right]$$

$$= ir^2(\pi)$$

Theorem: 4

Cauchy theorem. (or) cauchy's theorem for a rectangle.

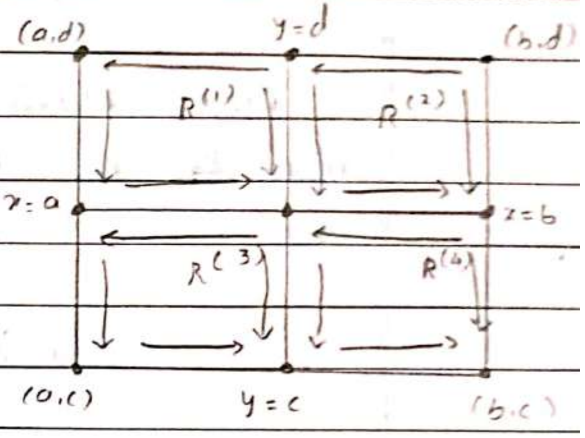
statement:-

If the function $f(z)$ is analytic on 'R'
Then $\int_{\partial R} f(z)\, dz = 0$

Let R be the rectangle formed by the lines $a \le x \le b$ & $c \le y \le d$

we divide the rectangle R into ~~four~~ equal points ~~parts~~

By bisciting the sides and call the indi~~tual~~vidual rectangle $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$



And let $\partial R^{(1)}, \partial R^{(2)}, \partial R^{(3)}$ and $\partial R^{(4)}$ denotes the boundary of this rectangles.

ie) $\partial R = \partial R^{(1)} + \partial R^{(2)} + \partial R^{(3)} + \partial R^{(4)}$

$$\Rightarrow \int_{\partial R} f(z)\, dz = \int_{\partial R_1} f(z)\, dz + \int_{\partial R_2} f(z)\, dz + \int_{\partial R_3} f(z)\, dz + \int_{\partial R_4} f(z)\, dz$$

let $\eta(R) = \int_{\partial R} f(z) \cdot dz$

$$\therefore \boxed{\eta(R) = \int_{\partial R_1} f(z)\, dz + \int_{\partial R_2} f(z)\, dz + \int_{\partial R_3} f(z)\, dz + \int_{\partial R_4} f(z)\, dz} \quad \text{—①}$$

For the integrals over the common sides cancel each other

For atleast one of the rectangle $R^{(k)}$

$$R^{(K)} \{K = 1, 2, 3, 4)$$

we get

$$\Rightarrow |\eta(R^{k})| \geq \frac{1}{4} |\eta(R)|$$

we denotes rectangle by $R^{(1)}$

ie) $|\eta(R_1)| \geq \frac{1}{4} |\eta(R_1)|$

Then again R_1 divided into 4 equal Part. Then R_2 be the rectangle of R_1 such that

$$|\eta(R_2)| \geq \frac{1}{4}, 2| \eta(R_1)|$$

This process can be repeted then final we get the sequence of nested rectangles $R \supset R_1 \supset R_2 \supset R_3 \supset \ldots$

with the property

$$|\eta(R_n)| \geq \frac{1}{4} |\eta(R_{n-1})|$$

and so that.

$$|\eta(R_n)| \geq \frac{1}{4} |\eta(R_{n-1})| \ \&$$

$$|\eta(R_n)| \geq \frac{1}{4^2} |\eta(R_{n-2})| \ \&$$

$$|\eta(R_n)| \geq \frac{1}{4^3} |\eta(R_{n-3})|$$

$$\vdots$$

and Hence $|\eta(R_n)| \geq \frac{1}{4^n}|\eta(R)|$ — ②

Since the rectangle we get each one is contained in Preceding one and those The area goes to zero as $n \to \infty$

There exist a Point which is common to all rectangle

let the point be $z^* \in R$

ie) R_n contained in $|z-z^*| < \epsilon$ as $n \to \infty$ choose δ small enough then $f(z)$ is defined and analytic in $|z-z^*| < \delta$

since, $f(z)$ is analytic at z^*

$$\Rightarrow f'(z^*) = \lim_{z \to z^*}\left[\frac{f(z)-f(z^*)}{z-z^*}\right]$$

for any given $\epsilon > 0$

$\exists$ a $\delta > 0$ such that ($\ni$)

$$\Rightarrow \left|\frac{f(z)-f(z^*)}{z-z^*} - f'(z^*)\right| < \epsilon \; \forall \; |z-z^*| < \delta$$

$$\Rightarrow \left|f(z)-f(z^*) - f'(z^*)(z-z^*)\right| < \epsilon |z-z^*| \quad \text{— ③}$$

$$\forall \; |z-z^*| < \delta$$

now,

$$\int_{\partial R_n} [f(z)-f(z^*)-f'(z^*)(z-z^*)]\,dz = \int_{\partial R_n} f(z)\,dz - f(z^*)\int_{\partial R_n} dz -$$

$$f'(z^*)\int_{\partial R_n} z\,dz + f'(z^*)(z^*)\int_{\partial R_n} dz$$

[∵ 1 & z are the derivatives of analytic fun. z & $\frac{z^2}{2}$ res. & R_n is closed]

[Since $\int_{\partial R_n} (1)\, dz = 0$ because [illegible] is the derivatives of $f(z) = z$ which is analytic and

$\int_{\partial R_n} z\, dz = 0$ because z is the derivatives of

$f(z) = \frac{z^2}{2}$ which is analytic]

$\therefore \int_{\partial R_n} dz = 0 ;\ \int_{\partial R_n} z\, dz = 0$

$$= \int_{\partial R_n} f(z)\, dz - 0 - 0 + 0$$

$$= \eta(R_n)$$

$$\int_{\partial R_n} [f(z) - f(z^*) - f'(z^*)(z - z^*)]\, dz = \eta(R_n)$$

$$\Rightarrow |\eta(R_n)| = \int_{\partial R_n} |f(z) - f(z^*) - (z - z^*) f(z^*)|\, |dz|$$

$$\therefore |\eta(R_n)| \leq \int_{\partial R_n} \epsilon\, |z - z^*|\, |dz| \quad (\text{by ③})$$

Now, $|z - z^*|$ is almost equal to the diagonal 'd_n' of R_n and 'L_n' denote the length of perimeter R_n

Then. $|\eta(R_n)| \leq \epsilon\, d_n \int_{\partial R_n} |dz|$

$$\therefore \boxed{|\eta(R_n)| \leq \epsilon\, d_n (L_n)}$$

where, d & L denotes the diagonal & perimeter of the original rectangle 'R'.

Then.

$$d_n = \frac{d}{2^n} \; ; \; L_n = \frac{L}{2^n}$$

$$\therefore |\eta(R_n)| \le \epsilon\left(\frac{d}{2^n}\right)\left(\frac{L}{2^n}\right)$$

$$\Rightarrow |\eta(R_n)| \le \epsilon\left(\frac{dL}{4^n}\right)$$

comparing ②

$$\Rightarrow \frac{1}{4^n}|\eta(R_n)| \le |\eta(R_n)| \le \epsilon \frac{dL}{4^n}$$

$$\therefore |\eta(R_n)| \le \epsilon dL$$

since,
'ϵ' is arbitrary

$$\therefore \eta(R) = 0$$

ie) $\int_{\partial R} f(z)\,dz = 0$

Theorem:5
State and prove Gowrset lemma.

Statement:
let $f(z)$ be a analytic on the set R' [region Ω] obtain from the rectangle R by omitting a finite number of interior points ξ_i and If $\lim_{z \to \xi_i} (z-\xi_i) f(z) = 0$ then $\int_{\partial R} f(z)\,dz = 0$

proof

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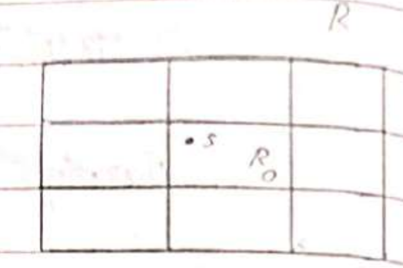
It is enough to consider the case one exceptional point s in R at and which $f(z)$ is not analytic

we divide the rectangle R into 9 rectangle as in figure



let R_0 be the rectangle with centre around the point 's:

Applying cauchy theorem for rectangle

$$\Rightarrow \int_{\partial R} f(z)dz = \int_{\partial R_0} f(z)dz \quad ①$$

now, Given that,

$$\lim_{z\to s} (z-s)f(z) = 0$$

$$\Rightarrow |z-s||f(z)| < \epsilon$$

$$\therefore \boxed{|f(z)| < \frac{\epsilon}{|z-s|}} \text{ in } R_0 \quad ②$$

$$① \Rightarrow \left|\int_{\partial R} f(z)\, dz\right| = \left|\int_{\partial R_0} f(z)\, dz\right|$$

$$< \int_{\partial R_0} |f(z)dz|$$

$$\leq \int_{\partial R_0} |f(z)|\, |dz|$$

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$$< \frac{\epsilon}{|z-s|} \int_{\partial R_0} |dz| \quad (\text{by } ②)$$

$$< \epsilon \int_{\partial R_0} \frac{|dz|}{|z-s|} = (8\epsilon)$$

$$< \epsilon \frac{4a}{(a/2)}$$

$$< \epsilon (4a)\left(\frac{2}{a}\right)$$

$$< 8\epsilon$$

$$\epsilon \frac{1}{a} 8a = 8\epsilon$$

$$< \epsilon$$

$$\left| \int_{\partial R} f(z)\, dz \right| < \epsilon$$

if 'R_0' is a square with centre 's'

$$\therefore \int_{\partial R} f(z)\, dz = 0$$

$|z-s| \geq a$, $z \in \partial R_0$ where '$2a$' is a side of R_0

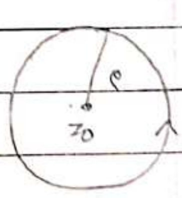
$\frac{1}{z-a} \leq \frac{1}{a}$

$\int_{\partial R_0} |dz| =$ length of $\partial R_0 = 8a$

Define: open (or) circular disc

An open disc is the region given by $|z-z_0| < \rho$ such regions are denoted by Δ

ie) $\Delta = |z-z_0| < \rho$

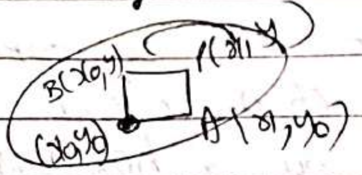


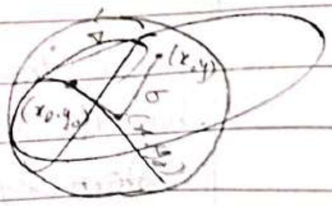
Theorem:-6

Cauchy's theorem for circular disc

statement:

If $f(z)$ is anaytic in an open disc Δ then $\int_{\gamma} f(z)\, dz = 0$ ∀ closed curve γ





Proof

let Δ be a open circular disc with centre (x_0, y_0) and radius 'r'

let $f(z)$ be an analytic function in Δ

let (x, y) be any point in Δ

let as define $F(z) = \int_{\sigma} f(z)\, dz$

where σ is consisting of horizontal line segment from (x_0, y_0) to (x, y_0) and the vertical line segment from (x, y_0) to (x, y)

$$\therefore \frac{\partial F}{\partial z} = f(z) \longrightarrow ①$$

$$\& \ \frac{\partial F}{\partial y} = i f(z) \longrightarrow ②$$

$$\Rightarrow F(z) = \int_{y} f(z)\, d(x+iy) \quad (z = x+iy)$$

$$= \int_{y} f(z)\, dx + i \int f(z)\, dy$$

$$= 0 + i \int_{y} f(z)\, dy$$

$$\Rightarrow \frac{\partial F}{\partial y} = i f(z)$$

(or)

$$\boxed{-i \frac{\partial F}{\partial y} = f(z)} \ — ③$$

on the other hand σ can be replaced by the path consisting of a vertical ~~system~~ segment followed by a horizontal

$$\Rightarrow f(z) - \int_{\chi} f(z)\, dz$$

$$= \int_{\chi} f(z)\, d(x+iy)$$

$$= \int_{\chi} f(z)\, dx + i\int_{\chi} f(z)\, dy$$

$$= \int_{\chi} f(z)\, dx + i(0)$$

$$= \int_{\chi} f(z)\, dx$$

$$\therefore \frac{\partial F(x)}{\partial x} = f(z) \longrightarrow ④.$$

③ & ④

$$\boxed{\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}}$$

which is the complex form of C-R equation for $F = u + iv$ also $f(z)$ is continuous

$$\Rightarrow \frac{\partial F}{\partial x} \text{ and } \frac{\partial F}{\partial y} \text{ is also continuous}$$

hence, $f(z)$ is anaytic in Δ

such that

$$f(z) = F'(z) = \frac{\partial F}{\partial x}$$

$$\Rightarrow f(z)\, dz = F'(z)\, dz$$

An exact differential hence the values of $\int_{\gamma} f(z)\, dz$ depend on the end point

Since γ is closed curve

$$\therefore \int_{\gamma} f(z)\, dz = 0 \quad \forall \gamma$$ [By cauchy the for rectangle]

Cauchy Integral formula :-

Index of a point with respect to closed curve :-

Define : winding number (or) Index of a point w.r.t. to γ

we define the index of a point with respect to 'γ' by a winding equation

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\, dz$$

(or)

$$n(\gamma, a)(2\pi i) = \int_{\gamma} \frac{f(z)\, dz}{z-a}$$

Theorem :-

If the piecewise differentiable closed curve γ does not passes through the point 'a' Then the value of $\int_{\gamma} \frac{f(z)}{z-a}\, dz$ is a multiple of $2\pi i$

proof

let γ be a piecewise differentiable closed curve is closed ~~by~~ curve is given by

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$z = z(t) \quad ; \alpha \leq t \leq \beta$

$\Rightarrow \frac{dz}{dt} = z'(t)$

$\Rightarrow dz = z'(t)\, dt$

now, $z(\alpha) = z(\beta)$

$$\int_{\gamma} \frac{dz}{z-a} = \int_{\alpha}^{\beta} \frac{z'(t)\, dt}{z(t)-a}$$

let

$$h(t) = \int_{\alpha}^{t} \frac{z'(t)}{z(t)-a}\, dt \quad — ①$$

since 'a' doesn't lie on 'γ'

$\therefore$ h(t) is defined by a continuous and closed curve 'γ' on the interval (α, β)

$$① \Rightarrow h(t) = \{\log[z(t)-a]\}_{\alpha}^{t}$$

$$= \log(z(t)-a) - \log(z(\alpha)-a)$$

$$= \log\left[\frac{z(t)-a}{z(\alpha)-a}\right]$$

$$\Rightarrow e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a} \Rightarrow e^{h(\beta)} = \frac{z(\beta)-a}{z(\alpha)-a}$$

[since, 'γ' is closed curve

$\Rightarrow z(\alpha) = z(\beta)$

$$e^{h(\beta)} = 1$$

$\therefore h(\beta) = 2n\pi i$ ∀ n is any integer

$$\therefore \int_{\alpha}^{\beta} \frac{z'(t)}{z(t)-a}\, dt = n(2\pi i)$$

hence, $\int_{\gamma} \frac{dz}{z-a}$ is a multiple of $(2\pi i)$

Problem:-1

Find the winding no. of a point 'a' w.r.to the circle 'c' with centre at 'a'

soln

we know that

The ~~wind~~ winding no. of 'γ'

$$n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a} \text{ —— ①}$$

~~sol~~

Given that, 'γ' is a circle [γ = r

$$\Rightarrow |z-a| = r$$

$$\Rightarrow z-a = re^{i\theta}$$

$$z = a + re^{i\theta}$$

$$\frac{dz}{d\theta} = 0 + r(ie^{i\theta})$$

$$\therefore dz = r(ie^{i\theta})d\theta$$

$$① \Rightarrow n(c, a) = \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{r(ie^{i\theta})d\theta}{a + re^{i\theta} - a}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} (i)\, d\theta$$

$$= \frac{1}{2\pi i} i\, [\theta]_0^{2\pi} = \frac{1}{2\pi}[2\pi - 0]$$

$n(C, a) = 1$

Property (1) :-

$$n(-\gamma, a) = -n(\gamma, a)$$

let, $n(\gamma, a) = {}^{1}/_{2\pi i}\int_\gamma \frac{dz}{z-a}$

$$\Rightarrow n(-\gamma, a) = \frac{1}{2\pi i}\int_{(-\gamma)} \frac{dz}{z-\alpha}$$

W.k.t

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)\, dz$$

$$\therefore n(-\gamma, a) = \frac{-1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

$$n(-\gamma, a) = -n(\gamma, a)$$

Property :2

If 'γ' lies inside a circle Then $n(\gamma, a) = 0$ for all point 'a' outside of same circle

Proof

Given 'γ' lies inside of a circle

ie, A point set 'γ' is closed in the open-set Δ

let $({}^{1}/_{z-a})$ is analytic in the disc

By Cauchy theorem for a circle is

$$\int_{\gamma} \frac{dz}{z-a} = 0$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a} = 0$$

$$n(\gamma, a) = 0$$

Note:

i) $n(\gamma, a) = n(\gamma, b) \quad (\forall\, a, b \in \gamma)$

ii) $n(\gamma, a) = \begin{Bmatrix} 0 & \text{if } a \notin \gamma \\ 1 & \text{if } a \in \gamma \end{Bmatrix}$

The Integral formula

Theorem: 8

suppose that $f(z)$ is analytic in an open disc Δ and let 'γ' be a closed curve in Δ, for any point 'a' not in γ Then $n(\gamma, a)\, f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}\, dz$

where $n(\gamma, a)$ is index of a point with respect to 'γ'

let $f(z)$ be a analytic function In a open disc Δ

let 'γ' be a closed curve in Δ

and let 'a' be a point not in γ

consider

$$F(z) = F'(z) = \frac{f(z) - f(a)}{z-a}$$

This function is analytic everywhere in Δ except at $z=a$

$$\Rightarrow \lim_{z \to a} [(z-a)F(z)] = \lim_{z \to a} \left[(z-a)\left(\frac{f(z)-f(a)}{z-a}\right)\right]$$

$$= \lim_{z \to a} f(z) - f(a)$$

$$= f(a) - f(a)$$

$$\therefore \lim_{z \to a} (z-a)F(z) = 0$$

we know that,

let 'f' be anaytic in 'Δ' obtained by omitting a finite no. of points δ_j from an open disc Δ

If $f(z)$ satisfies the condition

$$\lim_{z \to \delta_j} (z - \delta_j) f(z) = 0 \ \forall j$$

Then $\int_\gamma f(z)\,dz = 0$ for any closed curve $\delta \in \Delta$

$$\Rightarrow \int_\gamma F(z)\,dz = 0$$

$$\Rightarrow \int_\gamma \left[\frac{F(z) - f(a)}{z-a}\right] dz = 0$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-a}\,dz - \int_\gamma \frac{f(a)\,dz}{z-a} = 0$$

$$\Rightarrow \int_\gamma \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(a)\,dz}{z-a}$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)\,dz}{z-a} = \frac{1}{2\pi i}\int_{\gamma} \frac{f(a)\,dz}{z-a}$$

$$= f(a)\left[\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}\right]$$

$$\therefore \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)\,dz}{z-a} = n(\gamma, a) f(a)$$

Theorem: 9

Cauchy Integral formula.

statement.

If f(z) is analytic in an open disc Δ and If 'γ' be any closed curve Δ, then for any 'z' inside 'γ' we get

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)\,d\xi}{\xi - z}$$

1st we prove theorem ⑧

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)\,dz}{z-a} = n(\gamma, a) f(a) \quad ①$$

If the point 'a' is inside 'γ'

$\therefore n(\gamma, a) = 1$

$$① \Rightarrow f(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)\,dz}{z-a}$$

hence, (sub $a = z$; $z = \delta$

$$\therefore f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\delta)\, d\delta}{\delta - z}$$

~~proof~~ Problem.

find $\int_{|z|=1} \frac{e^z dz}{z}$

Given $\int_{|z|=1} \frac{e^z dz}{z}$ — ①

w.k. that C.I.F is

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)\, dz}{z-a} \quad — ②$$

comparing ① & ②

$\gamma : |z| = 1$

$f(z) = e^z$; $a = 0$

$f(a) = e^a$

$f(0) = e^0 = 1$

② ⇒ $2\pi i\, f(a) = \int_{\gamma} \frac{f(z)\, dz}{z-a}$

$$2\pi i\, f(0) = \int_{|z|=1} \frac{e^z dz}{z-0}$$

$$\therefore \int_{|z|=1} \frac{e^z dz}{z} = 2\pi i$$

2) $\int_{|z|=2} \frac{dz}{z^2+1}$

sol

$$\int_{|z|=2} \frac{dz}{z^2+1} \quad \text{—} ①$$

W.K.T the C.I.F is

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)\,dz}{z-a} \quad \text{—} ②$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$\times (z+i)(z-i)$

$$1 = A(z-i) + B(z+i).$$

sub $z = -i \Rightarrow 1 = A(-i-i) + B(0)$

$1 = A(-2i)$ $\therefore A = \frac{-1}{2i}$

sub $z = i \Rightarrow 1 = 0 + B(i+i)$

$1 = B(2i)$ $\therefore B = \frac{1}{2i}$

$$\therefore \frac{1}{z^2+1} = \frac{-1}{2i(z+i)} + \frac{1}{2i(z-i}$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$① \Rightarrow \int_{|z|=2} \frac{dz}{z^2+1} = \frac{1}{2i}\int_{|z|=2}\left[\frac{1}{z-i} - \frac{1}{z+1}\right]dz$$

$$= \frac{1}{2i}\int_{|z|=2}\left(\frac{1}{z-i}\right)dz - \frac{1}{2i}\int_{|z|=2}\left(\frac{1}{z+i}\right.$$

$a = i$; $f(z) = 1$

$f(a) = f(i) = 1$

$a = -i$

$f(z) = 1$

$f(a) = f(-i) = 1$

$$\Rightarrow 2\pi i\, f(a) = \int_{|z|=2} \frac{dz}{z-i}$$

$$\Rightarrow 2\pi i(1) = I_1$$

$$2\pi i f(a) = \int_{|z|=2} dz/_{z+i}$$

$$2\pi i(1) = I_2$$

$$② \Rightarrow \int_{|z|=2} \frac{dz}{z^2+1} = \frac{1}{2i}\left[2\pi i - 2\pi i\right]$$

$$= 0$$

Find $\displaystyle\int_{|z|=1} \frac{\cos z\, dz}{z(z-4)}$

Given

$$\int_{|z|=1} \left(\frac{\cos z\, dz}{z(z-4}\right) dz \text{ — } ①$$

$$\rightarrow \int_{|z|=1} \left(\frac{\frac{\cos z}{z-4}}{z}\right) dz \text{ - } ①$$

We know that

$$\int_{\gamma} f(z)\, dz = 2\pi i\, f(a) \longrightarrow ②$$

Comparing ① & ②

$$f(z) = \frac{\cos z}{z-4}, \quad \gamma : |z| = 1$$

& $a = 0$

$$② \rightarrow \int_{|z|=1} \frac{\cos dz}{z(z-4)} = 2\pi i\, f(a)$$

$$= 2\pi i\, f(0)$$

$$= 2\pi i \left(\frac{\cos 0}{0-4}\right)$$

$$= 2\pi i\left(-1/4\right)$$

$$= -\pi i/2,$$

Higher order derivatives

Theorem: 10

If $\phi(\gamma)$ is continuous on 'γ' then the function

$$F_n'(z) = \int_\gamma \frac{\phi(\gamma)}{(\gamma - z)^n} d\gamma$$ is analytic in each of the region determine bn 'γ' and its derivative is

$$F_n'(z) = n F_{n+1}(z)$$

Proof:-

Given

The $\phi(\delta)$ is continuous on 'γ'

and $F_n(z) = \int_\gamma \frac{\phi(\delta)}{(\delta - z)^n} d\delta$

For $n=1$

$$\Rightarrow F_1(z) = \int_\gamma \frac{\phi(\delta)}{(\delta - z)} d\delta$$

let us Prove the theorem by induction method.

let, $z_0 \notin \gamma$

Step: 1

To Prove $F_1(z)$ is is continuous at $z_0 \notin$

let 'z_0' be any Point not on 'γ'
and
let $|z - z_0| < \epsilon$ be a neighbourhood. of 'z_0'

which doesn't meet 'γ'.

$$\rightarrow |\delta - z| > \epsilon/2 \ \forall \ \delta \in \gamma$$

and

$$|\delta - z_0| > \epsilon/2 \ \forall \ \delta \in \gamma$$

$$F_1(z) - F_2(z_0) = \int_\gamma \frac{\phi(\delta)d\delta}{\delta - z} - \int_\gamma \frac{\phi(\delta)d\delta}{\delta - z_0}$$

$$= \int_\gamma \phi(\delta)\left[\frac{1}{\delta - z} - \frac{1}{\delta - z_0}\right] d\delta$$

$$= \int_\gamma \phi(\delta)\left[\frac{\delta - z_0 - \delta + z}{(\delta - z)(\delta - z_0)}\right] d\delta$$

$$= (z - z_0)\int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta - z)(\delta - z_0)} \quad — ①$$

$$|f_1(z) - f_1(z_0)| \le |z - z_0| \int_\gamma \frac{|\phi(\delta)|\,|d\delta|}{|\delta - z|\,|\delta - z_0|}$$

$$< |z - z_0|\left(\frac{4}{\epsilon^2}\right)\int_\gamma |\phi(\delta)|\,|d\delta|$$

$$< |z - z_0| \frac{4}{\epsilon^2} M \int_\gamma |d\delta| \quad \left(\text{let } \max|\phi(\gamma)| = 0\right.$$

$$< |z - z_0| \frac{4}{\epsilon^2} M(L)$$

$$< (\epsilon)\left(\frac{4}{\epsilon^2}\right) ML$$

$$< 4/\epsilon\,(ML)$$

$$< 4\epsilon ML$$

$\therefore |F_1(z) - f_1(z_0)| < \epsilon$

$\Rightarrow |F_1(z) - F_1(z_0)| = 0$

$\therefore F_1(z) = F_1(z_0)$

step(ii) ①⇒

$$\lim_{z \to z_0} \left[\frac{F_1(z) - F_1(z_0)}{z - z_0}\right] = \lim_{z \to z_0} \left[\int_\gamma \frac{\phi(\delta)\, d\delta}{(\delta - z_0)(\delta - z}\right.$$

$$= \lim_{z \to 0} \left[\int_\gamma \frac{\phi(\delta)\, d\delta}{(\delta - z_0)(\delta - z_0)}\right]$$

$$F_1'(z_0) = \int_\gamma \frac{\phi(\delta)\, d\delta}{(\delta - z_0)^2}$$

$\Rightarrow F_1'(z_0) = F_2(z_0)$

since, 'z_0' is arbitary

$F_1'(z) = F_2(z)$ —— ②

let us P.T the general case by the method of induction

if, $F_{n-1}'(z) = (n-1) F_n'(z)$

and $F_n(z)$ is analytic

TO Prove:-

$F_n(z)$ is continuous at 'z_0'

let $F_n(z) - F_n(z_0) = \int_\gamma \frac{\phi(\delta)\, d\delta}{(\delta - z)^n} - \int_\gamma \frac{\phi(\delta)}{(\delta - z_0)^n} d\delta$

$$= \int_\gamma \frac{(\delta-z_0)\phi(\delta)}{(\delta-z_0)(\delta-z)^n} - \int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta-z_0)^n}$$

$$= \int_\gamma \frac{(\delta-z) + (z-z_0)\,\phi(\delta)\,d\delta}{(\delta-z_0)(\delta-z)^n} - \int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta-z_0)^n}$$

$$= \int_\gamma \frac{(\delta-z)\,\phi(\delta)}{(\delta-z)^n(\delta-z_0)} + (z-z_0)\int_\gamma \frac{\phi(\gamma)\,d\delta}{(\delta-z)^n(\delta-z_0)} - \int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta-z_0)^n}$$

$$= \int_\gamma \frac{\phi(\gamma)\,d\delta}{(\delta-z)^{n-1}(\delta-z_0)} + (z-z_0)\int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta-z)^n(\delta-z_0)} - \int_\gamma \frac{\phi(\delta)\,df}{(\delta-z_0)^n} \;\text{———}\; \textcircled{7}$$

$$\lim_{z\to z_0}\left[F_n(z)-F_n(z_0)\right] = \lim_{z\to z_0}\left[\int_\gamma \frac{\phi(\gamma)\,d\delta}{(\delta-z)^{n-1}(\delta-z_0)} + (z-z_0)\int_\gamma \frac{\phi(\delta)\,d\delta}{(\delta-z)^n(\delta-z_0} - \int_\gamma \frac{\phi(\delta)}{(\delta-z_0)^n}\,df\right]$$

$$= \int_\gamma \frac{\phi(\delta)}{(\delta-z_0)^n}\,d\delta - \int_\gamma \frac{\Phi(\delta)}{(\delta-z_0)^n}\,df$$

$$\lim_{z\to 0} F_n(z) - F_n(z_0) = 0$$

$\lim_{z\to z_0} F_n(z) = F_n(z_0)$ => $F_n(z)$ is analytic at 'z_0'

o Prove: $F_n'(z) = n F_{n+1}(z)$

② $\Rightarrow$ $$\frac{F_n(z)-F_n(z_0)}{z-z_0} = \left(\frac{1}{z-z_0}\right)\int_\gamma \frac{\phi(\delta)\, df}{(\delta-z)^{n-1}(\delta-z_0)}$$

$$+ \left(\frac{z-z_0}{z-z_0}\right)\int_\gamma \frac{\phi(\delta)\, df}{(\delta-z)^{n}(\delta-z_0)} - \left(\frac{1}{z-z_0}\right)\int_\gamma \frac{\phi(\delta)}{(\delta-z_0)^{n}}$$

$$\therefore \frac{F_n(z)-F_n(z_0)}{z-z_0} = \frac{\psi_{n-1}(z)-\psi_{n-1}(z_0)}{z-z_0} \cancel{+\psi_n(z_0)} \underset{-}{\cancel{+}} \frac{1}{z-z_0}\int_\gamma \frac{\phi(\delta)}{(\delta-z_0}$$

where, $\psi_{n-1}(z) =$

$\psi_{n-1}(z_0) =$

& $\psi_n(z) =$

$$\Rightarrow \lim_{z\to z_0}\left[\frac{f_n(z)-f_n(z_0)}{z-z_0}\right] = \lim_{z\to z_0}\left[\frac{\psi_{n-1}(z) - \psi_{n-1}(z_0)}{z-z_0}\right]$$

$$- \lim_{z\to z_0}\left[\int_\gamma \frac{\phi(\delta)\, d\delta}{(\delta-z)^{n}(z-z_0)}\right]$$

By induction hypothesis

$$\psi_{n-1}'(z_0) = (n-1(\psi_n(z_0))$$

$$\lim_{z\to z_0}\left(\cancel{\psi_{n-1}(z) = (n-1)\psi_n(z_0)}\right.$$

$$\lim_{z\to z_0}\left(\frac{\psi_{n-1}(z)-\psi_{n-1}(z_0)}{z-z_0}\right)=(n-1)\psi_n(z_0)$$

$$① \Rightarrow F_n'(z_0)=(n-1)\psi_n(z_0)+\psi_n(z_0)$$

$$=n\psi_n(z_0)-\psi_n(z_0)+\psi_n(z_0)$$

$$F_n'(z_0)=n\psi_n(z_0)$$

$$\Rightarrow F_n'(z_0)=n\int_{\gamma}\frac{\phi(\delta)\,d\delta}{(\delta-z_0)^{n+1}}$$

$$F_n'(z_0)=nF_{n+1}(z_0)$$

Since, 'z_0' is arbitrary

hence $F_n'(z)=nf_{n+1}(z)$

Local propaties of analytic functions.

Theorem: 11

Cauchy Integral formula for n^{th} derivative

Statement:

let $f(z)$ be a analytic function in a region Ω. let 'γ' be a simple closed curve in Ω and if 'z' is any Point inside 'γ' then.

$$f^n(z)=\frac{n!}{2\pi i}\int_{\gamma}\frac{f(\delta)}{(z-\delta)^{n+1}}\,d\delta$$

we know that
The cauchy Integral formula is

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\delta)\, d\delta}{z-\delta} \quad \text{---} ①$$

also we know that,

if $\phi(\delta)$ is continuous on 'x'

$$\Rightarrow F_n(z) = \int_{\gamma} \frac{\phi(\delta)\, d\delta}{(\delta - z)^n} \quad \text{---} ②$$

is analytic theorem

$$F_n'(z) = n F_{n+1}(z) \quad \text{---} ③$$

$$① \Rightarrow f(z) = \frac{1}{2\pi i} f_1(z) \quad (\text{by } ②)$$

diff. w. r. to (z)

$$\Rightarrow f'(z) = \frac{1}{2\pi i} F_1'(z)$$

$$\Rightarrow F'(z) = \frac{1}{2\pi i} F_2(z) \quad (\text{by } ③)$$

agined w. r. to 'z'

$$\Rightarrow f''(z) = \frac{1}{2\pi i} F_2'(z) = \frac{1}{2\pi i} (2) F_3(z)$$

diff. w. r. to 'z'

$$\therefore F'''(z) = \frac{3!}{2\pi i} F_4(z)$$

$$\vdots$$

$$F^n(z) = \frac{n!}{2\pi i} F_{n+1}(z)$$

hence,

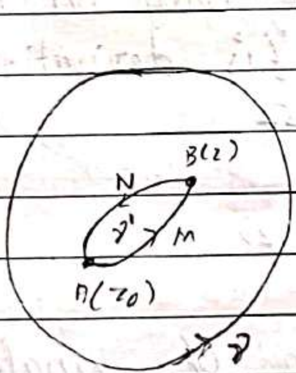
$$F^n(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\delta)}{(z-\delta)^{n+1}} d\delta$$

Theorem: n

Morera's theorem (or) converse of cauchy theorem.

statement:

If $f(z)$ is defined and continuous in a region Ω & if $\int_\gamma f(z)dz = 0$ $\forall$ closed curve γ in Ω Then $f(z)$ is analytic



Let z_0 be a fixed Point and z be a variable point in 'Ω'

Let 'γ'' be a closed curve (AMBNA)

Then by hypothesis

$$\int_{\gamma'} f(z)\,dz = 0$$

$$\Rightarrow \int_{AMBNA} f(z)\,dz = 0$$

$$\Rightarrow \int_{AMB} f(z)\,dz + \int_{BNA} f(z)\,dz = 0$$

$$\Rightarrow \int_{AMB} f(z)\,dz = -\int_{BNA} f(z)\,dz$$

$$= \int_{ANB} f(z)\,dz$$

ie) the integral depends only on the end point

we know that

"The integral $\int_{\gamma} f(z)\,dz$ with continuous function only on the end Point of ~~a~~ 'γ' iff If 'f' is derivative on the analytic function Ω

ie) $F'(z) = f(z)$

The derivative of analytic function is analytic

Hence $f(z)$ is analytic in 'Ω'

Theorem: 13

~~Cauy.~~ Cauchy's Estimate (or) Cauchy's Inequality

Statement:-

let 'f(z)' be an analytic inside and on the circle $|z-a| = r$ and let 'M' be the maximum value of $|f(z)|$ on the circle Then $|f^n(a)| \leq \frac{M(n!)}{r^n}$

Proof:-

we know that

The cauchy integral formula for the n^{th} derivative

$$\rightarrow f^n(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow |f^n(a)| = \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-a)^{n+1}}\right|$$

$$\leq \left|\frac{n!}{2\pi i}\right| \left|\int_\gamma \frac{f(z)\,dz}{(z-a)^{n+1}}\right|$$

$$< \frac{n!}{2\pi i}\int_\gamma \frac{|f(z)|\,|dz|}{|z-a|^{n+1}}$$

$$\leq \frac{n!}{2\pi}\int_\gamma \frac{M\,|dz|}{r^{n+1}}$$

$$\leq \frac{n!}{2\pi}\left[\frac{M}{r^{n+1}}\right]\int_\gamma |dz$$

$$\leq \frac{n!}{2\pi}\left(\frac{M}{\gamma^{n+1}}\right)(2\pi\gamma)$$

$$|f^n(a)| \leq \frac{n!M}{\gamma^n}$$

$$f^n(a) \leq \frac{M(n!)}{\gamma^n}$$

Liovillie's theorem.

statement '4 show that

A function which is analytic and bounded in the ~~well~~ whole plane must reduce to a constant

Proof:

By cauchy inequality

$$= |f^n(a)| \leq \frac{M(n!)}{\gamma^n}$$

when $n=1$

$$\Rightarrow |f'(a)| \leq \frac{M(1!)}{\gamma^{(1)}}$$

$$\therefore |f'(a)| \leq \frac{M}{\gamma}$$

by taking $\gamma \to \infty$

Then the above inequality becomes.

$$f'(a) = 0 \ \forall \ a \in \Omega$$

$\therefore$ f(a) is constant

hence, f(z) is constant.

Fundamental theorem of algebra.

Statement : 15

~~Every polynomi~~

~~degree Every~~ . Polynomial / of / Positive

Every Polynomial of Positive degree must have a root

proof:

let $P(z) > 0$ be a polynomial of degree > 0

let $P(z)$ has no root

$\Rightarrow P(z) \neq 0$

let, $P(z) = \frac{1}{f(z)}$

Then, $f(z)$ is analytic in the whole plane.

we know that

$P(z) \to 0$ as $|z| \to \infty$

$\therefore f(z) = \frac{1}{P(z)}$ is bounded

since, by Liouville's theorem,

$f(z)$ is constant.

$\Rightarrow \frac{1}{P(z)}$ is constant.

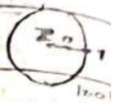
which is $\Rightarrow$ E to ①

$\therefore$ our assumption is wrong

Hence,
The equation $P(z)=0$ must have one root.

Removable singularities:

Singualarities (or) singular Point:-



A point 'a' is called a singular point (or) singularities of a function $f(z)$ if, $f(z)$ is not analytic at 'a' & F is analyti at some point of every disk $|z-a|<r$

Theorem: 16

If $f(z)$ is analytic in the region Ω obtained by omitting a point 'a' from the region Ω a necessary and sufficiant condi that there exist an analytic function Ω which consider a function $f(z)$ in Ω is that $\lim_{z\to a}(z-a)f(z)=0$ The extented function is uniquely determined

Proof:

Necessary condition:-

Let $\phi(z)$ be a analytic function in such that

$$\boxed{\phi(z)=f(z)}; \quad z\neq 0 \quad — ①$$

since $\phi(z)$ is analytic

$$\Rightarrow \lim_{z\to a}(z-a)\,f(z)=0 \quad — ②$$

PAGE NO

DATE :

$$\Rightarrow \lim_{z \to a} (z-a)\,\phi(z) = 0 \quad \text{③}$$

let, if possible, $\psi(z)$ is analytic function such that

$$\boxed{\psi(z) = f(z)} ; \quad z \neq 0$$

② & ③

$$\Rightarrow \phi(z) - \psi(z) = f(z) - f(z)$$
$$= 0$$

$$\therefore \boxed{\phi(z) = \psi(z)} \quad ; \quad z \neq a$$

Hence, the entended function is unique.

Sufficient condition:

we draw circle 'c' about 'a' so that 'c' lies completely with Ω Then by cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\delta)\, d\delta}{(\delta - z)} \quad ; \quad z \neq a$$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\int_c \frac{f(\delta)\, d\delta}{\delta - a} \quad [\text{'a' is inside 'c'}]$$

$$\text{let}, f(z) = \frac{f(z) - f(a)}{z - a} \quad ; \quad z \neq a$$

but, 'F' is not defined for 'a'

$$\Rightarrow \lim_{z \to a} (z-a)\,F(z) = \lim_{z \to a} [F(z) - f(a)]$$

Also

$$\lim_{z\to a} F(z) = \lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\right)$$

$$\boxed{\lim_{z\to a} F(z) = f'(a)} \quad : z = a$$

Hence there exist an analytic function which is equal to f(z) for $z \neq a$ and equal to f'(z) for $z = a$

Removable singularity

A point $z=a$ lying inside the region Ω is said to be a removable singularity of a function f(z) which is analytic except $z=a$

$$\lim_{z\to a} (z-a)\, f(z)$$

∴ Theorem: 17 (Taylor's theorem)

Statement:

If f(z) is analytic in a region Ω containg 'a' it is possible to write

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a$$

$$+ \frac{f^{n}(a)}{n!}(z-a)^n$$

where $F_n(z)$ ~~f(z)~~ is analytic in Ω and ~~$f_n(z)$=~~

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f(\delta)\, d\delta}{(\delta-a)^n(\delta-z)}$$

where, 'c' is any circle with centre 'a' & $c \in \Omega$

Proof:

let $f(z)$ is analytic in a region 'Ω' containing 'a'

let us define a analytic function.

$$f_1(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

if $z \neq a$

$$\Rightarrow f_1(z) = \frac{f(z)-f(a)}{z-a}$$

$$\Rightarrow (z-a) f_1(z) = f(z) - f(a)$$

$$\therefore \boxed{f_{\cancel{0}}(z) = (z-a) f_1(z) + f(a)} \quad \text{—— ①}$$

|||'y we define $f_2(z)$

$$f_2(z) = \begin{cases} \dfrac{f_1(z)-f_1(a)}{z-a} & \text{if } z \neq a \\ f_1'(a) & \text{if } z = a \end{cases}$$

if $z \neq a$

$$\Rightarrow f_2(z) = \frac{f_1(z)-f_1(a)}{z-a}$$

~~$\Rightarrow f_2$~~

$$\Rightarrow (z-a) f_2(z) = f_1(z) - f_1(a)$$

$$f_1(z) = (z-a)f_2(z) + f_1(a) \text{ —— ②}$$

Continuing in this way

$$f_2(z) = (z-a)f_3(z) - f_2(a) \text{ —— ③}$$

$$f_3(z) = (z-a)f_4(z) - f_3(a) \text{ —— ④}$$

$$\vdots$$

$$f_{n-1}(z) = (z-a) \cdot f_n(z) + f_{n-1}(a) \text{ —— ⑤}$$

From ① to ⑤

$$\Rightarrow f(z) = (z-a)f_1(z) + f(a)$$

$$= (z-a)\left[(z-a)f_2(z) + f_1(a)\right] + f(a)$$

$$= (z-a)^2\left[f_2(z)\right] + (z-a)f_1(a) + f(a)$$

$$\vdots$$

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \dots + (z-a)$$

$f_n($

Diff. 'k' times & sub $z = a$

$$\Rightarrow f^k(a) = k!\, f_k(a)$$

$$\Rightarrow f_k(a) = \frac{f^k(a)}{k!}$$

$$f(z) = f(a) + (z-a) f'(a) + (z-a)^2 \frac{f''(a)}{2!} + \dots + (z-a)^{n-1} \frac{f^{n-1}(z)}{(n-1)}$$

$$+ f_n(z) \cdot (z-a)^n$$

Also

$f_n(z)$ is analytic in Ω

let 'c' be a circle about a such that it lies in 'Ω'

Then by cauchy integral formula

$$f_n(z) = \frac{1}{2\pi i} \int_c \frac{f_n(\delta)\, d\delta}{(\delta - z)} \longrightarrow ⑦$$

by Taylor's series

$$f_n(\delta) = \frac{1}{(\delta - z)^n} \left[f(\delta) - f(a) - \frac{f'(a)(\delta - a)}{1!} + \dots + \frac{f^{n-1}(a)(\delta-a)^{n-1}}{(n-1)!} \right]$$

$$f_n(z) = \frac{1}{2\pi i} \int_c \frac{1}{(\delta-a)^n(\delta-z)} \left[f(\delta) - f(a) - \frac{f'(a)(\delta-a)'}{1!} + \dots + \frac{f^{n-1}(a)(\delta-a)^{n-1}}{(n-1)!} \right] d\delta$$

$$= \frac{1}{2\pi i} \int_c \frac{f(\delta)\, d\delta}{(\delta-a)^n(\delta-z)} - \frac{1}{2\pi i} \int_c \frac{f(a)\, d\delta}{(\delta-a)^n(\delta-z)} - \frac{1}{2\pi i} \int_c \frac{f'(a)\, d\delta}{(\delta-a)^{n-1}(\delta-z)}$$

$$- \frac{1}{2\pi i (n-1)!} \int_c \frac{f^{n-1}(a)\, d\delta}{(\delta-a)(\delta-z)}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(\delta)\,d\delta}{(\delta-a)^{n}(\delta-z)} - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{f^{n-k}(a)}{(n-k)!}\int_c \frac{d\delta}{(\delta-z)(\delta}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(\delta)\,d\delta}{(\delta-z)(\delta-a)^n} - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{f^{n-1}(a)}{(n-k)!}F_k(a)$$

$$\text{where } F_k(a) = \int_c \frac{d\delta}{(\delta-z)(\delta-a)^k} \quad — ①$$

sub $k=1$

$$F_1(a) = \int_c \frac{d\delta}{(\delta-z)(\delta-a)^1}$$

$$= \int_c \frac{1}{(z-a)}\left[\frac{1}{(\delta-z)} - \frac{1}{(\delta-a)}\right]d\delta$$

$$\Rightarrow F_1(a) = \frac{2\pi i}{z-a}\left[\frac{1}{2\pi i}\int_c \frac{d\delta}{(\delta-z)} - \frac{1}{2\pi i}\int_c \frac{d\delta}{(\delta-a)}\right]$$

$$= \frac{2\pi i}{z-a}\left[n(\gamma, z) - n(\gamma, a)\right]$$

Since $n(\gamma, z) = n(\gamma, a)$.

$$= \frac{2\pi i}{z-a}(0)$$

$$F_1(a) = 0$$

next

$$F_2(a) = \int_{\gamma} \frac{d\delta}{(\delta-z)(\delta-a)^2}$$

$$= F_1'(a) = 0$$

$$F_2(a) = 0$$

III'y

$$F_3(a) = F_4(a) = \cdots F_{n-1}(a) = 0$$

⑤=> $$f_n(z) = \frac{1}{2\pi i} \int_{c} \frac{f(\delta)}{(\delta-a)^n(\delta-z)} \, d\delta$$

Hence the proof.

Theorem: 16

If f(z) is analytic function f(a) and all its derivatives vanish then. f(z)=0.

let 'γ' be circle with centre 'a' and radius 'R'.

such that 'γ' is in 'Ω'

Then by Taylor's theorem, we get

$$f(z) = \sum (z-a)^n f_n(z) \quad \forall n$$

$$f(z) = (z-a)^n \frac{1}{2\pi i} \int_{c} \frac{f(\gamma)\, d\delta}{(\delta-a)^n(\delta-z)}$$

$$|f(z)| = \left|(z-a)^n \frac{1}{2\pi i} \int_{c} \frac{f(\delta)\, d\delta}{(\delta-a)^n(\delta-z)}\right|$$

$$\leq \frac{1}{2\pi}|z-a|^n \left| \int_C \frac{f(\delta)\,d\delta}{(\delta-a)^n(\delta-z)} \right|$$

$$\leq \frac{|z-a|^n}{2\pi} \max_{\delta \in \gamma} |f(\delta)| \left(\frac{1}{|\delta-a|^n |(\delta-a)-(z-a)|} \right)$$

$$\leq \frac{1}{2\pi}\left[\frac{|z-a|}{R}\right]^n \left[\frac{M}{R-|z-a|}\right] 2\pi R$$

since $\frac{|z-a|}{R} < 1$; $\left[\frac{|z-a|}{R}\right]^n \longrightarrow 0$ as $n \rightarrow$

Hence $\boxed{f(z)=0}$ $\forall$ z inside γ.

Zero of order 'n'

A function f(z) is analytic at 'a' is said to have a zero of order 'n'
If $\boxed{f(z) = (z-a)^n f_n(z)}$
Where $f_n(z)$ is analytic and $f_n(a) \neq$

Theorem : 19

Let a function f(z) be analytic at a point 'a' which is zero of order 'n' There is a neighbourhood of 'a' at which f(z) has no other zero's unless 'a' is identically zero.

(or)

A zero of an analytic function which is not identically zero (or) isolated.

Proof

Let the function f(z) have a zero of a order 'n' show that $f(z) = (z-a)^n f_n(z)$

where $f_n(z)$ is analytic and $f_n(a) \neq 0$

Since.

$f_n(z)$ is continuous at 'a'

for given $\epsilon > 0$

There exit a number of δ

such that

$|f_n(z) - f_n(a)| < \epsilon$ whenever $|z-a| < \delta$

if, $\epsilon = \frac{|f_n(a)|}{2}$

and

'δ_0' is the corresponding value of 'δ'

$\Rightarrow |f_n(z) - f_n(a)| < \frac{|f_n(a)|}{2}$ —— ①

if $f_n(z) \neq 0$ at any point in the neighbourhood of $|z-a| < \delta_0$

because if, $f_n(z) = 0$

① $\Rightarrow f_n(a) < \frac{|f_n(a)|}{2}$

which is $\Rightarrow\Leftarrow$

Hence

The zero of an analytic function which is not identically zero cor) isolated.

Meromorphic function.

→ A function $f(z)$ which is analytic in the region Ω ~~except~~ ~~expat~~ except for its poles is said to be meromorphic function.

Theorem: 20 [weierstrass theorem)

statement :-

An analytic function comes arbitrarily close to any complex value every neighbourhood of an essential singula

case (i)

let α be any number

$$\Rightarrow \lim_{z \to a} |z-a|^{\alpha} \cdot |f(z)-A| = \infty$$

Hence 'a' is not a essential singularity of $|f(z)-A|$

case (ii) let $B > 0$ be any no

$$\Rightarrow \lim_{z \to a} |z-a|^{B} |f(z)-A| = 0$$

$$\Rightarrow \lim_{z \to a} |z-a|^{\beta} |f(z)-A| \geq \lim_{z \to a} |z-a|^{B}$$

$$|f(z)-A| - \lim_{z \to a} |z-a|^{\beta}$$

$$\Rightarrow 0 \geq \lim_{z \to a} |z-a|^{\beta} |f(z)-A|$$

$$\therefore \lim_{z \to a} |z-a|^{B} |f(z)-A| = 0$$

Since,

$$\lim_{z \to a} |z-a|^{\beta} |A| = 0$$

Hence it would act to a essential as $f(z)$

Theorem-21

maximum principle (or) modulus principle

statement:

If $f(z)$ is defined & continuous on a closed, bounded set E, and analytic on the interior of E'. Then the maximum modulus of $f(z)$ on E is assumed on the boundary of E

proof:

Given that,

$f(z)$ be analytic in the closed & bounded set E

let 'z_0' be the interior point of E'

let 'γ' be a circle with centre at 'z_0' and radius 'r' which is contain in 'E'

$$\text{so, } |z-z_0| = r$$

$$\rightarrow z-z_0 = re^{i\theta}$$

$$z = z_0+re^{i\theta}$$

$$dz = r(i)e^{i\theta}\,d\theta$$

by cauchy Integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z).dz}{z-z_0}$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}(re^{i\theta})(i)\,d\theta$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \quad \text{—— ①}$$

ie) The value of the function at the centre 'z_0' is arithmetic mean of its value on the circle.

Also.

$$① \Rightarrow |f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right|$$

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)+re^{i\theta})|\,d\theta$$

If $f(z_0)$ is maximum than.

$$|f(z_0) + re^{i\theta}| \leq |f(z_0)| \forall \theta$$

If true by the continuity on the whole arc.

$$② \Rightarrow |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z)|\,d\theta$$

$$< \frac{1}{2\pi} |f(z)|(2\pi)$$

$$|f(z_0)| < |f(z))$$

which is ($\Rightarrow \Leftarrow$) contradiction

$\therefore$ $|f(z_0)| = |f(z)|$ $\forall$ z of all sufficient p

ie) $f(z)$ must reduce to constant.

$|z-z_0| = r$

Hence $|f(z)|$ must attain.

its maximum value on the boundary

Theorem-22

Schwartz's lemma (or) Inequality theo

statement.

If $f(z)$ is analytic for $|z|<1$ and satisfies the condition $|f(z)| \leq 1$ & $f(0)=0$ Then $|f(z)| \leq |z|$ & $|f'(0)| \leq 1$ if $|f(z)|=|z|$

or if $|f'(0)|=1$ for some $z \neq 0$, if $f(z)=cz$ with a cons absolute value let $f(z)$ be analytic $|z| < 1$

let $|f(z)| < 1$ & $f(0) = 0$

let us defined the function as followed

$$f_1(z) = \begin{cases} f(z) & ; z \neq 0 \\ f'(0) & ; z = 0 \end{cases}$$

$$|z| = r < 1$$

$$\Rightarrow |f_1(z)| = \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}$$

by maximum modulus theorem.

$$\Rightarrow |f_1(z)| \leq \frac{1}{r} \text{ on } 'c'$$

That. limit. as $r \to 1$

$$\Rightarrow |f_1(z)| \leq 1 \quad \text{—①}$$

$$|f(z)| \leq 1$$

$$\Rightarrow \left| \frac{f(z)}{z} \right| \leq 1$$

$$\therefore |f(z)| \leq |z|$$

① $\Rightarrow |f_1(z)| \leq 1 \quad \forall |z| < 1$ at $z = 0$

$$|f_1(0)| \leq 1$$

$$\Rightarrow |f'(0)| \leq 1$$

Next, if $|f(z)| = |z|$

then $\left| \frac{f(z)}{z} \right| = 1$

$$\Rightarrow |f_1(z)| = 1$$

Again by maximum principle the

$|f_1(z)|$ must attain its maximum value on the boundary (not at any interior point

ie $f_1(z) = c$

$\Rightarrow \frac{f(z)}{z} = c$ ——— ②

② $\Rightarrow \left|\frac{f(z)}{z}\right| = |c|$

$|f(z)| = |c||z|$ (ib $|c| = 1$

Hence,

$|f(z)| = |z|$

Unit - II
complex integration

The general form of cauthy theorem.

Define chain:-

let, $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_n$ form a subdivision of arc 'γ'

$$\int f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \ldots + \int_{\gamma_n} f\,dz$$

$\gamma_1 + \gamma_2 + \ldots \gamma_n$

Then the sum $(\gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_n)$ of arc is called a chain

cycle

A chain is a cycle If it can be represented as a sum of closed curve.

simply connected

A region is simply connected. If it is complement w.r.to the extended plane is connected

Eg:-

A disk, A half plan & ||'e of strip are connected.

Theorem: (1)

A region 'γ' is simply connected $\Leftarrow\Rightarrow n(\gamma, a) = 0$ $\forall$ cycles 'γ' in 'Ω' and all points 'a' not in 'Ω'.

proof.

$$n(\gamma, a) = \frac{1}{2\pi i}\int\frac{dz}{z-}$$

necessary condition:-

let 'Ω' be simply connected

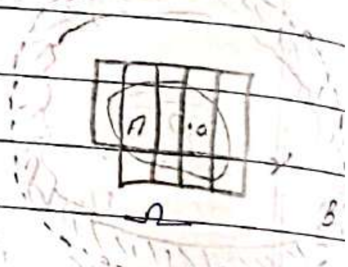
⇒ The complement of 'Ω' w.r. to the extended complex plane is connected.

let 'γ' be any cycle in 'Ω'

let 'a' be the point not in 'Ω'

Then 'a' lies in 'Ω^c' which continuo the point at infinity. 'a' lies in the unbounded region determined by 'γ'

$$\therefore \boxed{n(\gamma, a) = 0}$$



sufficient condition:-

if $n(\gamma, a) = 0$ ∀ cycle $\gamma \in \Omega$ & a ∉

To P.T. Ω is s.c.

first we assume that

'Ω' is not simply connected

by the definition of simply connected

⇒ Ω^c is not connected

⇒ Ω^c is separable.

let the complement of Ω can be represented as the union of two disjoint closed sets [ie] A∪B]

one of these sets containing infinity (B) and the other (A) is bounded

The sets A and B have a shortest distance $\delta > 0$ covers the whole plane with the squares 'Q' of side $(\delta/\sqrt{2})$

let $a \in A$ be a point

which is taken as the cent

The bounda

we If the square 'Q' as closed of the cycle $|\gamma = \sum_{j} \partial Q_j|$

Where the sum ranges over all squares 'Qj' in which a common Point with 'A'

Since, the length of the size of any 'Qj' $< \frac{\delta}{\sqrt{2}}$

'γ' does not meet at 'a'

Since any side which meets a common side of two squares is ~~[illegible]~~ included in the some direction as the size of two adjacent squares so that, the sum of ① it doesn't appear also 'γ' doesn't meet 'B'

$$\Rightarrow \gamma \notin A \cup B = \Omega$$

$$\Rightarrow \gamma \in \Omega$$

Since, 'Q' lies with in the region by $\gamma \neq 0$

which is ⇒⇐ to the hypotheses that 'Ω' is connected

Hence 'Ω' is connected

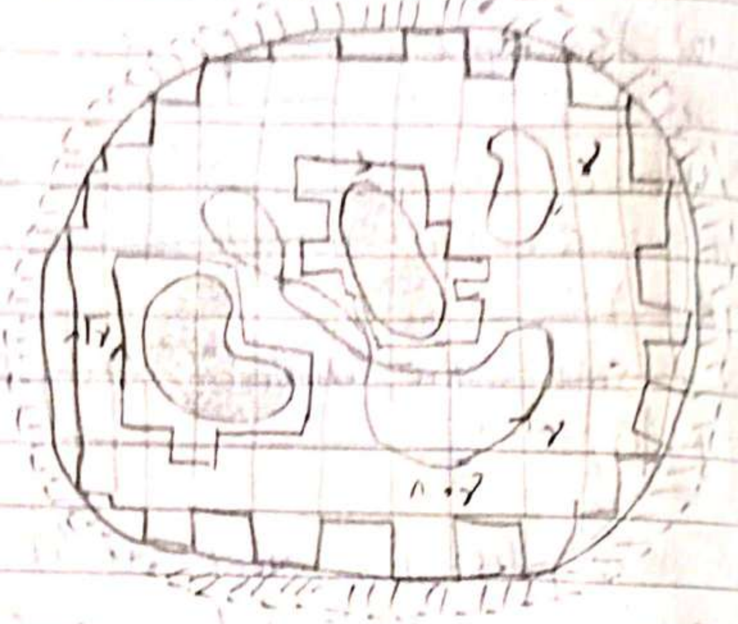
imp. 10m. Theorem(2) The general form of Cauchy theorem

Statement:

If f(z) is analytic in 'Ω'.

Then

$\int_{\gamma} f(z)\, dz = 0$ for every cycle 'γ' is which homologoes to zero in 'Ω'



case (i)

let Ω be bounded

Given, $f(z)$ is analytic in Ω

Given $\delta > 0$, we cover the plane by a net of sequares of sides 'δ' and we denoted by Q_j $\forall j \in J$

The closed sequares in the set which are contained in 'Ω'

since 'Ω' is bounded

The set 'j' is bounded and if 'δ' is very small its non-empty

The union of sequare 'Q_j' $\forall J$

consists of closed regions whose oriented boundaries make up the cycles

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j$$

clearly

Γ_δ is the sum of oriented line segments which are side of exactly 1 Q_j

let us denoted by Ω_δ is the interior of the union of $(\cup Q_j)$

let 'γ' be a cycles which is homologoes to zero in 'Ω'

choose 'δ' so that,
'γ' is contain in 'Ω_δ'

let 'δ' be the a point in $(\Omega - \Omega_\delta)$ it belong's to atleast one Q which is not in Q_j

There is a point in δ_0 in $Q \notin \Omega$ we can joint δ_0 by a line segment which lies in Q and therefore it doesn't meet (Ω_δ) [Since 'δ' is consider as a point set is contained in 'Ω_δ']

$$\therefore n(\gamma, \delta) = n(\gamma, \delta_0) = 0$$

in particular

$$n(\gamma, \delta) = 0 \quad \forall \text{ points } \delta \in \Gamma_\delta$$

If 'f' is analytic in 'Ω' and if 'z' lies in the interior of 'Q_{j_0}'

$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\delta)}{\delta - z}\, d\delta = \begin{cases} f(z) & \text{if, } j = j_0 \\ 0 & \text{if, } j \neq j_0 \end{cases}$$

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$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\delta)\,d\delta}{\delta - z}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\delta)\,d\delta}{\delta - z} \quad \forall z$$

$$\int_\gamma f(z)dz = \int_\gamma \left[\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\delta)\,d\delta}{\delta - z}\right]$$

[by changing the order of integration

$$\int_\gamma f(z)dz = \int_{\Gamma_\delta}\left[\frac{1}{2\pi i}\int_\gamma \frac{dz}{\delta - z}\right] f(\delta)\,d$$

$$= -\int_{\Gamma_\delta}\left[\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\delta}\right] f(\delta)$$

$$= \int_{\Gamma_\delta}\left[-n(\gamma,\delta)\right] f(\delta)\,d\delta$$

$$\therefore \int_\gamma f(z)\,dz = 0 \quad (n(\gamma, a) = 0)$$

case (ii)

let 'Ω' be unbounded also 'Ω^c' with a disc $|z| < R$ ('R' is large)

Any point 'a' in the complement of

ie $a \in \Omega^c$

Then it lies either in the Ω^c (or) lies outside the disc

In either case $n(\gamma, a) = 0$ so that,

$\gamma \sim 0 (\text{mod } '\Omega^c')$

now the proof is done under case (i) is applicable to 'Ω^c' and the theorem is true for unbounded 'Ω'

Hence by case (i) case (ii) This theorem is true orbitrary 'Ω'

Define: Homologes:-

A cycle is an open set 'Ω' is said to be homologes.

If $n(\gamma, a) = 0 \ \forall \ a \in \Omega^c$

$P(x,y)dx + Q(x,y)dy = 0$

Locally Exact differential

A differential $(Pdx + qdy)$ is called locally Exact in Ω if it is exact in some neighbourhood of each point in 'Ω'

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Theorem (3).

If $pdx + qdy$ is locally exact in 'Ω' then $\int pdx + qdy = 0 \ \forall$ cycle $\gamma \sim 0$ in Ω

$\frac{\partial u}{\partial x} = P$

$\frac{\partial u}{\partial y} = Q$

Proof:-

let The distance from 'γ' to 'Ω^c' is 'ρ'

$u = \int P dx$

let 'γ' is given by $z = z(t)$.

Here, $z(t)$ is uniformly continuous on $[a,b]$

let the closed interval of the length $< \delta$

This 'δ' is determied by $z(t) - z'(t) < \rho$

for $|t - t'| < \delta$

let γ_i (sub arcs) of γ associated with in the sub interval of $[a,b]$ each will lies with in a disc of radius which in 'Ω'

The end point of 'γ_i' can be σ_i consisting of horizondal and vertical segmant and lies completely with a disc

since $(Pdx+qdy)$ is exact in the disc

$$\Rightarrow \int_{\gamma_i} Pdx+qdy = \int_{\sigma_i} Pdx+qdy$$

$$\Rightarrow \int_{\Sigma\gamma_i} Pdx+qdy = \int_{\Sigma\sigma_i} Pdx+qdy$$

$$\Rightarrow \int_{\gamma} Pdx+qdy = \int_{\sigma} Pdx+qdy \quad \text{———} ①$$

$(\text{since}, \Sigma\gamma_i = \gamma, \Sigma\sigma_i = \sigma)$

In order to prove the theorem its sufficient to prove that $\int_{\sigma} Pdx+qdy$

Extending the segment of the polygon to infinite line.

we get, The plane divided into some finite rectangle R_i and some unbounded region R_j which may be recovered as infinite rectangles

let $a_i \in R_i$ and

$\sigma_0 = \sum_i n(\sigma, a_i) \partial R_i$ —— ② to a cycle

where the sum takes over all the rectangles
its clear that,

$$n(\partial R_i, a_k) = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$$

If, a_j' is the point in the interior of R_j'

$$② \Rightarrow n(\sigma_0, a_i) = \sum_i n(\sigma, a_i) n(\partial R_i, a_i)$$

$$\therefore n(\sigma_0 a_i) = n(\sigma, a_i) \rightarrow ③$$

also

$$n(\sigma_0, a_j') = \sum_i n(\sigma, a_i) n(\partial R_i, a_j')$$

$$\Rightarrow n(\sigma_0, a_j') = 0 \longrightarrow ④$$

$$\Rightarrow n(\sigma_0, a_r') = 0 \longrightarrow ⑤$$

because the (a_j') lies in the rectangle (R_j') which lies in the bounded region determined by 'σ'

from ③ ④ & ⑤

$$\Rightarrow n(\sigma - \sigma_0, a) = 0 \quad \forall a = a_j'$$

This shows that $\sigma = \sigma_0$ upto the segment that cancel each other

let, σ_{ik} be a common side two adjacent rectangles R_i & R_k

let the orientation to shoose R_i lies to the left of ~~of~~ 'σ_{ik}'

suppose the reduced expression $(\sigma-\sigma_0)$ contains ~~multiples~~ multiples of σ_{ik}. Then the cycles $[\Gamma=\sigma-\sigma_0-c\partial R_i]$ is a cycle contains (σ_{ik}). $\therefore a_i, a_k$ belongs to the same determined by $n(\Gamma, a_i) = n(\Gamma, a_k)$

$$\Rightarrow n(\sigma-\sigma_0-c\partial R_i; a_i) = n(\sigma-\sigma_0-c\partial R_i$$

$$\Rightarrow n(\sigma-\sigma_0, a_i) - c\,n(\partial R_i, a_i) = n(\sigma-\sigma_0, a_k) - c\,n(\partial R_i, a_k)$$

$$0 - c(1) = 0 - c(0)$$

$$c = 0$$

$(\sigma-\sigma_0)$ will not contain any com of any two finite rectangles

Thus, every side of a finite rectangles occurs with co-efficient zero $(\sigma-\sigma_1)$.

$$\Rightarrow \sigma = \sum_i n(\sigma, a_i)\,\partial R_i$$

next we will prove that all R_i whose co-efficient $n(\sigma, a_i)$ is different from '0' are in 'Ω'

if 'a' (finite) in the closed rectan R_i not in 'Ω'

Then $n(\gamma, a) = 0$

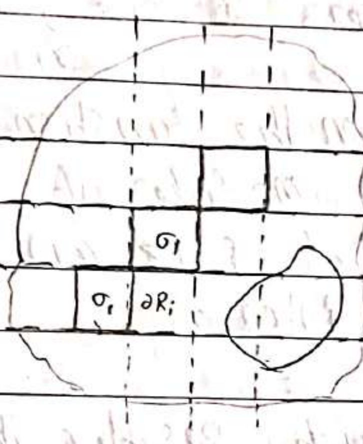
[since $\sigma \sim 0$ and Ω]

The line segment between a & a_i does intersect 'σ' & hence $n(\sigma, a_i) = n(\sigma, a)$

$$\therefore \int p\,dx + q\,dy = \int p\,dx + q\,dy = \sum n($$

$$= \sum n(\sigma, a_i) \int_{\partial R_i} Pdx + qdy$$

$\therefore \int_{\sigma} Pdx + qdy = 0 \ \forall$ Roctangle $R_i \in \Omega$ by local exactness.



Define: multiple connected region.

A region which is not simply connected is called multiple connected region.

Finite connectivity and Infinite connectivity

A region 'Ω' is called finite connectivity (n) in the complement of ~~the~~ Ω as exactly n components and infinite connectivity if the complement as infinitely many components.

homology basis $\quad n(\gamma, a) \neq 0$

If the cycles $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}$ its cannot be homology to zero.
unless all vanish (c_i) is zero

The circumferences of the cycles $\gamma_1, \gamma_2 \ldots \gamma_{n-1}$ are called the homology basis for the region Ω

modules of Periodicity

An analytic function $f(z)$ in Ω

$$\int_{\gamma} f(z)dz = c_1\int_{\gamma_1} f(z)dz + c_3\int_{\gamma_2} f(z)dz + \cdots + c_{n-1}\int_{\gamma_{n-1}} f(z)\,dz$$

The numbers of $P_i = \int_{\gamma_1} f\,dz$

depends only on the function not on 'γ' They are called modules of Periodicity of the differential $f\,dz$ (or) The Periods of the indefinite integral

v.i **Theorem: 4, (cauchy Residue theorem)**

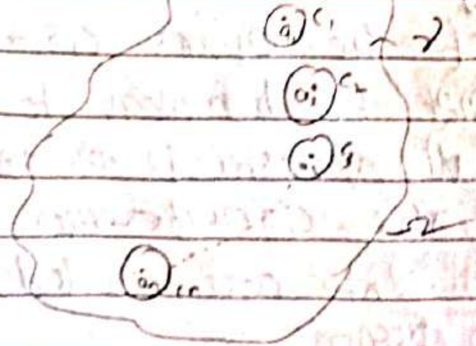
Statement:

let $f(z)$ be analytic except for isolated singularities a_j in a region 'Ω' Then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_{j} n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z$$

for any cycle 'γ' which is homologes to in 'Ω' and don't posses through any of the Point 'a_j'

proof:



case (i)

let There is only a finite no. of isolated singularities $(a_1, a_2 \ldots a_n)$ in 'Ω' The region is obtained by excluding The point a_j. will be denoted by ~~Ω_j~~ 'Ω''

Let 'γ' be the cycle in 'Ω'' which is homologes to zero with respect to 'Ω'

let c_j be the circle with ~~to~~ centre a_j, and radius $r > 0$

consider the cycle

$$\Gamma = \gamma - \sum_j n(\gamma, a_j)\, c_j$$

Now,

$$n(\Gamma, a_k) = n(\gamma, a_k) - n\Big(\sum_j n(\gamma, a_j) c_j, a_k\Big)$$

$$= n(\gamma, a_k) - \sum_j n(\gamma, a_j)\, n(c_j, a_k)$$

(since, $n(c_j, a_k)$ for $j = k$)

$$= n(\gamma, a_k) - n(\gamma, a_k)\, n(c_k, a_k)$$

$$= n(\gamma, a_k) - n(\gamma, a_k)(1)$$

$$n(\Gamma, a_k) = 0$$

let, $a \notin \Omega$

$$n(\Gamma, a) = n(\gamma, a) - ~~n~~\Big(\sum n(\gamma, a_j)\Big) c_j, a)$$

$$= n(\gamma, a) - \sum n(\gamma, a_j)\, n(c_j, a)$$

$$= 0 \quad \Big[\text{since } \gamma \sim 0 \ (\text{mod } \Omega)\ \&\ n(c_j, a) = 0\Big]$$

∴ Γ is a cycle in 'Ω' which is homologes to '0' (mod Ω) which doesn't Passes through, a_js

we know that
[Given form of cauchy's theorem]

$$\int_{\Gamma} f(z)dz = 0$$

$$\Rightarrow \int f(z)\,dz = 0$$

$$\gamma - \sum n(\gamma, a_j)\, c_j$$

$$\Rightarrow \int_{\gamma} f(z)\,dz = \int f(z)\,dz \sum_j n(\gamma, a_j$$

$$= \sum_j n(\gamma, a_j) \int_{c_j} f(z)\,dz$$

$$\therefore \frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j)\frac{1}{2\pi i}\int_{c_j} f(z)\,d$$

$$= \sum_j n(\gamma, a_j)\left(\frac{1}{2\pi i}\right)(P_j)$$

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum n(\gamma, a_j)\, R_j \qquad \left[R_j = \frac{P_j}{2\pi}\right.$$

case(ii)

If there are ~~man~~ infinity many isolated singularity in 'Ω'

The set of all point 'a' $n(\gamma,a)=0$ is open and contains all points outside of a larges circle

The complement is consequently is compact set and hence it cannot contains more than a finite number of isolated point a_j

$\therefore n(\gamma,a_j) \neq 0$

only for a finite no. of singularities and for this a_j case (i) applied

Hence the proof

Rouche's theorem:5

Statement:

let γ be the homologes to 'o' and such that $n(\gamma,z)$ is either 0 (or) 1 for any point z not on 'γ'. If f(z) and g(z) are analytic in 'Ω' ~~sta~~ satisfies the following inequaly

$$|f(z)-g(z)| < |f(z)| \text{ on } \gamma$$

Then, f(z) and g(z) have the same no. of zeros enclosed by 'γ'

Proof

If $f(z)=0$ on γ

Then the given equality

$$\Rightarrow |0 - g(z)| < 0$$

$$\therefore |g(z)| < 0$$

which is impossible

If, $g(z) = 0$ on 'γ'

Then the given inequality

$$\Rightarrow |f(z) - 0| < |f(z)|$$

$$\Rightarrow |f(z)| < |f(z)|$$

which is also impossible.

hence, neigther $f(z)$ nor $g(z)$ has zero on 'γ'

also, $|f(z) - g(z)| < |f(z)|$ on 'γ'

$$\Rightarrow \left|1 - \frac{g(z)}{f(z)}\right| < 1 \text{ on '}\gamma\text{'}$$

sub $F(z) = g(z)/f(z)$

$$|1 - F(z)| < 1 \text{ on '}\gamma\text{'}$$

are thus contained in open disk of centre and radius is one

let ~~F~~ Γ is the image of the cycle 'γ' with respect to the map $w = f(z)$

If Γ lies in a disk which doesn't contain the origin then $n(\Gamma, a)$

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{dw}{w-a} = 0 \Rightarrow \int_{\Gamma} \frac{F'(z)}{F(z)} = 0.$$

1 Find the no. of zero's of $F(z) = z^7 - 4z^5 + z^2 - 1$ which lies inside the circle $|z| = 1$

sol:

Given

$$F(z) = z^7 - 4z^5 + z^2 - 1 \quad \text{——} \; ①$$

The co-efficiant of $F(z)$ is $1, -4, 1, -1$

here (-4) is the max. absolute value

$$\text{let} \quad g(z) = -4z^5$$

$$\Rightarrow h(z) = z^7 + z^2 - 1$$

$$\therefore f(z) = g(z) + h(z)$$

on $|z| = 1$

$$\Rightarrow |g(z)| = |-4z^5|$$

$$= |-4(1)^5|$$

$$|g(z)| = 4$$

next

$$|h(z)| = |z^7 + z^2 - 1|$$

$$= |1 + 1 + 1|$$

$$|h(z)| = 3$$

$|g(z)| > |h(z)|$, $f(z)$ & $g(z)$ have same no. of zero's inside 'c'

$g(z)$ has zero of order 5

hence,

It has 5 zero inside 'c'

$\therefore f(z)$ has also 5 zero's inside.

2 $f(z) = z^8 - 5z^5 + 1$; $|z| = 1$

Given

$$f(z) = z^8 - 5z^5 + 1 \quad \text{——} \; ①$$

The co-efficient of $f(z)$ is $1, -5, 1$

here (-5) is max. absolute value

$$\text{let} \quad g(z) = -5z^5$$

$$\Rightarrow h(z) = z^8 + 1$$

$$f(z) = g(z) + h(z)$$

on $|z| = 1$

$$\Rightarrow |g(z)| = |-5z^5|$$

$$= |-5(1)^5|$$

$$|g(z)| = 5$$

next.

$$|h(z)| = |z^8 + 1|$$

$$= |1+1|$$

$$|h(z)| = 2$$

$$\Rightarrow |g(z)| > |h(z)|$$

$f(z)$ & $g(z)$ have same no. of zero's inside 'C'

$g(z)$ has zero of order 5

hence

It has 5 zero inside 'C'

$f(z)$ has also 5 zero's inside

3 $z^6 + z^3 - 6z + 9 = F(z)$

Given

$$f(z) = z^6 + z^3 - 6z + 9$$

The co-efficient of $f(z)$ is $1, 1, -6, 9$

here 9 is the max absolute value

let $g(z) = 9(z^0)$

$$\Rightarrow h(z) = z^6 + z^3 - 6z$$

$$F(z) = g(z) + h(z)$$

$|z| = 1$

$$\Rightarrow |g(z)| = |9| = 9$$

next $|h(z)| = |z^6 + z^3 - 6|$

$$= |1 + 1 + 6|$$

$$= 8$$

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$\Rightarrow |g(z)| > |h(z)|$

f(z) & g(z) same no. of zero's inside 'c'

g(z) hase zero of order

hence.

It has no zero inside 'c'

$\therefore$ f(z) has no zero's inside 'c'

4 $f(z) = z^8 - 5z^5 - 2z + 1$ $|z| = 1$

Given $f(z) = z^8 - 5z^5 - 2z + 1$

The co-efficien f(z) is 1, -5, -2, 1

here (-5) is the max absolute vale

let $g(z) = -5z^5$

$h(z) = z^8 - 2z + 1$

$\therefore f(z) = g(z) + h(z)$

on $|z| = 1$

$|g(z)| = |-5z^5| = |-5(1)^2| = 5$

$|h(z)| = |z^8 - 2z + 1| = |1+2+1| = 4$

$\Rightarrow |g(z)| > |h(z)|$

f(z) & g(z) have same no. of zeros inside 'c'

g(z) has zero of order 5

hence It has 5 zero inside 'c'

f(z) has also 5 zero's inside.

5 $f(z) = z^4 - 5z + 1$ which lies in the annulus $1 < |z| < 2$.

Given

$$f(z) = z^4 - 5z + 1 \text{ —— } ①$$

let C_1 & C_2 be the two circle

ie) $|z| = 1$; $|z| = 2$

on $|z| = 1$

$g_1(z) = -5z$

$|g_1(z)| = |-5z| = 5$

$|h_1(z)| = |z^4+1| = |1+1| = 2$

$|g_1(z)| > |h_1(z)|$

$g_1(z)$ & $h_1(z)$ have the same no. of zeros inside 'c'

let $g_1(z)$ has zero of order 1

∴ $f(z)$ has one zero's inside 'c_1'

on $|z| = 2$

$g_2(z) = z^4$, $h_2(z) = -5z+1$

$|g_2(z)| = |z|^4 = z^4 = 16$

& $|h_2(z)| = |-5z+1|$

$= |5(z)+1|$

$= 1$

$|g_2(z)| > |h_2(z)|$

but $g_2(z)$ has zero of order 4

$f(z)$ has 4 zero's inside 'c_2'

The no. of zero's of $f(z)$ in $1 < |z| < 2$ has $(4-1) = 3$

The calculus of residues

The residue of f(z) at an isolated singularity 'a' is the unique complex no. R which makes $(f(z)-R)/(z-a)$ The derivative of a single valued analytic function in an $0<|z-a|<\delta$

∴ The residue at $z=a$ at $f(z)$ is denoted by $\boxed{R = \operatorname*{Res}_{z=a} f(z)}$

case (i)

let $z=a$ be a single pole of $f(z)$

$$\therefore \operatorname*{Res}_{z=a} = \lim_{z\to a} (z-a) f(z)$$

case (ii)

If $f(z)$ ~~is of the~~ has a pole of order m

$$\therefore \operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

case (iii)

If $f(z)$ of the form $f(z) = \frac{\phi(z)}{\psi(z)}$

Then.

$$\operatorname*{Res}_{z=a} = \frac{\phi(a)}{\psi(a)}$$

Residue at infinity

If $f(z)$ has an isolated singularity at $z=\infty$ (or) $z=-\infty$ is analytic Then, The residue at $z=\infty$ is defined by

$$\operatorname*{Res}_{z=\infty} = \frac{-1}{2\pi i} \int_c f(z)\,dz$$ where c is

closed contours which encloses an the finite singularity of $f(z)$

The integral being taken in the positive direction

Evolvation of Definite Integral

Type:1

The integral of the form.

$\int_0^{2\pi} F(\cos\theta . \sin\theta) d\theta$ where the integrals rational function of $\cos\theta$ + $\sin\theta$

Then the integral is evaluated by the substitution method

$$ie) \quad z = e^{i\theta}$$

$$d(z) = ie^{i\theta} d\theta$$

$$\cos\theta = 1/2 (e^{i\theta} + e^{-i\theta}) = 1/2 (z + 1/z)$$

$$\sin\theta = 1/2i (e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - 1/z)$$

The integral becomes.

$$\int_0^{2\pi} F(\cos\theta \ \sin\theta) d\theta = \int_c F\left[\frac{1}{2}(z + 1/z) \cdot \frac{1}{2i}(z - 1/z)\right.$$

where. c is $|z| = 1$

$$= \int_c f(z) dz$$

$= 2\pi i$ (Sum of res. of $f(z)$ at it poles with in 'c')

1 Evaluate $\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta}$ $(a>0)$

Given,

let $I = \int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta}$ $(a>0)$

$= \int_0^{\pi} \frac{d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$.

$I = \int_{\theta=0}^{\pi} \frac{d\theta}{\frac{2a^2+1-\cos 2\theta}{2}}$.

$2\theta = t \Rightarrow \theta = 0 \Rightarrow t=0$

$2d\theta = dt \Rightarrow \theta = 0 \Rightarrow t = 2$

$d\theta = \frac{dt}{2}$

$I = \int_{t=0}^{2\pi} \frac{dt/2}{\frac{2a^2+1-\cos t}{2}}$

sub $z = e^{it}$

$dz = ie^{it}\,dt$

$dt = \frac{dz}{ie^{it}}$

$dt = \frac{dz}{iz}$

$I = \int_C \frac{dz/iz}{2a^2+1-\frac{1}{2}\left(\frac{z^2+1}{z}\right)}$

[where 'c' is the unit circle $|z|=1$]

$I = \frac{1}{iz} \int_C \frac{dz \cdot 2z}{24a^2+2z-(z^2+1)}$

$= \frac{2}{i}\int_C \frac{dz}{4a^2+2z-z^2-1}$

$= \frac{2}{i}\int_C \frac{dz}{-z^2+2z+4a^2-1}$

$$= \frac{2}{i}\int_C \frac{dz}{-(z^2-4za^2-2z+1)}$$

$$= \frac{2}{-i}\int_C \frac{dz}{z^2-2z(2a^2+1)+1}$$

$$= 2i\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$\frac{1}{i}$

$\frac{1}{-i}$

$$I = 2i\int_C f(z)\,dz \quad \text{①}$$

where,

$$f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$$

Pole of $f(z)$ is given by

$$z^2-2(2a^2+1)z+1=0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-2(2a^2+1)) \pm \sqrt{(-2(2a^2+1))^2-4(1)(1)}}{(2)}$$

$$= \frac{2(2a^2+1) \pm \sqrt{(-4a^2-2)^2-4}}{2}$$

$$= \frac{2(2a^2+1) \pm \sqrt{4(2a^2-1)^2-4}}{2}$$

$$= \frac{2(2a^2+1) \pm \sqrt{4(2a^2-1)^2-4}}{2}$$

$$= \frac{2\left[(2a^2+1) \pm \sqrt{(2a^2-1)^2-1}\right]}{2}$$

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$$\alpha = (2a^2+1) + \sqrt{(2a^2-1)^2-1}$$

$$\beta = (2a^2+1) - \sqrt{(2a^2-1)^2-1}$$

Clearly, $|\alpha| > 1$ & $|\beta| < 1$

$\therefore$ $z = \beta$ is lies inside 'c'

$$\underset{z=\beta}{\text{Res}}\, f(z) = \lim_{z\to\beta} (z-\beta)\left(\frac{1}{(z-\alpha)(z-\beta)}\right)$$

$$= \lim_{z\to\beta}\left(\frac{1}{z-\alpha}\right)$$

$$= \frac{1}{\beta-\alpha}$$

$$= \frac{1}{(2a^2+1)-\sqrt{(2a^2-1)^2-1}-(2a^2+1)-\sqrt{(2a^2-1)^2-1}}$$

$$= \frac{1}{-2\sqrt{(2a^2-1)^2-1}}$$

$$= \frac{1}{-2\sqrt{(4a^4\cancel{+1}-4a^2\cancel{-1}}} = \frac{1}{-2\sqrt{4a^2(\cancel{2}a^2-1)}}$$

$$= \frac{1}{-2(2a)\sqrt{a^2-1}}$$

$$\underset{z=\beta}{\text{Res}}\, f(z) = \frac{1}{-4a\sqrt{a^2-1}}$$

W.K.T

The cauchy residue theorem is

$$\int_c f(z)dz = 2\pi i \; [\text{sum of its residues at the pole inside c}$$

$$= 2\pi i \left[\frac{-1}{4a\sqrt{a^2+1}}\right]$$

$$= \frac{-\pi i}{2a\sqrt{a^2+1}}$$

$$① \Rightarrow I = 2i\left[\frac{-\pi i}{2a\sqrt{a^2+1}}\right].$$

$$I = \frac{\pi}{a\sqrt{a^2+1}}.$$

2 show that $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{\lfloor n}$ where 'n' is a +ve integer

sol:-

let

$$I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta$$

$$= R.P\int_0^{2\pi} e^{\cos\theta}\, e^{i(\sin\theta - n\theta)}\,d\theta$$

$$= R.P\int_0^{2\pi} e^{(\cos\theta + i\sin\theta)}\, e^{-in\theta}\,d\theta$$

let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \therefore d\theta = \frac{dz}{iz}$

assume c is $|z| = 1$

$$I = \int_c e^z\left(\frac{1}{z^n}\right)\left(\frac{dz}{iz}\right)$$

$z = e^{i\theta}$

$z = \cos\theta -$

$e^{\cos\theta + i\sin\theta} = e^z$

$z^n =$

$1/z^n =$

$$= R\cdot P \text{ of } \frac{1}{i}\int_C \frac{e^z}{z^{n+1}}\, dz$$

$$\boxed{I = R\cdot P \frac{1}{i}\int_C f(z)\, dz} \text{ — ①}$$

where, $f(z) = e^z / z^{n+1}$

$z=0$ is a pole of order $(n+1)$

The pole of order (m)

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

sub $m = n+1$

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(n+1-1)!} \lim_{z\to 0} \frac{d^{n+1-1}}{dz^{n+1-1}}\left[(z-0)^{n+1} f(z)\right]$$

$$= \frac{1}{n!} \lim_{z\to 0} \frac{d^n}{dz^n}\left[(z-0)^{n+1} f(z)\right]$$

$$= \lim_{z\to 0} \frac{1}{\lfloor n} \frac{d^n}{dz^n}\left[\cancel{z^{n+1}}\left(\frac{e^z}{\cancel{z^{n+1}}}\right)\right]$$

$$= \frac{1}{n!} \lim_{z\to 0} \frac{d^n}{dz^n}(e^z)$$

$$= \frac{1}{n!} \lim_{z\to 0} (e^z)$$

$$= \frac{1}{n!}(e^0)$$

$$\operatorname*{Res}_{z\to 0} f(z) = \frac{1}{n!}$$

Applying C.R.T

$\int_C f(z)dz = 2\pi i$ [sum of its residue of at the Pole iside 'c']

$= 2\pi i \left(\frac{i}{n!}\right)$

$\int_C f(z)dz = 2\pi i/n!$

$① \Rightarrow I = R.P \frac{1}{i} \int_C f(z)dz$

$I = R.P \left(\frac{1}{i}\right)\left(\frac{2\pi i}{n!}\right)$

$= RP \left(\frac{2\pi}{\lfloor n}\right)$

$I = \frac{2\pi}{\lfloor n}$

Modal: 2

Evaluation of the ∮ integral of the form $\int_{-\infty}^{\infty} f(z)\, dz$ where, $f(z)$ is analytic in the upper half of the form expect at a finite no. of tools on the real axis. The above type of integral is evaluated by $\oint F(z)$ around a center at 'c' consisting of a sami circle Γ of radius 'r'. Γ_R is large enogh to include all the poles of $f(z)$ and the part of real axis from $-R$ to $+R$

By cauchy residue theorem.

$$\int_c f(z)\, dz = 2\pi i \text{ [Sum of residue of } f(z) \text{ at the pole with in 'c']}$$

$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 2\pi i \sum R^{+}$$

Where, $\sum R^{+}$ denote the sum of residue of pole in the upper ~~of~~ half of the plane.

It can be so. that.

$$\lim_{z \to \infty} z f(z) = 0$$

Then, $\lim_{R \to \infty} \int_{\Gamma} f(z) dz = 0$

also. $\int_{-R}^{R} f(x)\, dx \longrightarrow \int_{-\infty}^{\infty} f(x) dx$ as $R \to \infty$

hence,

$$\int_{-\infty}^{\infty} f(x)\, d\theta = 2\pi i \sum R^{+}$$

Evaluate : $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\, dx$

let $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} = \int_c f(z) dz$

where,

$$f(z) = \frac{x^2}{(x^2+a^2)^3}$$

'c' is the contour consisting of large semi circle Γ of radius 'R' along the part of the real axis from $-R$ to $+R$

$\therefore$

$\int_C f(z)dz = 2\pi i$ [sum of rasidue of the si ~~at~~ its pole is 'c']

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \sum R^{+} \quad —$$

Since,

$$\lim_{z\to\infty} z f(z) = 0$$

$$\Rightarrow \lim_{z\to\infty} z\left(\frac{z^2}{(z^2+a^2)^3}\right) = \lim_{z\to\infty} \frac{z^3}{(z^2+a^2)^3}$$

$$= \lim_{z\to\infty} \frac{z^3}{z^6\left(1+\frac{a^2}{z^2}\right)^3} \quad (z=0)$$

$$= \frac{1}{\infty} = 0$$

$$\Rightarrow \lim_{z\to\infty} z\left(\frac{z^2}{(z^2+a^2)^3}\right) \to 0 \text{ as } R\to\infty$$

take $\lim_{R\to\infty}$ in ①

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = 2\pi i \sum R^{+} \longrightarrow ②$$

now, The Pole of $f(z) = \frac{z^2}{(z^2+a^2)^3}$ is atmost every where

$$\Rightarrow (z^2+a^2)^3 = 0$$

$$\Rightarrow z^2+a^2 = 0 \quad \text{(Thrice)}$$

$$z^2 = -a^2 \quad \text{(Thrice)}$$

$$z = \pm ai \quad \text{(Thrice)}$$

$z = \pm ai$

each of order 3 & $(z = ai)$ is alone upper half of the plane.

$\therefore \Sigma R^+$ Residue at $z = ai$ (Pole of order $m=3$) is

$$\underset{z=ai}{\operatorname{Res}} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-ai)^3 \frac{z^2}{(z-ai)^3(z+ai)^3}\right]$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z^2}{(z+ai)^3}\right)$$

$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{(z+ai)^3(2z) - z^2(3(z+ai)^2(1))}{(z+ai)^6}\right]$$

$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{2z(z+ai) - 3z^2}{(z+ai)^4}\right)$$

$$= \frac{1}{2!}\left[\frac{(z+ai)^4[4z+2ai-6z] - (2z^2+2aiz-3z^2)\,4(z+ai)^3}{(z+ai)^8}\right.$$

$$= \frac{1}{2!}\left[\frac{(z+ai)(2ai-2z) - 4(-z^2+2aiz)}{(z+ai)^5}\right]$$

$$= \frac{1}{2!}\left[\frac{(ai+ai)(2ai-2ai) - 4(a^2+2a^2)}{(ai+ai)^5}\right.$$

$$= \frac{1}{2!}\left[\frac{2ai(0)+4a^2}{(2ai)^5}\right.$$

$$= \frac{1}{2!}\left[\frac{4a^2}{(2ai)^5}\right] = \frac{2a^2}{2!(2ai)^5}$$

$=$

$$= \frac{\cancel{4}^{2} a^2}{2(32 a^8 i)} = \frac{-\cancel{2} a^2 i}{32 a^5}$$

$$\operatorname{Res}_{z=ai} f(z) = \frac{-i}{16 a^3}$$

$$\int_C f(z)dz = 2\pi i \left[\frac{-i}{16 a^3}\right]$$

$$= \frac{-2\pi (i)^2}{16 a^3}$$

$$\int_C f(z)dz = \frac{\pi}{8 a^3}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx = \frac{\pi}{8a^3}$$

2. Evaluate - The integral is of the form

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} \quad : |a| > 1$$

let

$$I = \int_0^{\pi/2} \frac{dx}{a+\sin^2 x} \; ; \; |a| > 1$$

$$= \int_0^{\pi/2} \frac{dx}{a+\left(\frac{1-\cos 2x}{2}\right)}$$

$$= \int_0^{\pi/2} \frac{dx}{\frac{2a+(1-\cos 2x)}{2}}$$

$$= 2\int_0^{\pi/2} \frac{dx}{(2a+1) - \cos 2x}$$

$$= \int_{|z|=1} \frac{dz/iz}{(2a+1) - \frac{1}{2}(z+\frac{1}{z})}$$

$$= -i \int_{|z|=1} \frac{dz}{z-2(2a+1)z+1}$$

here, $f(z) = z^2 - 2(2a+1)z+1$

$$\therefore z = (2a+1) \pm \sqrt{(2a+1)^2-1}$$

$$\alpha = (2a+1) + \sqrt{(2a+1)^2-1}$$

$$\beta = (2a+1) - \sqrt{(2a+1)^2-1}$$

$$R_\alpha = \lim_{z\to\beta} (z-\beta)\left(\frac{1}{(z-\alpha)(z-\beta)}\right)$$

$$= \frac{1}{\beta-\alpha} = \frac{-1}{2\sqrt{(2a^2+1)^2-1}}$$

$$(1) \Rightarrow I = \frac{1}{i}\, 2\pi i \left(\frac{-1}{2\sqrt{(2a^2+1)^2-1}}\right)$$

$$= \frac{-\pi}{2\sqrt{a^2(a+1)}}$$

3 Evaluate $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$ (or) $\int_0^\infty \frac{x \sin mx}{x^2+a^2}\,dx$.

let $\int_c e^{imz} f(z)\,dz = \int_c \frac{e^{imz}}{z^2+a^2}\,dz$

where c is the contour consisting of a semi-circle of radius 'R'

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large enough to include all the Poles of the integral in upper half of the Plane and also part of real axis from -R to R

$$\Rightarrow \int_C e^{imz} f(z)dz = 2\pi i \sum R^+$$

$$\Rightarrow \int_{-R}^{R} e^{imx} f(x)dx + \int_{\Gamma} e^{imz} f(z)dz = 2\pi i \Sigma$$

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{z^2+a^2} = 0$$

By Jordan lemma.

$$\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z) dz = 0$$

also,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x) dx = \int_{-\infty}^{\infty} e^{imx} f(x) dx$$

hence,

$$\int_{-\infty}^{\infty} e^{imx} f(x) dx = 2\pi i \sum R^+ \longrightarrow ①$$

poles of $\left(\frac{e^{imz}}{z^2+a^2}\right)$ are given by

$$z = \pm ai$$

here, $z = ai$ is lies inside 'c'

$$\therefore \operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai} (z-ai) \left[\frac{e^{imz}}{(z-ai)(z+ai)}\right]$$

$$= \frac{e^{-ma}}{2ai}$$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = 2\pi i\left[\frac{e^{-ma}}{2ai}\right]$$

$$= \frac{\pi}{a}\left(e^{-ma}\right)$$

$$\int_{-\infty}^{\infty} \frac{(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \frac{\pi}{a}\,e^{-ma} \longrightarrow ②$$

Equating real & im part of ②

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{a}\,e^{-ma}; \quad \int_{-\infty} \frac{\sin mx}{x^2+a^2}\,dx = 0$$

but, $$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = 2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx$$

$$\therefore 2\int^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \pi/a\; e^{-ma}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}\,e^{-ma}$$

Diff. w.r.to. 'm'

$$③ \Rightarrow -\int_{0}^{\infty} \frac{\sin mx(x)}{x^2+a^2}\,dx = \frac{\pi}{2a}\,e^{-ma}(-a)$$

$$\int_{0}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi}{2}\,e^{-ma} \longrightarrow ④$$

4 Evaluate $\int_0^\infty \frac{\sin mx}{x} dx$

$$\int_C e^{imz} f(z)dz = \int_C \frac{e^{imz}}{z} dz$$

where 'c' is the contour consisting of a semi-circle 'Γ' of radius 'R' large enough to include all the poles of the integral in upper half of the plane and also part of real axis from -R to +R.

$$\Rightarrow \int_C e^{imz} f(z)dz = 2\pi i \sum R^+$$

$$\int_{+R}^{R} e^{imx} f(x)\, dx + \int_\Gamma e^{imz} f(z)\, dz = 2\pi i \sum R$$

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty}\left(\frac{1}{z}\right)$$

$$=0$$

by Jordan lemma

$$\lim_{R\to\infty} \int_\Gamma e^{imz} f(z) = 0$$

Also,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x)\, dx = \int_{-\infty}^{\infty} e^{imx} f(x) dx$$

here,

$$\int_{-\infty}^{\infty} e^{imx} f(x) dx = 2\pi i \sum R^{+} \text{ — } ①$$

The Poles of $\left(f(z) = \frac{e^{imz}}{z}\right)$ is $z=0$

$\therefore z=0$ is only lies inside 'c'

$$\therefore \left[\underset{z=0}{Res} f(z)\right] = \lim_{z\to0} (z-0)f(z)$$

$$= \lim_{z\to0} (z)\left(\frac{e^{imz}}{z}\right)$$

$$= \lim_{z\to0} e^{imz}$$

$$= e^{0}$$

$$= 1$$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = 2\pi i(1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\cos x + i\sin x}{x}\right) dx = 2\pi i$$

$$\Rightarrow 2\int_{0}^{\infty} \left(\frac{\cos x + i\sin x}{x}\right) dx = 2\pi i$$

Equating real and imaginary Part

$\cos x = 0$, $2 \sin x = 2\pi$

$$\sin x = \frac{2\pi}{2}$$

$$\sin x = \pi$$

$$\int_{0}^{a} \frac{\sin mx dx}{x} = \pi$$

15m **Argument principle.**

Statement:

If $f(z)$ is meromorphic in 'Ω' with t 0's $\overset{(zeros)}{}$ a_j and the poles b_K Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \sum n(\gamma, a_j) - \sum_K n(\gamma$$

for every cycle 'γ' which is homologes to '0' in 'Ω' and doesn't passes through any of the zero's and poles.

proof:

Given the $f(z)$ is meromorphic in 'Ω'

let $f(z)$ be a zero's of order 'h' at a_j by the definition of zero

$$f(z) = (z-a_j)^h f_n(z) \longrightarrow ①$$

where, $f_n(z)$ is analytic and not equal to zero.

oplying 'log' on both side.

$$\log f(z) = \log\left[(z-a_j)^h f_n(z)\right]$$

$$= \log[z-a_j)^h + \log(f_n(z)$$

$$\log f(z) = h\log(z-a_j) + \log f_n(z)$$

differentiate on both side.

$$\Rightarrow \frac{f'(z)}{f(z)} = h\left(\frac{1}{z-a_j}\right) + \frac{f_n'(z)}{f_n(z)}$$

∴ The function $\left(\frac{f'(z)}{f(z)}\right)$ has a simple pole at $z = a_j$ of order 'h'

Ill'y

$z = b_k$, is a pole of order for $f(z) = \frac{g_n(z)}{(z-b_k)^h}$

~~There~~

where $g_n(z)$ is analytic at b_k and $g_n(z) \neq 0$

Taking log on both side.

$$\log f(z) = \log\left(\frac{g_n(z)}{(z-b_k)^h}\right)$$

$$= \log g_n(z) - \log (z-b_k)^h$$

$$= \log g_n(z) - h \log (z-b_k)$$

Ding on both side.

$$\frac{f'(z)}{f(z)} = \frac{g_n'(z)}{g_n(z)} - \frac{h}{(z-b_k)}$$

∴ The function $\left(\frac{f'(z)}{f(z)}\right)$ has a simple pole at $z = b_k$ with residul $(-h)$

By C.R.T, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = \left(\sum_j n(\gamma, a_j) \times \text{rasidue at } z = a_j\right)$

$+ \left(\sum_k n(\gamma, b_k) \times \text{residul at } z = b_k\right)$

$$= \sum_j n(\gamma, a_j)(h) + \sum_k n(\gamma, b_k)(-h)$$

$$= \sum_j n(\gamma, a_j)\, h - \sum_k n(\gamma, b_k)(h)$$

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = \text{Total no. of zero} - \text{total no.}$$

Note:-

If $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = 0$

$A \Rightarrow 0 =$ Total no. of zero of $f(z)$ $-$ Total no poles of $f(z)$

Total no. of zero's of $f(z)$ inside 'γ' = Total no. of poles of $f(z)$ inside 'γ'

$\therefore$ F and g have same no. of zero's and poles inside 'γ'

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

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Unit - III

Harmonic function and Power series expansion.

Define: Harmonic (or) potential function.

A real valued function $u(z)$ (or) $u(x,y)$ is defined and single valued in a region 'Ω' is said to be Harmonic in a region 'Ω' is a potential function in 'Ω' In 'Ω', are a potential function in 'Ω'

If it is continuous together with its partial derivatives of the first two order. and satisfies the laplace equation

ie) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Ex $f(z) = z^2$

$= (x+iy)^2$

$= x^2 - y^2 + i(2xy)$

$= u + iv.$

$= 2x + 2y$

$\nabla^2 u = 2 - 2. = 0$

Note.

The polar form $[(r,\theta)]$ of laplace equation is,

$$r\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Theorem: 1

If 'u' is harmonic in 'Ω' Then,

$f(z) = \frac{\partial u}{\partial x}\overset{\to u_x}{} - i\frac{\partial u}{\partial y}\overset{\to u_y}{}$ is analytic.

proof:

let $f(z) = U + iV$

Where $U = \frac{\partial u}{\partial x} = u_x$ and $V = -\frac{\partial u}{\partial y} = u_y$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} \text{ and } \frac{\partial v}{\partial x} = \frac{-\partial^2 u}{\partial x \partial y}$$

$$\rightarrow \partial u/\partial y = \frac{\partial^2 u}{\partial x \partial y} ; \frac{\partial v}{\partial y} = \frac{-\partial^2 u}{\partial y^2}$$

Since 'u' is harmonic

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

W.K.T

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Hence, u & v satisfies the C-R eqn and the partial derivatives of x & y is continuous

$\therefore f(z) = (u_x - iu_y)$ is analytic

Defn: conjugate differentiale of du (*du)

If u is harmonic in 'Ω'

$${}^*du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

hence *du is called the conjugate differentiale of du(*du)

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Theorem:2

If 'u' is harmonic in 'Ω'. Then

$\int_{\gamma}^{*} du = 0$ ∀ cycle 'γ' which is homologous to 'Ω'

Proof:

Given that 'u' is harmonic in 'Ω'

Then the function $f(z) = U + iV$

where $U = \frac{\partial u}{\partial x}$ and $V = -\frac{\partial u}{\partial y}$

$$\Rightarrow f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ is analytic in Ω

now,

$$f(z)\,dz = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\cdot(dx + i\,dy)$$

$$= \frac{\partial u}{\partial x} dx + i\frac{\partial u}{\partial y} dy - i\frac{\partial u}{\partial y} dx$$

$$- i^2 \frac{\partial u}{\partial y} dy$$

$$= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + i\left[\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx\right]$$

$$\Rightarrow f(z)dz = du + i^{*}du$$

$$\Rightarrow \int_{\gamma} f(z)dz = \int_{\gamma} du + i\int_{\gamma}^{*} du$$

By cauchy's theorem.

$$\Rightarrow \int_{\gamma} f(z)dz = 0$$.

since, $\int_{\gamma} du$ is an integral of the exact differential du vanish along all the cycles.

$\therefore \int_{\gamma} {}^{*}du = 0 \quad \forall$ cycles γ

which is homologes to '0' in 'Ω'

Theorem: 3 $\frac{3}{2}$

Interpretation of conjugate differential (*du) is $\int_{\gamma} {}^{*}du = 0$

If 'γ' is regular curve with the equation $z = z(t)$

The tangent determined by the angle $\alpha = \arg z'(t)$ and we can write

$$\therefore dx = |dz| \cos\alpha$$

$$dy = |dz| \sin\alpha$$

$$z = |z| e^{i\alpha}$$

$$dx + i dy = |z| [\cos\alpha + i \sin\alpha]$$

The normal which points to the direction is $\beta = \alpha - \pi/2$ and $\cos\alpha = -\sin\beta$ and $\sin\alpha = \cos\beta$

$$\therefore {}^{*}du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= \left[-\frac{\partial u}{\partial y} \cos\alpha + \frac{\partial u}{\partial x} \sin\alpha\right] |dz|$$

$$= \left[\frac{\partial u}{\partial y} \sin\beta + \frac{\partial u}{\partial x} \cos\beta\right] |dz|$$

$$= \left[\frac{\partial u}{\partial x} \frac{dx}{dn} + \frac{\partial u}{\partial y} \frac{dy}{dn}\right] |dz|$$

$${}^{*}du = \frac{\partial u}{\partial n} |dz|$$

(Here, $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\left(\frac{dy}{|dz|}\right) + \frac{\partial u}{\partial y}\left(\frac{dx}{|dz|}\right)$

$$= \frac{\partial u}{\partial x} \cdot \frac{dx}{dn} + \frac{\partial u}{\partial y}\left(\frac{dy}{dn}\right))$$

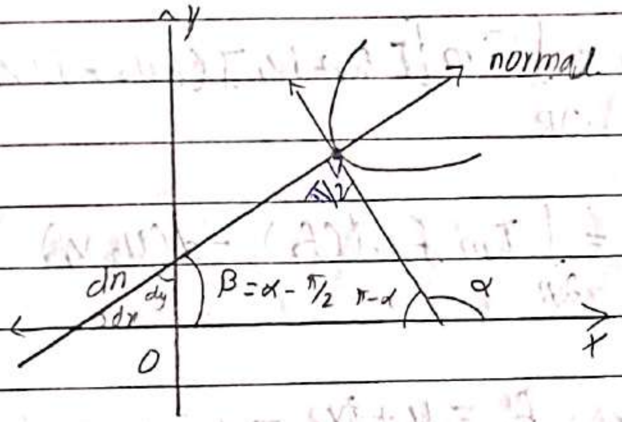
$$\frac{\partial u}{\partial n}|dz| = {}^{*}du$$

(here, $\frac{\partial u}{\partial n}$ is the nml. derivative of 'u')

$$\therefore \int_{\gamma} {}^{*}du = 0$$

$$\Rightarrow \int_{\gamma}^{*} \frac{\partial u}{\partial n} |dz| = 0$$

$\left(\frac{\partial u}{\partial n}\right)$ is the rate of change in the direction $\perp^{r}$ is zero



Theorem: 4

If u_1 & u_2 are harmonic function in the region 'Ω', Then

$$\int_{\gamma} (u_1 {}^{*}du_2 - u_2 {}^{*}du_1) = 0, \text{ for every}$$

cycle 'γ' which is homologoes to zero in 'Ω'

Proof

It is subficient to prove that
The resul of that $\gamma = \partial R$
where R is the rectangle in 'Ω'

In R,

u, and u_2 have the single valued conjugate function v_1 and v_2.

$$\therefore u_1^* du_2 - u_2^* du_1 = u_1 dv_2 - u_2 dv_1$$
$$= u_1 dv_2 + u_2 dv_1 + v_1 du_2 - v_1 d$$
$$= u_1 dv_2 + v_1 du_2 - d(u_2 v_1)$$

here, $d(u_2 v_1)$ is an Exact differential

and

$$u_1 dv_2 + v_1 du_2 = \text{image part of} \{(u_1 + iv_1)(du_2 + idv_2)\}$$

$$= \int_{\partial R} Im\{[u_1 + iv_1](du_2 + iuv_2 - d(u_2 v_1)]\}$$

$$= \int_{\partial R} Im\, f_1\, d(f_2) - d(u_2 v_1)$$

where, $f_1 = u_1 + iv_1$ & $f_2 = u_2 + iv_2$ $\Big\}$ are analytic in R.

The first integral $f_1 d(f_2)$ is vanish by cauchy theorem and so dous the integral of its imaginary part of the second integ of vanish.

Since, $d(u_2 v_1)$ is exact differential

$$\therefore \int_{\partial R} (u_1^* du_2 - u_2^* du_1) = 0$$

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Theorem: ^{5} (Mean value Theorem) → mean value property

Statement:

The arithimatic mean of a harmonic function over concentric circle $|z| = r$ is a linear function of $\log r$.

$$\frac{1}{2\pi}\int_{|z|=r} u\, d\theta = \alpha \log r + \beta$$ and If 'u' is harmonic in a disc • $\alpha = 0$ and the arithimetic mean is constant.

Proof:-

If u_1 and u_2 are harmonic function in 'Ω'

Then we get,

$$\int_{\gamma} (u_1 {}^*du_2 - u_2 {}^*du_1) = 0 \longleftarrow ① \ \forall \text{ cycles [illegible]}$$

sub $u_1 = \log r$; $u_2 = u$ be harmonic function in $|z| \leq \rho$ for Ω

let $\Omega : 0 < |z| < \rho$ and

let $\gamma : C_1 - C_2$, where C_1 & C_2 are circles.

C_i is a circle with $|z| = r_i < \rho$ in C_1 $|z| = r_1$ & in C_2 $|z| = r_2$ where $r_2 < r_1 < \rho$

$|z| = r$

$z = re^{i\theta}$

$dz = ir e^{i\theta} d\theta$

$|dz| = r d\theta$ on $|z| = r$

$$\Rightarrow r\left[\frac{\partial u}{\partial r}\right] d\theta = r\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\right] d\theta$$

$$= r\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right] d\theta$$

$$= \frac{\partial u}{\partial x}(r\cos\theta)\, d\theta + \frac{\partial u}{\partial y}(r\sin\theta)\, d\theta$$

$$= \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$$

$$\therefore r\left[\frac{\partial u}{\partial r}\right] d\theta = {*du} \; //$$

$$\Rightarrow {*du_1} = r \frac{\partial u_1}{\partial r} d\theta$$

$$= r\left[\frac{\partial(\log r)}{\partial r}\right] d\theta$$

$$= r\left(\frac{1}{r}\right) d\theta$$

$$*du_1 = d\theta$$

W.K.T

$$\int_{\gamma} (u_1 {*du_2} - u_2 {*du_1}) = 0$$

$$\int_{C_1 - C_2} (u_1 {*du_2} - u_2 {*du_1}) = 0$$

$$\therefore \int_{C_1} (u_1 {*du_2} - u_2 {*du_1}) = \int_{C_2} (u_1 {*du_2} - u_2 {*du_1})$$

$$\int_{C_1} \log r \left(r \frac{\partial u}{\partial r}\right) d\theta - u\, d\theta = \int_{C_2} \log r \left(r \frac{\partial u}{\partial r}\right) d\theta - u\, d\theta$$

$$\Rightarrow \int_{C_1} \left[r \frac{\partial u}{\partial r} \log r - u \right] d\theta = \int_{C_2} \left[r \frac{\partial u}{\partial r} \log r - u \right] d\theta$$

$$\Rightarrow \int_{|z|=r} r \frac{\partial u}{\partial r} \log r\, d\theta - \int_{|z|=r} u\, d\theta = -B_1 \longrightarrow ②$$

Also

$$\int_{\gamma} {}^*du = \int_{C_1 - C_2} r \frac{\partial u}{\partial r} \, d\theta$$

$$\Rightarrow \int_{C_1} {}^*du = \int_{C_2} {}^*du = \text{constant}$$

$$\Rightarrow \int_{|z|=r} r \frac{\partial u}{\partial r} \, d\theta = \alpha_1 \text{ (say)}$$

$$\Rightarrow \int_{|z|=r} r \frac{\partial u}{\partial \theta} \, d\theta = \alpha_1 (\log r)$$

$$\Rightarrow \int_{|z|=r} u \, d\theta = \alpha_1 \log r + \beta_1$$

$$\Rightarrow \frac{1}{2\pi} \int_{|z|=r} u \, d\theta = \alpha \log r + \beta .$$

where, $\alpha = \frac{\alpha_1}{2\pi}$ and $\beta = \frac{\beta_1}{2\pi}$

in particular u is harmonic in the disk

Then

$$\int_{|z|=r} {}^*du = 0$$

$$\therefore \alpha_1 = 0 \Rightarrow \alpha = 0$$

Hence,

$$\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = \beta \text{ (constant)}$$

Theorem: 6 [Maximum Principle for Harmonic fun

Statement :-

A non-constant harmonic function neither a maximum nor in a minimum in its region & definition consequently the maximum and the minimum on a closed bounded E are taken boundary of E

Proof :-

we will the prove the theorem by using maximum modules theorem.

$|u(z)|$

since, $U(z)$ is continuous.

Also, $|U(z)|$ is continuous closed and boundary in Ω

$$\text{ie)} \ \boxed{|u(z)| \leq M} \ \forall z \in \Omega \ \& \ M > 0 \quad ①$$

if possible.

$|u(z)|$ attains its maximum value at some interior point $z_0 \in \Omega$

$$\text{ie)} \ \boxed{|u(z_0)| = M} \quad ②$$

construct a circular disc,

$|z - z_0| \leq r$ contained in Ω

By mean value property.

$$v(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

$$\Rightarrow |u(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \right.$$

by ① & ②

$$|u(z)| \le |u(z_0)| \text{ on } |z - z_0| \le r$$

sub $z = z_0 + re^{i\theta}$

$$\Rightarrow |u(z_0 + re^{i\theta})| \le |u(z_0)|$$

$$\int_0^{2\pi} \overset{|u(z_0+re^{i\theta})|}{\cancel{|(u(z+re^{i\theta})|}} \, d\theta \le \int_0^{2\pi} |u(z_0)| \, d\theta$$

$$\le 2\pi \, |u(z_0)|$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} |u(z_0 + re^{i\theta})| \, d\theta \le |u(z_0)| \text{ — ④}$$

by ③ & ④

$$\Rightarrow |u(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |u(z_0 + re^{i\theta})| \, d\theta$$

$$2\pi \, |u(z_0)| = \int_0^{2\pi} |u(z_0 + re^{i\theta})| \, d\theta$$

$$\text{ie} \Rightarrow \int_0^{2\pi} |u(z_0)| \, d\theta = \int_0^{2\pi} |u(z_0 + re^{i\theta})| \, d\theta$$

$$\therefore \int_0^{2\pi} \left[|u(z_0)| - |u(z_0 + re^{i\theta})| \right] d\theta = 0$$

Since, The integral is continuous and non-negative

$$\Rightarrow |u(z_0)| - |u(z_0 + re^{i\theta})| = 0$$

$$\Rightarrow |u(z_0)| = |u(z_0 + re^{i\theta})|$$

$$\therefore \boxed{|u(z_0)| = |u(z)|} \ \forall\ z \in |z-z_0|<\gamma$$

by continuity

$$\therefore |u(z)| = \text{constant}$$
$$\Rightarrow u(z) = \text{constant}$$
which is $\Rightarrow\Leftarrow$

$\therefore$ $|u(z)|$ cannot attain its maximum in the interior of Ω
i.e $|u(z)|$ attains its maximum value only on the boundary.

Theorem: 7 [Poisson's [formula) Theorem)]

statment

suppose that $u(z)$ is harmonic for $|z|<R$ and $u(z)$ is continuous for $|z|<R$ Then

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2}\, u(z)\, d\theta \ \forall\ |a|<R$$

Proof

consider the linear transformation

$$z = S(\delta) = \frac{R^2\delta + Ra}{\bar{a}\delta + R} \longrightarrow ①$$

$$\Rightarrow \bar{a}\delta z + Rz = R^2\delta + Ra$$

$$\Rightarrow \bar{a}z\delta - R^2\delta = aR - zR$$

$$\therefore \delta = \frac{aR - zR}{\bar{a}z - R^2}$$

$$\Rightarrow \delta = \frac{R(a-z)}{\bar{a}z - R^2}$$

$$\therefore \boxed{\delta = \frac{R(z-a)}{R^2 - \bar{a}z}} \quad \text{——} \ ①$$

The transformation of ① maps.

$|\delta| \le 1$ onto $|z| \le R$ for $|\delta| \le 1$

$$\Rightarrow \frac{R|z-a|}{|R^2 - \bar{a}z|} \le 1$$

$$\Rightarrow R|z-a| \le |R^2 - \bar{a}z|$$

$$\Rightarrow R^2|z-a|^2 \le |R^2 - \bar{a}z|^2$$

$$\Rightarrow R^2(z-a)(\bar{z}-\bar{a}) \le (R^2 - \bar{a}z)(R^2 - a\bar{z})$$

$$\Rightarrow R^2 z\bar{z} - R^2 z\bar{a} - R^2 a\bar{z} + R^2 a\bar{a} \le R^4 - \cancel{R^2 a\bar{z}} - \cancel{R^2 \bar{a}z} + a\bar{a}z\bar{z}$$

$$\Rightarrow R^2|z|^2 - |a|^2|z|^2 \le R^4 - R^2|a|^2$$

$$\Rightarrow |z|^2\,\cancel{[R^2 - |a|^2]} \le R^2\,\cancel{[R^2 - |a|^2]}$$

$$|z| \le R$$

Also $\delta = 0$ corresponds to $z = a$

The function $u(z)$ (or) $u[S(\delta)]$ is harmonic in $|\delta| \le 1$ and the mean value of the function at 'a' is given by

$$u(a) = \frac{1}{2\pi} \int_{|\delta|=1} u[S(\delta)]\, d\psi$$

where $\psi = \arg(\delta) = \frac{d\delta}{i\delta}$

$$\Rightarrow \delta^2 = e^{i\theta}$$

$$\Rightarrow u(a) = \frac{1}{2\pi}\int_0^{2\pi} u[S(\delta)]\, d\,[\arg(\delta)] \text{ —— ③}$$

$$② \Rightarrow \delta = \frac{R(z-a)}{R^2-\bar{a}z}$$

$$\Rightarrow \log\delta = \log R + \log(z-a) - \log(R^2-\bar{a}z)$$

Diff. w. r. to 'z'

$$\Rightarrow \frac{d\delta}{\delta} = 0 + \frac{1}{z-a}\,dz - \frac{1}{R^2-\bar{a}z}(-\bar{a}\,dz)$$

$$\therefore \frac{d\delta}{\delta} = \left[\frac{1}{z-a} + \frac{\bar{a}}{R^2-\bar{a}z}\right] dz \text{ —— ④}$$

on the

$|z| =$

$$\Rightarrow z = Re^{i\theta};\ dz = Re^{i\theta}(i\,d\theta)$$

$$\boxed{dz = z i\,d\theta}$$

$$\frac{d\delta}{\delta} = \left[\frac{1}{z-a} + \frac{\bar{a}}{R^2-\bar{a}z}\right](zi)\,d\theta$$

[where $R^2 =$

$$\Rightarrow \frac{d\delta}{i\delta} = \left[\frac{z}{z-a} + \frac{\bar{a}z}{z\bar{z}-\bar{a}z}\right] d\theta$$

$$= \left[\frac{z}{z-a} + \frac{\bar{a}z}{z(\bar{z}-\bar{a})}\right] d\theta$$

$$= \left[\frac{z\bar{z} - \cancel{z\bar{a}} + \cancel{\bar{a}z} - a\bar{a}}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$\frac{d\delta}{i\delta} = \frac{R^2-|a|^2}{|z-a|^2}\,d\theta \longrightarrow ⑤ \text{ on the circle } |\delta|=1$$

$$⑤ \Rightarrow d\psi = \frac{R^2-|a|^2}{|z-a|^2}\,d\theta$$

$$\Rightarrow u(a) = \frac{1}{2\pi}\int_0^{2\pi} u[s(\delta)]\left(\frac{R^2-|a|^2}{|z-a|^2}\right)d\theta$$

$$\boxed{u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(z)\left[\frac{R^2-|a|^2}{|z-a|^2}\right]d\theta}$$

Schwarz's formula.

By poisson formula

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right)u(z)\,d\theta$$

let $\delta = e^{i\theta}$

$\frac{d\delta}{d\theta} = ie^{i\theta}$

$\therefore \frac{d\delta}{i\delta} = d\theta$

put, $a=z$ & $z=\delta$

$$\therefore u(z) = \frac{1}{2\pi}\int_{|\delta|=R} Re\left(\frac{\delta+z}{\delta-z}\right)u(\delta)\frac{d\delta}{i\delta}$$

$$u(z) = Re\left[\frac{1}{2\pi i}\int_{|\delta|=R}\left(\frac{\delta+z}{\delta-z}\right)u(\delta)\frac{d\delta}{\delta}\right]$$

The expression with in the bracket is an analytic function of 'z' for $|z| \le R$ it follows that, $u(z)$ is the real part of $f(z)$

$$\therefore f(z) = \frac{1}{2\pi i}\int_{|\delta|=R}\left(\frac{\delta+z}{\delta-z}\right)\frac{d\delta}{\delta} + c$$

where 'c' is an arbitrary constant This is known as schwarz's function formula.

positive linear function (or) poisson integral

for any piecewise continuous function $u(\theta)$ in $0 \le \theta \le 2\pi$

$$\therefore P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(\theta)\, d\theta$$

is called the poission integral of 'u'

weierstrass's theorem.

statement.

suppose that $f_n(z)$ is analytic in the region Ω_n and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω uniformly on every compact subset of 'Ω'.

Then $f(z)$ is analytic in Ω moreover $f_n'(z)$ converges uniformily in $f'(z)$ on every compack subset 'Ω'.

let, $c : |\delta - z| = r$ & $|z-a| < r$

since $f_n(z)$ is analytic in 'c'

by cauchy integral formula

$$f_n(z) = \frac{1}{2\pi i}\int_c \left(\frac{f_n(\delta)}{\delta - z}\right) d\delta$$

$$\lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_C \lim_{n\to\infty}\left(\frac{f_n(\delta)}{\delta - z}\right) d\delta$$

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$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_C \frac{f(\delta)}{\delta - z}\, d\delta$$

This formula shows that.

f(z) is analytic in the disc By the definition of dirivative

$$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\delta)}{(\delta - z)^2}\, d\delta$$

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_C \lim_{n\to\infty} \frac{f_n(\delta)}{(\delta - z)^2}\, d\delta$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\delta)}{(\delta - z)^2}\, d\delta$$

$$\lim_{n\to\infty} f'_n(z) = f'(z)$$

Hence, The converges of sequence of function is uniform for $|z-a| < \rho < r$

Any compact subset of Ω can be covered by a finite no. of such closed disc and therefore, the converges is uniform on every compact subset of Ω

Hurwitz's

i) A ~~Hurwitz~~ theorem [statement]:

If the function fn(z) are analytic and not equal to zero in a region 'Ω' and if fn(z) is converges to {f(z)} uniformly on every compact subset of 'Ω' then f(z) is either identically zero (or) never equal to zero in 'Ω'

proof:

If fn(z) is not equal to zero

By weierstrass theorem.

f(z) is analytic and hence its zero's are isolated

let $z_0 \in \Omega$, then

∃ a positive no. of 'r' such that f(z) is defined and non-zero for $0 \leq |z-z_0| \leq r$

Then by the principle of minimum modulus |f(z)| has +ve minimum on the circle $C: |z-z_0| = r$

Since,

$f_n(z) \neq 0$ in Ω $\forall n$

and $f_n(z)$ is analytic

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{f_n(z)}\right) = \frac{1}{\lim_{n\to\infty} f_n(z)}$$

$$= \frac{1}{f(z)}$$

ie) $\left(\frac{1}{f_n(z)}\right)$ is converge uniformly to $\left(\frac{1}{f(z)}\right)$ on c' and

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_c \frac{f_n'(z)}{f_n(z)}dz = \frac{1}{2\pi i}\int_c \frac{\lim_{n\to\infty} f_n'(z)}{\lim_{n\to\infty} f_n(z)}dz$$

$$= \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz$$

By the Principle of argument

$\frac{1}{2\pi i}\int_c \frac{f_n'(z)}{f_n(z)}dz$ = The no. of zeros of $f_n(z)$ is enclosed by 'c'

= 0 (by hypothesis)

The no. of zeros of $f_n(z)$ is enclosed by c = 0

$\therefore f(z_0) \neq 0$ (z_0 is lies inside c)

Since, 'z_0' is arbitrary

$\therefore f(z) \neq 0$ on 'Ω'

Reflection Principle Theorem:

Statement

let $f(z)$ be analytic in a domain 'Ω' containing a segment of x axis and b symmetric to that axis. $\overline{f(z)} = f(\bar{z})$ $\forall z \in$ $\Leftrightarrow$ $f(x)$ is real for each point on th segment of x axis

proof:

suppose $f(x)$ is real on the segment of x axis contained in 'Ω'

let as denote $g(z) = \overline{f(z)}$ — ①

and write $f(z) = u(x,y) + iv(x,y)$ — ②

$$g(z) = p(x,y) + iq(x,y) \quad \text{—} ③$$

clearly

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y) \longrightarrow ④$$

so, (by ③ & ④)

$$p(x,y) = u(x,-y)$$

$$q(x,y) = -v(x,-y)$$

by C-R equ.

$$u_x = v_y \ \& \ u_y = -v_x$$

$$\Rightarrow p_x = q_y \text{ and } p_y = -q_x$$

$\therefore$ p and q satisfies C-R eqn and also the 1st order partial derivative exist and it has continuous

$\therefore$ $g(z)$ is analytic in 'Ω'

$V(x,0) = 0$

Given that. f(x) is real on the segment of x axis

$\Rightarrow g(x) = P(x,0) \pm u(x,0)$

$\Rightarrow \boxed{g(x) = f(z)}$ on the segment of real axis

ie) $g(z) = f(z)$ on the segment of real axis

we know that

The uniqueness theorem for the function is

$g(z) - f(z)$ which is identically zero on the line segment of the x-axis and hence,

$g(z) - f(z) = 0 \;\forall\; z \in \Omega$

$\Rightarrow g(z) = f(z)$ (or) $f(z) = f(\bar{z}) \;\forall\; z \in \Omega$

conversely,

we assume that, $f(z) = f(\bar{z})$ is true $\forall\; z \in \Omega$

$\Rightarrow u(x,y) + iv(x,y) = u(x,-y) - iv(x,-y)$

in particular

$u(x,0) + iv(x,0) = u(x,0) - iv(x,0)$

$2iv(x,0) = 0$

$\boxed{V(x,0) = 0}$

hence, f(x) is real on the part of x-axis contained in the Domain in 'Ω'

? Schwartz theorem

statement:

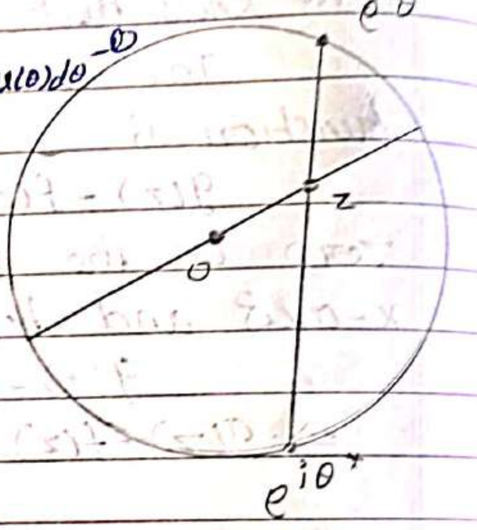
The function $P_u(z)$ of harmonic for $|z|<1$ and $\lim_{z\to e^{i\theta_0}} P_u(z) = U(\theta_0)$

provided that 'u' is continuous at 'θ_0'

proof:-

we know that

$$P_u(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)d\theta \quad -①$$



Now,

$$\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)d\theta \text{ is}$$

an analytic function
and its real part in harmonic in $|z|<1$

let C_1 and C_2 be complementary resid

of the unit circle $|z|=1$

let 'u_1' be the function ~~[illegible]~~ which coinsides with u on C_1 and vanishes on C_2 and u_2 be the corresponding func on C_2

Then

$$PU = PU_1 + PU_2$$

since PU_1 can be consider has a line integral over C_1 and its harmonic every where except on the arc C_1

Now,

$$\Rightarrow Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = \frac{1}{2}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z} + \frac{e^{-i\theta}+\bar{z}}{e^{-i\theta}-\bar{z}}\right]$$

$$= \frac{1}{2}\left[\frac{(e^{-i\theta}-\bar{z})(e^{i\theta}+z) + (e^{-i\theta}+\bar{z})(e^{i\theta}-z)}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-i\theta}e^{i\theta}+\cancel{z}e^{-i\theta}-\bar{z}e^{i\theta}-z\bar{z}+e^{-i\theta}e^{i\theta}-e^{-i\theta}z+\bar{z}e^{i\theta}-z\bar{z}}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right]$$

$$= \frac{1}{2}\left[\frac{2e^{-i\theta}e^{i\theta}-2z\bar{z}}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right]$$

$$= \frac{1}{2}\left[\frac{2e^{-i\theta+i\theta}-2z\bar{z}}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right] = \frac{1}{2}\left[\frac{2e^{0}-2z\bar{z}}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right]$$

$$= \frac{1}{2}\left[\frac{2-2z\bar{z}}{\underbrace{(e^{i\theta}-z)}_{z}\underbrace{(e^{-i\theta}-\bar{z})}_{\bar{z}}}\right]$$

$z\bar{z} = |z|^2$

$|e^{i\theta}-z)(e^{-i\theta}-\bar{z}) = |e^{i\theta}-z)^2$

$$= \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$ vanishes on $|z|=1$ for $z \neq e^{i\theta}$

$\therefore$ PU_1 is zero on open arc 'c_2'

and since, it is continuous. $PU(z) \to 0$ as $z \to e^{i\theta}$ on C_2 ~~next~~

next we will prove

$$\lim_{z \to e^{i\theta_0}} PU(z) = U(\theta_0) :-$$

we can take $U(\theta_0) = 0$

Then for any $\epsilon > 0$. we can find C_1 and
such that,

∵ $e^{i\theta_0}$ is an interior point of 'C_2' and $|U(\theta)| < \epsilon/2$ for $e^{i\theta} \in C_2$

under this condition.

$$|U_2(\theta)| < \epsilon/2 \ \forall \theta \text{ and}$$

hence, $|PU_2(z)| < \epsilon/2 \ \forall |z| < 1$

but, 'U_1' is continuous and vanishes at $e^{i\theta_0}$ and such that,

$$|PU_1(z)| < \epsilon/2 \text{ for } |z - e^{i\theta_0}| < \delta$$

$$\Rightarrow |PU(z)| \leq |PU_1(z)| + |PU_2(z)|$$

$$< \epsilon/2 + \epsilon/2$$

$$= 2\epsilon/2$$

$|PU(z)| < \epsilon$ when $|z| < 1$ and $|z - e^{i\theta}|<$

$$\lim_{z \to e^{i\theta_0}} P_U(z) = 0 = U(\theta_0)$$

Laurent's series

A function $f(z)$ defined in the annulus $R_1 < |z-a| < R_2$ is said to have a laurent's expansion in this annulus

If there is a laurent's series $\Sigma an(z-a)^n$ which continuous in this annalus and whose sum is equal to $f(z)$ at any point of the annulus.

Laurent's Theorem.

Statement:

Any analytic function $f(z)$ in an annulus $R_1 < |z-a| < R_2$ has a laurent's expansion in the given annulus (or)

Any analytic function $f(z)$ whose region of define contain an annulus $R_1 < |z-a| < R_2$ then it can always be developed in a general power series of the form

$$f(z) = \sum_{-\infty}^{\infty} A_n (z-a)^n.$$

$$\text{where, } A_n = \frac{1}{2\pi i} \int_{|\delta-a|} \frac{f(\delta)}{|\delta-a|^{n+1}} d\delta$$

Proof:

Choose two numbers r_1 and r_2 such that

$$R_1 \leq r_1 \leq r_2 \leq R_2$$

The oriented boundary of on compack annulus $r_1 \leq r \leq r_2$ is the difference of the circle 'c' of centre

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~~a~~ and radius 'r' in the positive sence and the circle 'c' of centre and radius 'r' to in the positive sence

let $f(z)$ be analytic in annules $R_1 < |z-a| < R_2$

let C_1 and C_2 be the circles $|z-a| = R$ and $|z-a| = R_2$ respectively

let C be a circle $|z-a| = r$ such that $R_1 < r < R_2$

Define the function $f_1(z)$ and $f_2(z)$ as follows

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0$$

$$+ \dots + \frac{f^n(z_0)(z-z_0)^n}{n!}$$

$$f_1(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\delta)\,d\delta}{(\delta-z)} \text{ if } |z-a| < r < R_2$$

and

$$f_2(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\delta)\,d\delta}{(\delta-z)} \text{ if } R_1 < r < |z-a$$

Since $f(z)$ is analytic b/w C_1 and C_2

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_{(C_2-C_1)} \frac{f(\delta)}{(\delta-z)}\,d\delta$$

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$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)\,ds}{(s-z)} - \frac{1}{2\pi i} \int_{C_1} \frac{f(s)\,ds}{(s-z)}$$

$$\boxed{f(z) = f_1(z) + f_2(z)} \text{ --- } ①$$

By the definition of $f_1(z)$ and $f_2(z)$

we get $f_1(z)$ is analytic in the disc $|z-a| < R_2$

$\therefore$ If it can be expand as a Taylor series about $z=a$

$$f_1(z) = f_1(a) + \frac{f_1'(a)}{1!}(z-a) + \frac{f_1''(a)(z-a)^2}{2!} + \dots$$

$$\therefore \boxed{f_1(z) = \sum_{n=0}^{\infty} \frac{f_1^n(a)}{n!}(z-a)^n}$$

$$\Rightarrow f_1(z) = \sum_{0}^{\infty} A_n (z-a)^n \longrightarrow ②$$

where, $A_n = \frac{f_1^n(a)}{n!} \longrightarrow ③$

but, $f_1 = \frac{1}{2\pi i} \int_{C_2} \frac{f(\delta)}{(\delta - z)} d\delta$

$$f_1^n(z) = \frac{n!}{2\pi i} \int_{C_2} \frac{f(\delta)}{(\delta - z)^{n+1}} d\delta$$

sub $z = a$

$$f_1^n(a) = \frac{n!}{2\pi i} \int_{C_2} \frac{f(\delta)}{(\delta - a)^{n+1}} d\delta$$

$$\frac{f_1^n(a)}{n!} = \frac{1}{2\pi i} \int_{C_2} \frac{f(\delta)}{(\delta - a)^{n+1}} d\delta$$

$$A_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\delta)}{(\delta - a)^{n+1}} d\delta \text{ --- } ①$$

To find the develpment of $f_2(z)$ we assume that transformation

$$\delta = a + \frac{1}{\delta'}$$

$$\therefore z = a + \frac{1}{z'}$$

This transformation carries $|\delta - a| =$ into $|\delta| = \frac{1}{r}$ with -ve orientation.

$$f_2\left(a + \frac{1}{z'}\right) = \frac{-1}{2\pi i} \int_{|\delta'| = \frac{1}{r}} \frac{f\left(a + \frac{1}{\delta'}\right)\left(\frac{-1}{\delta'^2}\right) d\delta'}{\left(\frac{1}{\delta'} - \frac{1}{z'}\right)}$$

$$= \frac{1}{2\pi i} \int_{|\delta'| = \frac{1}{r}} \frac{f\left(a + \frac{1}{\delta'}\right)(\delta' z')}{(z' - \delta')(\delta')^2} d\delta'$$

$$= \frac{1}{2\pi i} \int_{|\delta'|=\frac{1}{r}} \frac{f(a+1/\delta')}{(z'-\delta')} \left(\frac{z'}{\delta'}\right) d\delta' \text{ — ⑤}$$

$\therefore$ $f_2(z)$ is analytic outside the circle $|\delta - z| = r$. $f_2(a+\frac{1}{z'})$ is analytic in the disc $|\delta'| < \frac{1}{r}$

By taylors series.

$$f_2(z) = \sum_{n=1}^{\infty} B_n (z')^n \text{ — ⑥}$$

$$= \sum B_n \left(\frac{1}{z-a}\right)^n$$

Where,

$$B_n = \frac{1}{2\pi i} \int_{|\delta'|=1/\delta} \frac{f(a+\frac{1}{\delta'})}{(\delta')^{n+1}} d\delta'$$

$$B_n = \frac{-1}{2\pi i} \int_{|\delta-a|=r} \frac{f(\delta)}{(\delta-a)^{n+1}} \left(\frac{-d\delta}{(\delta-a)^2}\right)$$

$$= \frac{1}{2\pi i} \int_{|\delta-a|=r} \frac{f(\delta)}{(\delta-a)^{-n+1}} d\delta$$

$$B_n = A_n$$

$$① \Rightarrow f(z) = f_1(z) + f_2(z) = \sum_{0}^{\infty} A_n (z-z_0)^n + \sum_{1}^{\infty} B_n (z-a)^{-n}$$

$$f(z) = f_1(z) + f_2(z)$$

$$f_1(z)+f_2(z)=\sum_{0}^{\infty} An(z-z_0)^n+\sum_{1}^{\infty} An(z-a)$$

$$=\sum_{0}^{\infty} An(z-z_0)^n+\sum_{-1}^{\infty} An(z-a)$$

$$f(z)=\sum_{n=-\infty}^{\infty} An(z-a)^n$$

Where,

$$An=\frac{1}{2\pi i}\int_{|z-a|=r}\frac{f(\gamma)}{(\delta-a)^{n+1}}\,d\delta.$$

Unit - IV
Entire function.

definition: Entire function (or) Integral function.

A function which is analytic in whole complex plane is said to be Entire function.

Def: order of Entire function.

let f(z) be an entire function then.

$$M(r) = \max. |f(z)|$$

on the circle $|z| = r$

Then the order of f(z) is defined by

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log[\log M(r)]}{\log r}$$

λ is the smallest no. such that, $M(r) \leq e^{r(\lambda+\epsilon)}$ for any given $\epsilon > 0$ as seen as 'r' is subbicienly large.

Note:-

1) $\overline{\lim_{r\to\infty}}$ → superior limit

2) $\underline{\lim_{r\to\infty}}$ → inberior limit.

Theorem: 1

Hadamard's three circle them:

10m

Statement:

suppose that $f(z)$ is analytic in the circlater region $r_1 < |z| < r_2$ and continuous on the closed annulus if $M(r)$ denotes the maximum of $|f(z)|$. for $z = r$ Then Show that $M(r) \leq M(r_1)^{\alpha} \cdot M($

where

$$\alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}$$

proof:

consider the function $z^{\alpha} f(z)$

This is multivalued function in the compact madules.

But it is The modulus value of $|z^{\alpha} f(z)|$ is single valued function

Hence,

$|z^{\alpha} f(z)|$ attains the maximum value on the boundary either on $|z| = 1$ (or) $|z| = r_2$

Hence, if $r_1 \leq r \leq r_2$

$$r^{\alpha} m(r) \leq \max [r_1^{\alpha} m(r_1) \cdot r_2^{\alpha} m(r_2)]$$

choose 'α' so hat,

$$r_1^{\alpha} m(r_1) = r_2^{\alpha} m(r_2) \longrightarrow ①$$

Then, $r^{\alpha} m(r) \leq r_2^{\alpha} M(r_2)$

Taking 'log' on both sides.

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$$\log[r^{\alpha} m(r)] \leq \log[r_2^{\alpha} M(r_2)] \longrightarrow ②$$

$$① \Rightarrow \frac{M(r_1)}{M(r_2)} = \left(\frac{r_2}{r_1}\right)^{\alpha}$$

$$r_1^{\alpha} M(r_1) = r_2^{\alpha} M(r_2)$$

$$① \Rightarrow \log\left[\frac{M(r_1)}{M(r_2)}\right] = \log\left(\frac{r_2}{r_1}\right)^{\alpha}$$

$$\log(M(r_1)) - \log M(r_2) = \alpha \log(r_2/r_1)$$

$$\frac{\log(M(r_1)) - \log M(r_2)}{\log(r_2/r_1)} = \alpha \longrightarrow ③$$

sub ③ in ②

$$② \Rightarrow \alpha \log r + \log M(r) \leq \alpha \log r_2 + \log M(r_2) \longrightarrow ④$$

$$\left[\frac{\log(M(r_1)) - \log M(r_2))}{\log(r_2/r_1)}\right] \log r + \log M(r) \leq \left[\frac{\log(M(r_1)) - \log M(r_2)}{\log(r_2/r_1)}\right] \log r_2 + \log M(r_2)$$

$$\frac{[\log(M(r_1)) - \log M(r_2)] \log r + \log M(r) \log(r_2/r_1)}{\log(r_2/r_1)} \leq \frac{(\log M(r_1) - \log M(r_2)) \cancel{\log} r_2 + \log M r_2 \log(r_2/r_1)}{\log(r_2/r_1)}$$

$$[\log(M(r_1)) - \log M(r_2)] \log r + \log M(r) \log(r_2/r_1) \leq [\log M(r_1) - \log M(r_2)] \log r_2 + \log M r_2 \log(r_2/r_r$$

$\log M(r_1)\log r - \log M(r_2)\log r + \log M(r)\log(r_2/r_1)$

$\le \log M(r_1)\log r_2 - \log M(r_2)\log r_2 + \log M(r_2)\log($

$\log M(r)\log\left(\frac{r_2}{r_1}\right) \le \log M(r_1)\log r_2 - \log M(r_2)\log r_2 + \log M($
$\log(r_2/r_1) - \log M(r_1)\log r + \log M($

$\log M(r)\log(r_2/r_1) \le (\log r_2 - \log r)\log M(r_1) + \left[\log\left(\frac{r_2}{r_1}\right) - \log r\right.$
$\left. + \log r\right]\log M(r_2)$

$\le \log M(r_1)\left[\log\left(\frac{r_2}{r}\right)\right. + \log M(r_2)\left[\log\left(\frac{r_2}{r_1}\right) - \log\right.$
$+\log$

$\le \log M(r_1)\log\left(\frac{r_2}{r}\right) + \log M(r_2)\left[\log\left(\frac{r_2}{r_1}\right) + \log\left(\frac{r}{r}\right.\right.$

$\le \log M(r_1)\log\left(\frac{r_2}{r}\right) + \log M(r_2)\left[\log\left(\frac{r_2}{r_1}\times\frac{r}{r_2}\right)\right]$

~~log M(r) log~~

$\log M(r)\log(r_2/r_1) \le \log M(r)\log\left(\frac{r_2}{r}\right) + \log M(r_2)\log(r/r_1)$

$\Rightarrow \log M(r)^{\log(r_2/r_1)} \le \log M(r)^{\log(r_2/r)} + \log M(r_2)^{\log(r}$

$\log\left[M(r)^{\log(r_2/r_1)}\right] \le \log\left[M(r_1)^{\log(r_2/r)} \cdot M(r_2)^{\log(}\right.$

$M r^{\log(r_2/r_1)} \le \cancel{\log}[M r_1]^{\log(r_2/r)} M(r_2)^{\log(r/}$

$M(r) \le M r_1^{\left[\frac{\log(r_2/r)}{\log(r_2/r_1)}\right]} m(r_2)^{\left[\frac{\log(r/r_1)}{\log(r_2/r_1)}\right]}$

~~$M(r) \le M M_1$~~

$$M(r) \le M(r_1)^{\alpha} M(r_2)^{(1-\alpha)}$$

Theorem (Jenson's formula) 2

Statement:

let $f(z)$ be analytic in the disc $|z| < \rho$ and let $a_1, a_2 \ldots a_n$ be the zero's of $f(z)$ in the open disc $|z| < \rho$. let $z=0$ be not a zero of $f(z)$ then

$$\log|f(0)| = -\sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})| d\theta$$

Proof:-

let $f(z)$ be analytic in the disc $|z| < \rho$

let $a_1, a_2 \ldots a_n$ be the zero's of $f(z)$ in the open disc $|z| < \rho$

let $z=0$ be not a zero of $f(z)$

$$F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \overline{a_i} z}{\rho(z-a_i)} \longrightarrow ①$$

is free from zero's in the disc

$\therefore F(z)$ as zero's at $a_1, a_2 \ldots a_n$

we get.

$F(z)$ is analytic in the disc $|z| < \rho$

Also

$$|F(z)| = \left| f(z) \prod_{i=1}^{n} \frac{\rho^2 - \bar{a}_i z}{\rho(z-a_i)} \right|$$

$$|F(z)| \leq |f(z)| \prod_{1}^{n} \frac{|z\bar{z} - \bar{a}_i z|}{\rho|z-a_i|}$$ on |z|

$$\leq |f(z)| \prod_{1}^{n} \frac{|z||\bar{z} - \bar{a}_i|}{\rho|z-a_i|}$$

$$= |f(z)| \prod_{1}^{n} \frac{\rho|\bar{z} - a_i|}{\rho|z-a_i|}$$

$$|F(z)| = |f(z)| \prod_{1}^{n} (1)$$

$|z| = |x+iy|$

$|\bar{z}| = |x-iy| =$

$$\therefore |F(z)| = |f(z)|$$

$\therefore |\bar{z}| = |z|$

also we know that,

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(\rho e^{j\theta})| \, d\theta$$

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})| \, d\theta \quad — ②$$

$$\therefore ① \Rightarrow F(z) = f(z) \prod_{1}^{n} \frac{\rho^2 - \bar{a}_i z}{\rho(z-a_i)}$$

$$\therefore |F(0)| = |f(0)| \prod_{1}^{n} \frac{\rho}{|a_i|}$$

$$\Rightarrow \log|F(0)| = \log\left[|f(0)| \prod_{1}^{n} \frac{\rho}{|a_i|}\right]$$

$$\therefore \log|F(0)| = \log|f(0)| + \sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) \rightarrow ③$$

Sub ③ in ②

$$\log|f(0)| + \sum \log\left(\frac{\rho}{|a_i|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|F(\rho e^{i\theta})|\, d\theta$$

$$\boxed{\log|f(0)| = -\sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\, d\theta}$$

Theorem: [Poisson Jenson's formula (3)

statement:

let $f(z)$ be analytic in disc $|z|<\rho$ and let $a_1, a_2 \ldots a_n$ be the zero's of $f(z)$ in the open disc $|z|<\rho$

let $z=0$ be not a zero of $f(z)$

Then,

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{\rho^2 - \bar{a}_i z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right) \log|f(\rho e^{i\theta})|\, d\theta$$

Provided that, $f(z) \neq 0$

let $f(z)$ be analytic in disc $|z|<\rho$

let $a_1, a_2 \ldots a_n$ be the zero's of $f(z)$ in the open disc $|z|<\rho$

let $z=0$ be not a zero of $f(z)$

$$\text{If } F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}$$

is free from zero's in the disc
$f(z)$ as zero's $a_1, a_2, \ldots, a_n$
we get

$F(z)$ is analytic in the disc
$|z| < \rho$

Also,

$$|F(z)| = \left| f(z) \prod_{i=1}^{n} \frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)} \right|$$

$$\log|F(z)| = \log\left| f(z) \prod_{i=1}^{n} \frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)} \right|$$

$$\Rightarrow \log|F(z)| = \log|f(z) + \sum_{i=1}^{n} \log\left| \frac{\rho^2 - \overline{a_i}z}{\rho(z-a} \right.$$

using poission formula

$$\log|F(z)| = \frac{1}{2\pi} \int_0^{2\pi} Re\left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right) \log|f(\rho e^{i\theta}$$

From ① & ②

$$\log|f(z)| + \sum_{i=1}^{n} \log\left| \frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)} \right| = \frac{1}{2\pi} \int_0^{2\pi} Re\left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right) \log|$$

Now,

$\log |f(z)| =$.

$$|F(z)| = |f(z)| \prod_{1}^{n} \left| \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)} \right| \text{ on } |z| = \rho$$

$$\Rightarrow |F(\rho e^{i\theta})| = |f(\rho e^{i\theta})| \prod_{1}^{n} \left| \frac{z\bar{z} - \bar{a}_i z}{\rho(z - a_i)} \right|$$

$$= |f(\rho e^{i\theta})| \prod_{1}^{n} \frac{|z| \, |\bar{z} - \bar{a}_i|}{\rho(z - a_i)}$$

$$= |f(\rho e^{i\theta})| \prod_{1}^{n} \frac{\rho\,(\overline{z - a_i})}{\rho(z - a_i)}$$

$$\therefore |F(\rho e^{i\theta})| = |f(\rho e^{i\theta})| \longrightarrow ④$$

③ $\Rightarrow$

$$\boxed{\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)} \right| + \frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{Re}\left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right) \log |f(\rho e^{i\theta})| \, d\theta}$$

Theorem: 4

An entire function of fractional order assumes every finite value infinitely many times.

Proof:

Let 'a' be any constant

Then f and $(f - a)$ have the same order.

It is enogh to prove that f as infinitely any zero's

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Suppose 'f' has only a finite number of zero's

we can ~~find~~ divide by a polynomial and obtain the function of the same order without zero's.

It must be of the form $e^{g(z)}$ where $g(z)$ is the polynomial

$\therefore$ ~~O(g) =~~

$$\text{order}[e^{g(z)}] = \deg[g(z)]$$

which is an integer

This is contraduction

$f(z)$ as infinitely many zeros

Ex//

find the order of $e^{(e^r)}$

sol:

Here, $M(r) = e^{(e^r)}$

$\Rightarrow \log m(r) = e^r$

$\Rightarrow \log[\log m(r)] = r$

$$\Rightarrow \frac{\log[\log M(r)]}{\log r} = \frac{r}{\log r}$$

$$\lim_{r\to\infty} \frac{\log[\log m(r)]}{\log r} = \lim_{r\to\infty}\left(\frac{r}{\log r}\right)$$

By L'Hospital rule.

$$= \lim_{r\to\infty}\left(\frac{1}{1/r}\right) = \lim_{r\to\infty}(r)$$

$= \infty$

$\therefore$ The order of $e^{(e^r)}$ is ∞

2 Find the order of $e^{(az)}$

W.K.T

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$e^{az} = 1 + \frac{az}{1!} + \frac{a^2z^2}{2!} + \cdots$$

$$|e^{az}| \leq 1 + \frac{a|z|}{1!} + \frac{a^2|z|^2}{2!} + \cdots$$

$$\Rightarrow |e^{az}| \leq 1 + \frac{ar}{1!} + \frac{a^2r^2}{2!} + \cdots \quad \because |z| \leq r.$$

$$\therefore |e^{az}| \leq e^{ar}$$

hence,

$$m(r) = \max(|e^{az}|)$$

$$m(r) = e^{ar}$$

$$\Rightarrow \log m(r) = ar$$

$$\frac{\log[\log m(r)]}{\log r} = \frac{\log(ar)}{\log r}$$

$$\lim_{r \to \infty} \left[\frac{\log(\log m(r))}{\log r}\right] = \lim_{r \to \infty}\left(\frac{\log ar}{\log r}\right)$$

$$= \lim_{r \to \infty} \frac{\log a + \log r}{\log r}$$

$$= \lim_{r \to \infty} \frac{\log a}{\log r} + 1$$

$$= 0 + 1$$

$$\rho = 1$$

The order of $e^{(az)}$ is 1

3 find the order of cos z

W.k.T

$$\cos z = \frac{e^{z}+e^{-z}}{2} \qquad |z| \leq r$$

$$\Rightarrow \cos z = \frac{e^{r}+e^{-r}}{2}$$

$$m(r) = \max(\cos z)$$

$$= \frac{e^{r}+e^{-r}}{2}$$

$$m(r) = e^{r}\left[\frac{1+e^{-2r}}{2}\right]$$

$$\log m(r) = \log\left[e^{r}\left(\frac{1+e^{-2r}}{2}\right)\right]$$

$$= \log e^{r} + \log\left(\frac{1+e^{-2r}}{2}\right)$$

$$= r + \log\left(\frac{1+e^{-2r}}{2}\right)$$

$$= r\left[1+\tfrac{1}{r}\log\left(\frac{1+e^{-2r}}{2}\right)\right]$$

$$\log[\log m(r)] = \log r\left[1+\tfrac{1}{r}\log\left(\frac{1+e^{-2r}}{2}\right)\right]$$

$$\Rightarrow \frac{\log[\log m(r)]}{\log r} = \frac{\log r}{\log r} + \frac{\log\left[1+\tfrac{1}{r}\log\left(\frac{1+e}{2}\right.\right.}{\log r}$$

$$\lim_{r\to\infty}\frac{\log[\log m(r)]}{\log r} = 1 + \lim_{r\to\infty}\frac{\log\left[1+\tfrac{1}{r}\log\left(\frac{1+e}{2}\right.\right.}{\log r}$$

$$= 1 + \tfrac{1}{\infty} = 1+0$$

The order of cos z is $= 1$

Genus

The Genus of an entire function is the smallest integer 'h' such that f(z) can be represented in the form a

$$f(z) = e^{g(z)} \prod_{n} \left[1 - \frac{z}{a_n}\right] e^{\left[\left(\frac{z}{a_n}\right) + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \frac{1}{3}\left(\frac{z}{a_n}\right)^3 + \dots \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right]}$$

where g(z) is a polynomial of degree $\leq h$

Equicontinuous:- (or) continuity

The function in a family $\mathcal{F}$ is said to be Equicontinuous on a set $E \subset \Omega$ if only if for each $\epsilon > 0$ there exist $\delta > 0$ such that $d((f(z), f(z_0)) < \epsilon$ whenever $|z - z_0| < \delta$ and $z, z_0 \in E$ $\forall$ the function $f \in F$

Normality (or) Normal

A family $\mathcal{F}$ is said to be normal in Ω if every sequence $\{f_n\}$ of functions $f_n \in \mathcal{F}$ contains a subsequence which converges uniformly on every compact subset of Ω

Totally boundedness. (or) totally bounded set

A set is compact iff it is totally boundedness.

locally bounded

Every sequence as a subsequence which converges uniformly on compact sets iff it is locally bounded.

Theorem (5)

A family $\mathcal{F}$ of an analytic function is normal w.r.to iff The function in $\mathcal{F}$ are uniformly bounded on every compact set

Proof

Necessary part

Assume that $\mathcal{F}$ is normal

To prove

The function in $\mathcal{F}$ are uniformly bounded on every compact set

∴ It is sufficient to show that
uniformly bounded
$\mathcal{F}$ is equicontinuous on every small disc

let 'c' be a boundary of a closed disc in 'Ω' of radius 'r'

∴ $f(z, z_0)$ inside 'c'

by using cauchy integral theorem

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\delta)}{\delta - z}\,d\delta \text{ and}$$

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(\delta)}{\delta - z_0}\,d\delta$$

$$\Rightarrow f(z) - f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(\delta)}{\delta - z}\,d\delta - \frac{1}{2\pi i}\int_c \frac{f(\delta}{\delta - z_0}$$

$$= \frac{1}{2\pi i}\int_c f(\delta)\,d\delta \left[\frac{1}{\delta - z} - \frac{1}{\delta - z_0}\right]$$

$$= \frac{1}{2\pi i} \int_C f(\delta)\, d\delta \left[\frac{\delta - z_0 - (\delta - z)}{(\delta - z)(\delta - z_0)} \right]$$

$$= \frac{1}{2\pi i} \int_C f(\delta)\, d\delta \left[\frac{z - z_0}{(\delta - z)(\delta - z_0)} \right]$$

$$\therefore f(z) - f(z_0) = \frac{z - z_0}{2\pi i} \int_C \left[\frac{f(\delta)}{(\delta - z)(\delta - z_0)} \right] d\delta$$

let, $|f(z)| \leq M$ on 'c',
z and z_0 is smaller disc concentric disc of radius $(r/2)$

$$\Rightarrow |f(z) - f(z_0)| = \left| \frac{z - z_0}{2\pi i} \int_C \left[\frac{f(\delta)}{(\delta - z)(\delta - z_0)} \right] d\delta \right.$$

$$\Rightarrow |f(z) - f(z_0)| \leq \frac{|z - z_0|}{2\pi i} \int_C \frac{|f(\delta)|}{|\delta - z||\delta - z_0|} |d\delta|$$

$$\leq \frac{|z - z_0|}{2\pi} \left(\frac{M}{(r/2)(r/2)} \right) \int_C |d\delta|$$

$$\leq \frac{M|z - z_0|}{2\pi} \left(\frac{4}{r^2} \right) (2\pi r)$$

$$|f(z) - f(z_0)| \leq \frac{4M}{r} |z - z_0| \quad \text{—} ①$$

if $|z-z_0| < \epsilon$

$\therefore$ ① $\Rightarrow |f(z)-f(z_0)| < \epsilon$

This proves equicontinuous of the smallest disc

conversely

let as assume that the function in $\mathcal{F}$ are uniformly bounded on every compact set

To prove $\mathcal{F}$ is normal

It is enough to prove that $\mathcal{F}$ is equi continuous on every compact set in 'Ω'

let E be a compact set 'Ω'

W.K.T

Each point of E is the centre of a disc of radius 'r'

Also, the open disc of radius $r/4$ form on open covering of E

Since, E is compact

$\Rightarrow$ E have a finite subcovering denote the corresponding ~~[illegible]~~ centre radius and boundarie are

δ_k, r_k, M_k

let $r = \min r_n$ and $M = \max M_k$

for a given $\epsilon > 0$
and let $\delta = \min\left[\frac{r}{4}, \frac{\epsilon r}{4m}\right]$

If $|z - z_0| < \delta$ and $|z - \delta_k| < \frac{r_k}{4}$

$$\therefore |z-\delta_k| = |(z-z_0) + (z_0 - \delta_k|$$
$$\leq |z-z_0| + |z_0 - \delta_k|$$
$$\leq \delta + r_k/4$$
$$< \frac{r_k}{4} + \frac{r_k}{4}$$

$$|z - \delta_k| < \frac{r_k}{2}$$

$$\therefore ① \Rightarrow |f(z) - f(z_0)| \leq \frac{4m_k\delta}{r_k}$$
$$\leq \frac{4m\delta}{r}$$
$$\leq \frac{4m}{r} \epsilon \frac{r}{4m}$$

$$|f(z) - f(z_0)| < \epsilon$$

Hence

$\mathcal{F}$ is equicontinuous on every compact set 'E' in 'Ω'.

By Arzela's theorem.

$\mathcal{F}$ is normal with respect to 'C'

Theorem: 6

A locally bounded family of analytic function has locally bounded derivative.

Proof:-

The cauchy raperesentation of derivative.

If 'c' is the boundary of the closed disc 'z' of radius 'r'

$\therefore$

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\delta)}{(\delta - z)} d\delta$$

$$f'(z) = \frac{1}{2\pi i}\int_c \frac{f(\delta)}{(\delta - z)^2} d\delta$$

Hence

$|f'(z)| < \frac{4m}{r}$ in the consentric disc of radius $r/2$

We see that f' are indeed locally bounded.

Theorem [Arzela's theorem (or) Arzela-Ascali The]

statement (7)

A family $\mathcal{F}$ of continuous function with value i~~mat~~ in a matric space 's' is in normal in the region Ω of the complex plane iff

i) $\mathcal{F}$ is equicontinuous on every compact set equicontinuous on Ω.

ii) for any $z \in \Omega$ the values $f(z)$ is in compact subset of 'S' for all $f \in \mathcal{F}$

Proof:

Necessary condition:

Let $\mathcal{F}$ be a family of continuous function with values ~~mo~~ in a metric space 'S' and be normal in the region 'Ω'

Assume that $\mathcal{F}$ is not equi continuous on E. Then $F \in \mathcal{F}$; $\epsilon > 0$ and a sequence of points. $\{z_n\}\{z_n'\}$ of E and function $f_n \in \mathcal{F}$

such that

$$|z_n - z_n'| \to 0$$

but,

$$d(f_n(z_n), f_n(z_n')) \geq \epsilon \longrightarrow ①$$

because 'E' is compact.

we can choose subsequence of $\{z_n\}\{z_n'\}$

which converge to a comman limit Points $z_n'' \in E$

because $\mathcal{F}$ is normal There exist subsequence of ~~normal~~ $\{F_n\}$ which is converges uniformly on E

we can choose on the three subsequence to have the same subscrit $\{n_k\}$

The limit of function on $\{f_{n_k}\}$ is unifo continuous on the compact set E.

Hence we can find k such that.

$$d(f_{n_k}(z_{n_k}), f(z_{n_k})) < \epsilon/3 \longrightarrow ②$$

$$d(f_{n_k}(z'_{n_k}), f(z'_{n_k})) < \epsilon/3 \rightarrow ③$$

$$d(f(z_{n_k}), f(z'_{n_k})) < \epsilon/3 \longrightarrow ④$$

②③,④

$$\therefore d(f_{n_k}(z_{n_k}), f_{n_k}(z'_{n_k})) < \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$< 3\epsilon/3$$

$$< \epsilon$$

we arrove at a contraduction

~~There~~ $\therefore$ our assumtion is wrong

Hence $\mathcal{F}$ is unquie continuous on E

we will show that closser of the set is compact

$(f(z) / f \in \mathcal{F}, z \in \mathcal{F}\}$ is compact

let $\{w_n\}$ be a sequece in the closser.
To each w_n we find $f_n \in \mathcal{F}$
show that distance of
$$d(f_n(z), w_n) < \frac{1}{n}$$

by the definition of normal.
$\exists$ a convergen sequence $\{f_{n_k}(z)\}$
and the Sequence $\{w_{n_k}\}$ is conveges
to the same value,

$\therefore$ The set is $\{f(z) \mid f \in \mathcal{F}, z \in \mathcal{F}\}$ is
compact.

subficient condition
let a family $\mathcal{F}$ have the both
propertys of (i) and (ii)
we absenve that $\mathcal{F}$ is every where
dense sequence of point δ_k in Ω
for the points rational co-ordinaties
difference from the sequene $\{f_n\}$
A subsequene which converges bet
all the points δ_k ~~To find the~~ conv.
To find the ~~converges~~ sequence which converge
one points which all the points of ~~and~~
converge (ii)
we can find a matric of
subscripts.

$$n_{11} < n_{12} < \cdots < n_{1j} < \cdots$$
$$n_{21} < n_{22} < \cdots < n$$
$$\vdots$$
$$\vdots$$
$$n_{k_1} < n_{k_2} < \cdots <$$
$$\vdots$$

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such that each row is contain in the Preseding

$$\lim_{j \to \infty} f_{n_{k_j}}(\delta_k) \text{ is exists } \forall k$$

The diagonal sequence $\{n_{jj}\}$ stricting increasing and it is subsequence of row is eqn ⑤

Hence $\{f_{n_j}\}$ is a subsequence of $\{f_n\}$ which converges at all points 'δ_k'

Let as consider a compact set $E \subset \Omega$ Assume that $\{f_i\}$ equi continuous on E

We will prove that the sequence $\{f_{n_j}\}$ is converges uniformly on 'E'.

$\epsilon > 0$ There exist $\delta > 0$ such that

$$d(f(z), f'(z)) < \epsilon/3 \quad \forall \, |z - z'| < \delta \; \forall \, z, z'$$

because 'E' is compact

It can be covered by a finite nomber of $(\delta/2)$ neighboorhods

we select point δ_k from each of this neighboorhoud.

$$\therefore d(f_{n_i}(\delta_k), f_{n_j}(\delta_k)\} < \epsilon/3 \quad \forall \, i, j \in \delta_k$$

$\forall \, z \in E$

one of the δ_k with in the distance of from z we get

$d \{ (f_{n_i}(z), f_{n_i}(\delta_k) \} < \epsilon/3$

and

$d (f_{n_j}(z), f_{n_j}(\delta_k)) < \epsilon/3$

By Triangular inequality

$$d (f_{n_i}(z), f_{n_j}(z)) \leq \epsilon/3 + \epsilon/3 + \epsilon/3$$
$$< 3\epsilon/3$$
$$< \epsilon$$

All value of $f(z)$ belong to a compact set and

Hence a complete subset of 'S'

It follows that sequence $\{f_{n_j}\}$

$\therefore \mathcal{F}$ is normal family

Theorem (8)

The family $\mathcal{F}$ totally bounded iff to every compact set $E \subset \Omega$ and $\epsilon > 0$ It is possible to find $f_1, f_2, \ldots f_n \in \mathcal{F}$ such that $\forall f \in \mathcal{F}$ satisfies $\rho(f, f_j) < \epsilon$ (or) $d(f, f_j) < \epsilon$ on E : $\forall f_j$

Let $\mathcal{F}$ be a totally bounded Then there exist $f_1, f_2, \ldots f_n \in \mathcal{F}$; $f \in \mathcal{F}$

$\Rightarrow \rho(f, f_j) < \epsilon \longrightarrow ①$

we know that

$$\rho(f, g) = \sum_{k=1}^{\infty} \delta_k(f, g) 2^{-k}$$

$$g = f_j \Rightarrow \rho(f, f_j) 2^{-k}$$

$$= \sum_{1}^{n} \delta_k(f, f_j)$$

$$\therefore \delta_k(f, f_j) < 2^k(\epsilon) \longrightarrow ②$$

(or)

$\delta(f, f_j) < 2^k \epsilon$ on E_k $\forall$ k

we can find $\delta(f, f_j)$ arbitarity same value on $\{f_j\}$ and the same is $d(f, f_j)$ also

$① \Rightarrow d(f, f_j) < \epsilon$ on E' $\forall$ f_j

conversely

we choose k_0,

$2^{-k_0} < \epsilon/2$

By assumption

we can find $f_1, f_2, f_3 \ldots f_n \ni$ $f \in \mathcal{F}$ satisfies the inequality

$$\delta(f, f_i) \leq d(f, f_j) < \epsilon/2k_0 \text{ on } E_{k_0}$$

$$\Rightarrow \delta_k(f, f_j) < \epsilon/2k_0 \ \forall \ k \leq k_0$$

$$Ⓐ \Rightarrow \rho(f, f_j < k_0\left(\frac{\epsilon}{2k_0}\right) + 2_0^{-k_0 - 1} + \ldots$$

$$= \epsilon/_2 + 2^{-k_0-1}[1 + 2^{-1} + 2^{-2} + \cdots]$$

$$= \epsilon/_2 + 2^{-k_0-1}(2^{-1})^{-1}$$

$$= \epsilon/_2 + 2^{-k_0-1}(2)$$

$$= \epsilon/_2 + 2^{-k_0} \Rightarrow \epsilon/_2 + \epsilon/_2$$

$$\rho(f, f_j) < \epsilon \quad \forall f_j$$

The family $\mathcal{F}$ is totally bounded.

Hadamard's Theorem $\textcircled{9}$

statement :-

The genus and the order of the entire function satisfies the double inequality $h \le \lambda \le h+1$

let $f(z)$ be an integral function assume that $f(z)$ is finite genus 'h'

we will Prove that

$$\lambda \le h+1$$

By weierstrass's Theorem

$$f(z) = e^{g(z)} \prod_{n \neq 0} (1 - z/a_n)\, e^{[z/a_n + 1/2(z/a_n)^2 + \frac{1}{m_n}[z/a_n]^{m_n}]}$$

where $g(z)$ is a Polynomial of degree $\le h$

let,

$$p(z) = \prod_{n\neq 0}\left(1 - z/a_n\right) e^{\left[z/a_n + \frac{1}{2}\left(z/a_n\right)^2 + \cdots + \frac{1}{m_d}\left(z/a_n\right)^{m\cdots}\right]}$$

It enough to point that, $P(z)$ is of order $\leq h+1$

$$P(z) = \prod_{n\neq 0} E_n\left(z/a_n\right) \longrightarrow \circledast$$

where

$$E_h(u) = (1-u)\, e^{\left[u + \frac{1}{2}u^2 + \cdots \frac{1}{h}u^h\right]} \quad -①$$

if $h = 0$

$$E_0(u) = (1-u)\, e^0$$

$$E_0(u) = (1-u) \longrightarrow \circledast \; \cancel{\circledast}$$

claim

$$\log |E_h(u)| \leq (2h+1)\, |u|^{h+1}$$

case (i) If $|u| < 1$

$$① \Rightarrow |\log E_h(u)| = \left|\log (1-u)\, e^{\left[u + \frac{1}{2}u^2 + \right.}\right.$$

$$\Rightarrow \log |E_h(u)| \leq \left| (-u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots - \frac{u^h}{h} \right.$$
$$+ u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots \frac{u^h}{h}$$

$$= \left| \frac{-u^{h+1}}{h+1} - \frac{u^{h+2}}{h+2} - \cdots \right|$$

$$\leq \frac{|u|^{h+1}}{h+1} + \frac{|u|^{h+2}}{h+2} + \cdots$$

$$= \frac{|u|^{h+1}}{h+1}\left[1+|u|+\frac{|u|^2}{2}\cdots\right]$$

$$\log|E_h(x)| = \frac{|u|^{h+1}}{h+1}\left[1-|u|\right]^{-1}$$

$$\log|E_h(x)| = \frac{|u|^{h+1}}{h+1}\left(\frac{1}{1-|u|}\right)$$

$$\log|E_h(u)| \le \frac{|u|^{h+1}}{1-u}$$

$$\therefore (1-u)\log|E_h(u)| \le |u|^{h+1} \quad \text{——②}$$

$$① \Rightarrow E_h(u) = (1-u)\,e^{\left(u+\frac{u^2}{2}+\cdots\frac{u^{h-1}}{h-1}\right)}e^{u^h/h}$$

$$= E_{h-1}(u)\,e^{u^h/h}$$

$$\frac{E_h(u)}{E_{h-1}(u)} = e^{u^h/h}$$

$$\Rightarrow \left|\frac{E_h(u)}{E_{h-1}(u)}\right| = \left|e^{u^h/h}\right|$$

$$\Rightarrow \log\left|\frac{E_h(u)}{E_{h-1}(u)}\right| = \log\left|e^{u^h/h}\right|$$

$$\Rightarrow \log\left|\frac{E_h(u)}{E_{h-1}(u)}\right| = \left|\frac{u^h}{h}\right|$$

$$\log|E_h(u)| - \log|E_{h-1}(u)| \le |u|^h$$

$$\log |E_h(u)| \leq \log |E_{h-1}(u)| + |u|^h \quad ③$$

we will prove that by induction method on

When $h=0$

$$\Rightarrow |E_0(u)| \leq |1-u| \leq 1+|u| \quad (\text{by} \oplus ④$$

$$\Rightarrow \log |E_0(u)| \leq \log |(1+u)| \leq |u|$$

$$\therefore \log |E_h(u)| \leq (2h+1) |u|^{h+1}$$

The claim is true for $h=0$

let us assume that the claim is true for all the values $\leq h$

$$\text{ie)}\ \log |E_{h-1}(u)| \leq (2h-1) |u|^h \quad ④$$

$\forall$ values of h

$$② \Rightarrow (1-|u|) \log |E_h(u)| \leq |u|^{h+1} \text{ for } |u| < 1$$

$$\Rightarrow \log |E_h(u)| - |u| \log |E_h(u)| \leq |u|^{h+1}$$

$$\Rightarrow \log |E_h(u)| \leq |u|^{h+1} + |u| \log |E_h(u)$$

$$\leq |u| \left[|u|^h + \log |E_h(u)| \right]$$

$$\leq |u| \left[|u|^h + |u|^h + \log E_{h-1}(u) \right]$$

$$\leq |u| \left[2|u|^h + \log E_{h-1}(u) \right]$$

$$\leq |u| \left[2|u|^h + (2h-1) |u|^h \right] \text{ by } ④$$

$$\leq |u|^{h+1} \left[2+2h-1 \right].$$

$$\log |E_h(u)| \leq (2h+1) |u|^{h+1}$$

case (ii) if $|u| \geq 1$

$$③ \Rightarrow \log|E_h(u)| \leq \log|E_{h-1}(u)| + |u|^h$$

$$\leq (2h-1)|u|^h + |u|^h \text{ (by ④)}$$

$$= 2h|u|^h - |u|^h + |u|^h$$

$$= 2h|u|^h$$

$$\leq 2h|u||u|^h$$

$$\leq 2h|u|^{h+1}$$

$$\therefore \log|E_h(u)| \leq (2h+1)|u|^{h+1}$$

∴ The claim is true

now,

$$|P(z)| = \prod_{n \neq 0} \left|E_h\left(\frac{z}{a_n}\right)\right|$$

$$= \sum_n \log\left|E_h\left(\frac{z}{a_n}\right)\right|$$

$$\leq \sum_h (+2h+1)\left|\frac{z}{a_n}\right|^{n+1} \text{ (by claim)}$$

$$|P(z)| = (2h+1)|z|^{h+1} \sum_n \left(\frac{1}{|a|^{h+1}}\right)$$

∴ The order of $p(z)$ is almost the order of $(h+1)$

$$\lambda \leq h+1 \quad —⑤$$

next to prove that $h \leq \lambda$

If, $f(z)$ is of finite order λ and let 'h' be the largest integer $\leq \lambda$

$$h+1 > \lambda$$

we will prove that 'h' is the genus of $f(z)$

ie) To prove, $\sum_n \frac{1}{|a_n|^{h+1}}$ is converges for the smallest integer 'h' let $\nu(\rho)$ denote the no. of zero's with $|a_n| \le \rho$

by Jenson's formula.

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta - \sum_{i=1}^{\nu(\rho)} \log\left(\frac{\rho}{|a_i|}\right.$$

laplase ρ by 2ρ end omit the terms

$$\log\left(\frac{2\rho}{|a_n|}\right) \text{ with } |a_i| \geq \rho$$

$$\Rightarrow \log f(0) = -\sum_{1}^{\nu(\rho)} \log\frac{2\rho}{|a_n|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d$$

$$\le -\sum_{1}^{\nu(\rho)} \log\left(\frac{2}{1}\right) + \sum_{1}^{\nu(\rho)} \log\frac{\rho}{|a_n|}$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \log|f(2\rho e^{i\theta})|\,d\theta$$

$$\le -\nu(\rho)\log 2 + \sum_{1}^{\nu(\rho)} \log\frac{\rho}{|a_n|} + \frac{1}{2\pi}\int_0^{2\pi} \log$$

$$|f(2\rho e^{i\theta})|\,d\theta$$

$$\log|f(0)| + \sum_{1}^{\nu(\rho)} \log\frac{\rho}{|a_n|} + \frac{1}{2\pi}\int_0^{2\pi} \log[f(2\rho e^{i\theta})]\,d\theta$$

$$\le -\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(2\rho e^{i\theta})|$$

$\Rightarrow \nu(\rho)$ is bounded above

$\Rightarrow \nu(\rho) \le M(r)$

The definition of ρ

$$\Rightarrow \lim_{r \to \infty} \frac{n(r)}{r^{\rho+\epsilon}} = \frac{1}{\infty} = 0 \longrightarrow (i)$$

$\forall\ \epsilon > 0$ arrange the zeros of $f(z)$ according to the absolute value

$$|a_1| \le |a_2| \le |a_3| \le \dots \le |a_n| \le \dots$$

$$(i) \Rightarrow \text{for } n \to \infty \Rightarrow \frac{n(|a_n|)}{|a_n|^{\rho+\epsilon}} < 1$$

$$\Rightarrow \frac{1}{n(|a_n|)} > \frac{1}{|a_n|^{\rho+\epsilon}}$$

$$\Rightarrow \frac{1}{|a_n|^{\rho+\epsilon}} < \frac{1}{n(|a_n|)} \longrightarrow (ii)$$

[It is clear that $n \le n(|a_n|)$ for $n \to \infty$]

$$\therefore (ii) \Rightarrow \frac{1}{|a_n|^{\rho+\epsilon}} < \frac{1}{n}$$

$$\Rightarrow \frac{1}{|a_n|} \le \left(\frac{1}{n}\right)^{\frac{1}{\rho+\epsilon}}$$

$$\Rightarrow \frac{1}{|a_n|^{\rho+1}} \le \left(\frac{1}{n}\right)^{\frac{\rho+1}{\rho+\epsilon}}$$

$$\Rightarrow \cancel{\frac{1}{|a_n|^{\rho+1}}}$$

$$\Rightarrow \sum \left(\frac{1}{|a_n|}\right)^{\rho+1} \le \sum \left(\frac{1}{n}\right)^{\frac{\rho+1}{\rho+\epsilon}}$$

Choose, $\rho+1 > \rho+\epsilon$

$$\therefore \frac{n+1}{\lambda+\epsilon} > 1$$

(since, $\sum \frac{1}{n^p}$ is convergerent for $p>1$)

$$\therefore \epsilon \left(\frac{1}{n}\right)^{\frac{n+1}{\lambda+\epsilon}}$$ is converges

hence,

$$\sum \left(\frac{1}{|a_n|}\right)^{n+1}$$ is convergent $\longrightarrow$ ①

Next, we will show that,

The Polynomial $g(z)$ is of degree $\leq n$:-

By poisson-Jensons formula

$$\log|f(z)| = -\sum_{i=1}^{\nu(\rho)} \log\left|\frac{\rho^2-\bar{a}_i z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right) \log|f(\rho e^{i\theta})|\, d\theta$$

$$\log|f(z)| = -\sum_{i=1}^{\nu(\rho)} \log|\rho^2-\bar{a}_i z| + \sum_{i=1}^{\nu(\rho} \log|\rho(z-a_i)|$$
$$+\frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right) \log|f(\rho e^{i\theta})|$$

Apply the operation $\left(\frac{\partial}{\partial x}\right) - i\left(\frac{\partial}{\partial y}\right)$ we get

$$\frac{f'(z)}{f(z)} = -\sum_{i=1}^{\nu(\rho)} \frac{-\bar{a}_i}{\rho^2-\bar{a}_i z} + \sum_{i=1}^{\nu(\rho)} \frac{\rho}{\rho(z-a_i)} + \frac{1}{2\pi}\int_0^{2\pi} (\rho\, \cancel{e^{i\theta}} - z$$

$$\pm \frac{\rho e^{i\theta}+z}{(\rho e^{i\theta}-z)^2} \log|f(e^{i\theta})|\, d\theta.$$

$$= \sum_i \frac{\overline{a_i}}{\rho^2 - \overline{a_i} z} + \sum_i \frac{1}{z - a_i} + \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta}}{(\rho e^{i\theta} - z)^2} \log|f(\rho e^{i\theta})|\, d\theta$$

$$= \sum \frac{\overline{a_i}}{\rho^2 - \overline{a_i} z} + \sum_i (z - a_i)^{-1} + \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta}}{(\rho e^{i\theta} - z)^2} \log|f(\rho e^{i\theta})|\, d\theta$$

Diff. 'h' time

$$\Rightarrow D^h\left(\frac{f'(z)}{f(z)}\right) = -h! \sum_i (a_i - z)^{-h-1} + h! \sum_i \overline{a_i}^{h+1} (\rho^2 - \overline{a_i} z)^{-h-1}$$

$$+ (h+1)! \int_0^{2\pi} 2\rho e^{i\theta} (\rho e^{i\theta} - z)^{-h-2} \log|f(\rho e^{i\theta})|\, d\theta \quad \text{— ⑩}$$

now,

$$\sum_{i=1}^{\vartheta(\rho)} \frac{\overline{a_i}^{h+1}}{(\rho^2 - \overline{a_i} z)^{h+1}} \le \sum_{i=1}^{\vartheta(\rho)} \frac{|\overline{a_i}|^{h+1}}{|\rho^2 - \overline{a_i} z|^{h+1}} \le \sum_i \frac{e^{h+1}}{(\rho^2 - \rho(\frac{\rho}{2}))^{h+1}}$$

Sub. $z = \frac{\rho}{2}$

$$= \sum_i \frac{\rho^{h+1}}{(\rho^2/2)^{h+1}}$$

$$= \sum_i (2^{h+1}) (e^{+h+1}) (\rho^{-2h-2})$$

$$\Rightarrow \left|\sum_i \frac{\overline{a_i}^{h+1}}{(\rho^2 - \overline{a_i} z)^{h+1}}\right| \le \frac{2^{h+1}}{\rho^{h+1}} \vartheta(\rho)$$

$$\left|\sum_i \frac{\overline{a_i}^{h+1}}{(\rho^2 - \overline{a_i} z)^{h+1}}\right| \to 0 \text{ as } \rho \to \infty \longrightarrow ⑪$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta} \log|f(\rho e^{i\theta})|\, d\theta}{(\rho e^{i\theta} - z)^{h+2}} \le \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta}}{|\rho e^{i\theta} - z|^{h+2}} \log|f(\rho e^{i\theta})|\, d\theta$$

Sub $z = \rho/2$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho(1)}{(\rho-\rho/2)^{h+2}} \log |f(\rho e^{i\theta})| \, d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho}{(\rho^2/2)^{h+2}} \log |f(\rho e^{i\theta})| \, d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{(2)\, 2^{h+2}}{\rho^{h+1}} \log |f(\rho e^{i\theta})| \, d\theta$$

$$= \frac{2^{h+2}}{\pi \rho^{h+1}} \int_0^{2\pi} \log |\rho (e^{i\theta})| \, d\theta$$

$$\therefore \left| \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+1}} \log |f(\rho e^{i\theta})| \, d\theta \right| \to 0 \text{ as } \rho \to \infty \longrightarrow ②$$

⑩ $\Rightarrow D^h\left(\frac{f'(z)}{f(z)}\right) = -h! \sum_i^{\nu(\rho)} \frac{1}{(a_i-z)^{h+1}} \longrightarrow$ ⑬ as $\rho \to \infty$

$$\Rightarrow f(z) = e^{g(z)} p(z)$$

$$\Rightarrow \log (f(z)) = \log [e^{g(z)} p(z)] = g(z) + \log p(z)$$

Diff,
$$\frac{f'(z)}{f(z)} = g'(z) + \frac{p'(z)}{p(z)}$$

Ill'y Diff , 'h' times

$$D^h\left[\frac{f'(z)}{f(z)}\right] = g^{h+1}(z) + D^h\left[\frac{p'(z)}{p(z)}\right] \longrightarrow ㉔$$

Since,
$$p(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{[z/a_n + (z/a_n)^2 + \cdots + \frac{1}{n}(z/a_n)^n}$$

$$\Rightarrow \log(P(z)) = \log\left(\prod_{n=1}^{\infty}(1-z/a_n)e^{[z/a_n+(z/a_n)^2+\cdots+\frac{1}{h}(z/a_n)^h]}\right.$$

Diff.

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty}\left[\frac{1}{(1-a/z_n)}\left(\frac{-1}{a_n}\right)+\cdots\frac{1}{h}(h)(z/a_n)\left(\frac{1}{a_n}\right)\right]$$

$$= -\sum_{n=1}^{\infty}\frac{1}{a_n(1-z/a_n)}+\cdots+\left(\frac{1}{a_n}\right)^h(z)^{h-1}$$

again Diff 'h' times.

$$\cancel{D^h\left[\frac{P'(z)}{P(z)}\right] = \sum_{n=1}^{\infty}\frac{(-1)^h(h)!}{(z-a_n)^{h+1+0}}}$$

$$\Rightarrow D^h\left[\frac{P'(z)}{P(z)}\right] = -h!\sum_{n=1}^{\infty}\frac{1}{(z+a_n)^{h+1}}$$

$$\therefore ⑭ \Rightarrow D^h\left[\frac{\overset{f}{P}'(z)}{\underset{f}{P}(z)}\right] = g^{h+1}(z)+h!\sum_{1}^{\infty}-(a_n-z)^{-h-1}$$

$$g^{h+1}(z) = h!\sum(a_n-z)^{-h-1}+D^h\left[\frac{\overset{f}{P}'(z)}{\underset{f}{P}(z)}\right]$$

$g^{h+1}(z)=0$ (using ⑬)

$\therefore$ $g(z)$ is a Poly. of degree $\le h$
'h' is the genus of $f(z)$

$$\boxed{h \le \lambda} \longrightarrow ⑮$$

by ⑤ & ⑮

$$h < \lambda \le h+1$$

Unit - 5

Conformal mapping, Dirichlets problem:

Theorem: 1

The Riemann mapping theorem.

Statement:

Given any simply connected region 'Ω' which is not the whole plane. and a point ~~z_0 not z_0~~ $z_0 \in \Omega$ there exist a unique analytic function $f(z)$ in Ω normlized by the conditions $f(z_0)=0$, $f'(z_0)>0$ such that $f(z)$ defines a 1-1 mapping of Ω onto the disc $|w|<1$

Proof:

Claim: 1

$f(z)$ is unique.

If $f_1(z)$ and $f_2(z)$ are two analytic function such that

$$\left.\begin{aligned} f_1(z_0)=0, &\quad f_1'(z_0)>0 \\ f_2(z_0)=0 &\quad f_2'(z_0)>0 \end{aligned}\right\} \longrightarrow ①$$

and

f_1, f_2 are 1-1, onto mapping $|w|<1$ to the region 'Ω'

$$\Rightarrow f[f_2^{-1}(w)]$$

is also 1-1, onto mapping $|w|<1$ to the region Ω

But any making of $|w|<1$ to itself is a bilinear form given by.

$$S(w) = e^{i\alpha}\left(\frac{w - w_0}{w\bar{w}_0 - 1}\right) \longrightarrow ⑤$$

let $S(w = f_1[f_2^{-1}(w)]$

Diff. w.r.to w.

⑥ $\Rightarrow$

$$s'(w) = \left[\frac{(w\bar{w}_0 - 1)(1) - (w - w_0)(\bar{w}_0)}{(w\bar{w}_0 - 1)^2}\right](e^{i\alpha})$$

$$= e^{i\alpha}\left[\frac{\cancel{w\bar{w}_0} - 1 - \cancel{w\bar{w}_0} + w_0\bar{w}_0}{(w\bar{w}_0 - 1)^2}\right]$$

$$s'(w) = e^{i\alpha}\left[\frac{w_0\bar{w}_0 - 1}{(w\bar{w}_0 - 1)^2}\right] \longrightarrow \Theta$$

hence, $s(0) = 0$; $s'(0) > 0$ (by ①)

$$\Rightarrow e^{i\alpha}\left[\frac{w\bar{w}_0 - 1}{(w\bar{w}_0 - 1)^2}\right] > 0$$

$w_0 = 0$ and $e^{i\alpha}(-1) > 0$

$$e^{i\alpha} < 0$$

hence, $\alpha = \pi$

⑥ $\Rightarrow$ $s(w) = (-1)\left[\frac{w - 0}{0 - 1}\right]$ $(w_0 = 0)$

$$s(w) = w$$

$$\Rightarrow f_1[f_2^{-1}(w)] = w$$

$\Rightarrow f_2^{-1}(w) = f_1^{-1}(w)$

$f_2 = f_1$

$\therefore f$ is unique.

claim: 2

Existence of 'f'

let $\mathcal{F}$ be a normal family of all function g with the following property

i) $g(z)$ is analytic and univalent (1-1) in Ω;

ii) $|g(z)| \leq 1$ in Ω

iii) $g(z_0) = 0$ and $g'(z_0) > 0$

first we will prove that $\mathcal{F} \neq \phi$:-

Since Ω is simply connected its possible to define a single valued prange of $\sqrt{z-a}$ in Ω is defined by $h(z)$

$h(z)$ does not take any value twice. and opposite value

The image of Ω under the map 'h'. covered a disc $|w - h(z_0)| < e$

W.K.T

It doesnot meet the disk

$|w + h(z_0)| < \rho$

$\Rightarrow |h(z) - (-h(z_0))| > \rho$

3(i)

$\Rightarrow |h(z) + h(z_0)| \geq \rho$

In Perticular

$|h(z_0) + h(z_0)| \geq \rho$

$\Rightarrow 2|h(z_0)| \geq \rho$

$\therefore |h(z_0)| \geq \rho/2$ —④

let $g_0(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \left(\frac{h(z_0)}{h'(z_0)}\right)\left(\frac{h(z)-h(z_0}{h(z)+h(z_0}\right.$

Since, h is analytic and 1-1

$\therefore g_0$ is analytic and 1-1

also, $g_0(z_0) = 0$

$\Rightarrow g_0(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \left(\frac{h(z_0)}{h'(z_0)}\right)\left(\frac{h(z)-h(z_0)+h(z}{h(z)+h(}\right.$

$= \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \left[\frac{h(z_0)}{h'(z_0)}\left(1 - \frac{2h(z_0)}{h(z)+h(z_0)}\right)\right]$

$|g_0(z)| \leq \frac{\rho}{4} \left[\frac{|h'(z_0)|}{|h(z_0)|^2}\right] \left[\frac{|h(z_0)|}{|h'(z_0)|}\right] \left[1 + \frac{2h(z_0)}{h(z)+}\right.$

$$\leq \frac{\rho}{4}\left[\frac{h'(z_0)}{h(z_0)^2}\right]\left[\frac{|h(z_0)|^2}{|h'(z_0)|}\right]\left[\frac{1}{|h(z_0)|}+\frac{2h(z_0)}{|h(z_0)||h(z)+h(z_0)|}\right]$$

$$= \frac{\rho}{4}\left[\frac{1}{h(z_0)}+\frac{2}{h(z)+h(z_0)}\right]$$

$$\leq \frac{\rho}{4}\left[\frac{1}{(\rho/2)}+\frac{2}{\rho}\right] \quad (\text{by} ④, ③)$$

$$\leq \rho/4\left[2/\rho + 2/\rho\right]$$

$$\leq \rho/4\left[4/\rho\right]$$

$$|g_0(z)| \leq 1$$

next,

$$g'(z) = \rho/4 \frac{h'(z_0)}{|h(z_0)|^2}\left(\frac{h(z)}{h'(z_0)}\right)\left[\frac{(h(z)+h(z_0))h'(z)-h(z)-h(z_0))}{(h(z)+h(z_0))^2}\right]$$

$h'(z)$

$$= \rho/4\left(\frac{|h'(z_0)|}{|h(z_0)|^2}\right)\left(\frac{h(z_0)}{h'(z_0)}\right)\left(\frac{2h(z_0)h'(z)}{[h(z)+h(z_0)]^2}\right)$$

sub $z = z_0$

$$g'(z_0) = \frac{\rho}{4}\left(\frac{|h'(z_0)|}{|h(z_0)|^2}\right)\left(\frac{h(z_0)}{h'(z_0)}\right)\left(\frac{2h(z_0)h'(z_0)}{[h(z_0)+h(z_0)]^2}\right)$$

$$= \rho/4\left(\frac{|h'(z_0)|}{|h(z_0)|^2}\right)\left(\frac{h(z_0)}{h'(z_0)}\right)\left(\frac{2h(z_0)h'(z_0)}{4(h(z_0))^2}\right)$$

$$= \frac{\rho}{8}\left[\frac{|h'(z_0)|}{|h(z_0)|^2}\right]$$

$\therefore g_0'(z_0) \geq 0$

$\Rightarrow g_0 \in \mathcal{F}$

$\therefore \mathcal{F} \neq \phi$

ent

To find f'

let $B = \text{lub}\{g'(z_0) / g \in \mathcal{F}\}$

clearly, $B > 0$

There exist a sequence of function $g_n \in \mathcal{F}$ such that

$g_n'(z_0) \longrightarrow B$

by the define of $\mathcal{F}$

Each function $g_n \in \mathcal{F}$ is bounded in $\mathcal{F}$

$\therefore \mathcal{F}$ is normal.

here $\{g_{n_k}\}$ which converges unique to the analytic function f on every convergent set

ie) $\lim_{k\to\infty} g_{n_k}(z) = f(z) \ \forall \ z \in \Omega$

$$\Rightarrow \lim_{k\to\infty} g_{n_k}(z_0) = f(z_0)$$

$$\Rightarrow \lim_{k\to\infty} (0) = f(z_0)$$

$$f(z_0) = 0$$

also.

$$f'(z_0) = \lim_{k\to\infty} g'_{n_k}(z_0) \longrightarrow ⑤$$

$$\therefore f'(z_0) \neq 0 \quad (\text{since, } g'_{n_k}(z_0) > 0)$$

again.

$$|f(z_0)| = \lim_{k\to\infty} |g_{n_k}(z_0)|$$

$$\Rightarrow |f(z_0)| \leq 1 \;\forall\, z \in \Omega \;\left[\text{since, } g_{n_k}(z_0) \leq 1\right]$$

All the required condition are true

$\therefore$ 'f' is 1-1

$$⑤ \Rightarrow f'(z_0) = \lim_{k\to\infty} g'_{n_k}(z_0)$$

$$f'(z_0) = B$$

$$f'(z_0) \longrightarrow 0$$

$\Rightarrow f(z_0)$ is not constant

choose a point $z \in \Omega$ and define

$$g_1(z) = g(z) - g(z_1) \ \forall\ g \in \mathcal{F}$$

Since, g is 1-1

$$g(z) \neq g(z) \ \forall\ z \neq z_1$$

$$\therefore g_1(z) \neq 0 \ \forall\ z \neq z_1$$

$$\Rightarrow g_1(z) \neq 0 \ \forall\ z \in \Omega - \{z_1\}$$

by Hurwitz therom.

"Every limit function $\{g_1(z)\} = \{g(z), g(z_$
is either identically zero"

$\therefore$ The function $f(z) - f(z_1)$ is either zero (or) never zero

$$\Rightarrow f(z) - f(z_1) \neq 0 \ (\forall\ z \neq z_1)$$

$$\Rightarrow f(z) \neq f(z_1)$$

$$\therefore f \text{ is } 1\text{-}1$$

f is onto:-

Assume that for element w_0, $|w|<1$ such that there exist no element in z is 'Ω' whose image is w_0

Since 'Ω' is simply connected its possible to define a single valued branch

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0}\, f(z)}} \quad \text{---} ①.$$

$$\left[\because f \text{ is } 1\text{-}1 \text{ and } |f(z)| < 1 \right]$$

Define. $G(z) = \dfrac{|F'(z_0)|}{F'(z_0)} \left[\dfrac{F(z) - F(z_0)}{1 - \overline{F(z_0)}\, F(z)} \right]$

$\Rightarrow G(z_0) = 0$

$$\Rightarrow G'(z) = \frac{|F'(z_0)|}{F'(z_0)} \left[\frac{\left(1 - \overline{F(z_0)}\, F(z)\right) F'(z) - \left(F(z) - F(z_0)\right)\left(-\overline{F(z_0)}\, F'(z)\right)}{\left(1 - \overline{F(z_0)}\, F(z)\right)^2} \right]$$

$$\Rightarrow G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} \rightarrow$$

$$\frac{1 - |F(z_0)|^2}{|F(z_0)|\left[1 - F(z_0)^2\right]^{-2}}$$

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} \quad (iii)$$

$\Rightarrow G'(z_0) > 0$

$\therefore G = F$

$\therefore$ ① $\Rightarrow F(z_0) = \sqrt{\dfrac{f(z_0) - w_0}{1 - \overline{w_0}\, f(z_0)}}$ $\left(\text{since } f(z_0) = 0 \right)$

$= \sqrt{-w_0}$

$$\boxed{|F(z_0)| = \sqrt{|w_0|}}$$

$$\Rightarrow [F(z)]^2 = \frac{f(z) - w_0}{1 - \overline{w_0}\, f(z)} \left[\frac{f'(z) - \overline{w_0}\, f(z)\, f'(z) + \overline{w_0}\, f(z)}{f'(z) - w_0 \overline{w_0} + f'(z) + |w_0|^2} \right]$$

$$\Rightarrow 2F(z)\, F'(z_0) = \frac{[1-\overline{w_0} f(z)] f'(z) - [f(z)-w_0](-\overline{w_0} f'(z))}{[1-\overline{w_0}\, f(z)]^2}$$

$$\Rightarrow 2F(z_0)\, F'(z_0) = \frac{f'(z_0) - |w_0|^2 f'(z_0)}{[1-\overline{w_0} f(z_0)]^2}$$

$$= f'(z_0)\,[1-|w_0|^2]$$

$$F'(z_0) = \frac{f'(z_0)\,[1-|w_0|^2]}{2F(z_0)} = \frac{f'(z_0)(1+}{2\sqrt{|w}}$$

$$|F'(z_0)| = \frac{(1-|w_0|^2)\,|f'(z_0)|}{2\,|F(z_0)|}$$

$$= \frac{[1-|w_0|^2]\,[f'(z_0)]}{2\,|F(z_0)|\,(1-|w_0|^2)}$$

$$= \frac{1-|w_0|^2\, f'(z_0)}{2\sqrt{|w_0|}\,(1-|w_0|)}$$

$$|F'(z_0)| = \frac{1-|w_0|^2}{2\sqrt{|w_0|}}\left(\frac{f'(z_0)}{1-|w_0|}\right)$$

111 $\Rightarrow$ $G'(z_0) = \frac{1+|w_0|}{2\sqrt{|w_0|}} \cdot f'(z_0)$

$\Rightarrow G'(z_0) > f'(z_0)$

which is $\Rightarrow \Leftarrow$ (since f' is l.u.b)

$\therefore$ F is onto the disc $|w|<1$

⑤

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3/ Theorem

5m A continuous function $u(z)$ which statisfies the condition mean value property is nessarayly harmonic

Proof

let $u(z)$ be a real valued continuous function in region Ω

W.K.T

Every harmonic fun satisfies mean value property

If $(u-v)$ satisfies the mean value property

Then

$(u-k)$ is also satisfies the mean value property

where uk is harmonic function

consider the disc $|z-z_0| \leq \rho$ is in Ω

By possion formula.

we can constract a function $v(z)$ which is harmonic in the disc $|z \neq z_0| \leq \rho$ is continuous and equal to $u(z)$ on $|z-z_0| < \rho$

ie) $u(z)=v(z)$ on $|z-z_0| \leq \rho$

Apply using min - principle to $u(z)$

we get,

$u(z) = v(z)$ on $|z - z_0| = \rho$

hence, $u(z)$ is harmonic in $|z - z_0| \leq r (r \leq$

Harnack's principle:-

Theorem:

Harnack's Inequality

statement

⑩ If $u(z)$ is harmonic in $|z| < \rho$ and continuous in $|z| \leq \rho$ then for all $r < \rho$ and $\rho - r/_{\rho + r}\, u(0) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(0$

Proof:

W.K.T [Poission formula.

$$u(z) = 1/_{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta})\, d\theta \quad \text{—}①$$

now, $|\rho e^{i\theta} - z| \leq |\rho e^{i\theta}| + |z| \leq \rho + r$

$$\Rightarrow |\rho e^{i\theta} - z|^2 \leq (\rho + r)^2$$

$$\frac{1}{|\rho e^{i\theta} - z|^2} \geq \frac{1}{(\rho + r)^2}$$

$$\Rightarrow \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \geq \frac{\rho^2 - r^2}{(\rho + r)^2} \quad (\rho - r)(\rho + r)$$

$$\Rightarrow \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \geq \frac{\rho - r}{\rho + r} \longrightarrow ②$$

also.

$$|\rho e^{i\theta} - z| \geq |\rho e^{i\theta}| - |z| \leq \rho - r$$

$$\Rightarrow |\rho e^{i\theta} - z|^2 \geq (\rho - r)^2$$

$$\frac{1}{|\rho e^{i\theta} - z|^2} \leq (\rho - r)^2$$

$$\Rightarrow \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho^2 - r^2}{(\rho - r)^2}$$

$$\Rightarrow \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + r}{\rho - r} \longrightarrow ③$$

② & ③

$$\frac{\rho - r}{\rho + r} \leq \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + r}{\rho - r}$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\rho - r}{\rho + r}\right) u(\rho e^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2}\right) u(\rho e^{i\theta})\, d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\rho + r}{\rho - r}\right) u(\rho e^{i\theta})\, d\theta$$

$$\Rightarrow \left(\frac{\rho - r}{\rho + r}\right)\left[\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta}\, d\theta)\right] \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2}\, u(\rho e^{i\theta}) d\theta \leq \left(\frac{\rho + r}{\rho - r}\right)\left[\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta}) d\theta\right]$$

$$\therefore \left(\frac{\rho - r}{\rho + r}\right) u(\theta) \leq u(z) \leq \frac{\rho + r}{\rho - r}\, u(\theta) \quad \text{where} \quad u(\theta = \tfrac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta}) d\theta$$

Theorem [Harnack's Principle]

Statement:

consider a sequence ~~to~~ of function $\{u_n(z)\}$ each defined at harmonic in a certain region in Ω_n. let Ω be a region such that every point in Ω has a nighbeurhood contained in all but a finite number of Ω_n and assume more over that in this neighbourhood $u_n(z) \leq u_{n+1}(z)$ has so as 'n' is sufficiently large $(n \to \infty)$ Then There is a only two Possiblity either $u_n(z) \xrightarrow{\text{tends to}}$ uniformly to infinity of every an compack subset of Ω (or) $u_n(z) \xrightarrow{\text{tends to}}$ harmonic limit function $u(z)$ in Ω, uniformly on compack sets

Proof:

let

$$\lim_{n \to \infty} u_n(z_0) = \infty$$

for atleast one point $z_0 \in \Omega$

By assumption There exist r and m The sequence~~to~~ of function $u_n(z)$ are harmonic and form a non-degress sequence in $|z - z_0| < r$ and $n \geq m$

When $u_n(z) - u_m(z) \geq 0 \ \forall \ n \geq m$ applying, Harnack's inequality

$$\frac{\rho-r}{\rho+r}\, u(0) \le u(z) \le \frac{\rho+r}{\rho-r}\, u(0)$$

$$\Rightarrow \frac{\rho-r}{\rho+r}\, (u_n-u_m)(z_0) \le u(z) \le \frac{\rho+r}{\rho-r}\, (u_n-u_m)(z_0)$$

$$\frac{\rho-\rho/2}{\rho+\rho/2}\, (u_n-u_m)\, z_0 \le u(z) \le \frac{\rho+\rho/2}{\rho-\rho/2}\, (u_n-u_m)(z_0)$$

$$\frac{\rho/2}{3\rho/2}\, (u_n-u_m)\, z_0 \le u(z) \le \frac{3\rho/2}{\rho/2}\, (u_n-u_m)(z_0)$$

$$\tfrac{1}{3}\, (u_n-u_m)\, z_0 \le u(z) \le 3\, (u_n-u_m)(z_0) \quad \hookrightarrow ①$$

let . consider,

$$\frac{1}{3}\, (u_n-u_m)\, z_0 \le (u_n-u_m)\, z$$

$$\rightarrow (u_n-u_m)\, z \ge \tfrac{1}{3}\, (u_n-u_m)(z_0)$$

Taking $\lim_{n\to\infty}$ on both sides

$$\lim_{n\to\infty} ((u_n-u_m)(z)) \ge \lim_{n\to\infty} \left[\frac{1}{3} (u_n-u_m)\, z_0\right]$$

$$\therefore \lim_{n\to\infty} (u_n-u_m)(z) = 0$$

(since, $\lim_{n\to\infty} u_n(z_0) = \infty$

uniformly in all compact subset of Ω

$\therefore \{u_n(z_0)\} \to \infty$, on all compact subsets of Ω

next,

suppose that,

$\lim_{n \to \infty} u_n(z_0) < \infty$

for atleast one point $z_0 \in \Omega$

① $\Rightarrow (u_n \cdot u_m)(z) \leq 3(u_n - u_m) z_0$

$\Rightarrow \{u_n(z)\}$ is bounded in $|z| < r = \frac{\rho}{2}$

also, The subset of Ω in which the $\lim u_n(z)$ is finite (or) infinite (or) both open since

'Ω' is connected

one of these two must be empty when the limit infinite of a single Points of 'Ω' it must be identically ~~#~~ infinite at all point 'Ω'.

when, $\lim_{n \to \infty} u_n(z)$ is finite

Then its uniformly finite over Ω

① $\Rightarrow \lim_{n \to \infty} \{u_n(z) - u_m(z)\} \leq 3 \lim_{n \to \infty} \{u_n(z_0) - u_m(z$

$\Rightarrow u(z) - u_m(z) \leq 3[u(z_0 - u_m(z_0)]$

$\therefore u(z) < \infty \ \forall z \in \Omega$

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[since, $\lim_{n \to \infty} u_n(z_0) = u(z)$ is given the limit]

also,

$$\lim_{n \to \infty} u_n(z_0) = u(z_0)$$

$$\Rightarrow \lim_{n \to \infty} u_n(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta$$

$$u_n(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta$$

$\therefore u(z)$ satisfies the mean value property

$u(z)$ is continuous

$\therefore u(z)$ is harmonic in 'Ω'

Theorem [mapping on a rectangle.]

Statement

If the 'Ω' is a region

$$F(w) = \int_0^w \frac{dw}{\sqrt{w(w-1)(w-e)}}$$ which on

elliptic integral. let F(w) ~~has~~ is real axis Then each square root is either positive or purely imaginary with +ve imaginary since $0 < w < 1$ There are one real and two imaginary square roots.

ie) F(w), decreases from 0 to $-k$

where, $$k = \int_0 \frac{dw}{\sqrt{t(1-t)(e-t)}} \quad \text{——} ①$$

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Proof
case (i) $1 < w < e$

There is only one imaginary square root
∴ Integral from 1 to ω is purely imaginary with negative imaginary part

Hence,
$F(\omega)$ follow a vertically segment from $-k$ to $(-k-ik)$

where, $K(\omega) = \int_0^e \frac{dt}{t(1-t)(e-t)}$

case (ii)
$w = e$

The integral is Positive & $F(\omega)$ will Trace a horicantal segment in the Positive direction

Since the image is to be a rectangle it must end at the point $(-ik)$

ie) one way is to express the length of the line segment by the integral

$$\int_0^{\infty} \frac{dt}{\sqrt{t(1-t)(e-t)}}$$

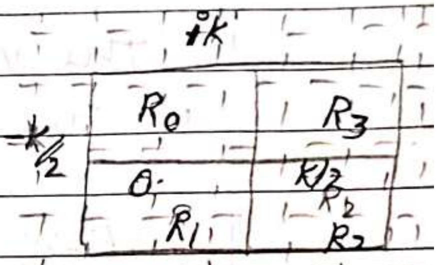
and to show by the change of integration variable

$$f = \frac{e-u}{1-u}$$

That the integral transform to the eqn ①

By cauchy's theorem which gives

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{t(1-t)(e-t)}} = 0$$



This integral over semi-circle with radius R tends to zero ~~are~~ as R tends to '∞'

ie) The real part is not equal to '0' we get

$-\infty < w < 0$ is mapped onto segment from $-ik$ to ∞

ie) The rectangle is completed

consider

$$F(w) = \int_{0}^{\infty} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$

It was clear that $\sqrt{(1-w^2)(1-k^2w^2)}$ is +ve real Points Parts.

ie) The rectangle will have ~~be~~ the vertices:
~~$-K/2$~~, $K/2$, ~~$K/2 + iK$~~, $-K/2 + iK$, $-K/2$, $K/2$, $\frac{K}{2} - iK$, $\frac{K}{2}$

Where, $$K = \int_{-1}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and $$K' = \int_{-1}^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}$$ where, $0 < k < 1$

The image of upper half plane is the rectangle (or) not as shown in the figure, Denoted by ~~as~~ Positive inverse function of F' by $w = f(z)$ is defined in R and it can be extended by continuity of ~~onto~~ one to one mapping on closed rectangle onto the closed half plane

~~The~~

The extension is given by

$$f(z) = \overline{f(z)} \;\forall\; z \in R, \text{ and } f(z) = \overline{f(K-\bar{z})}$$
$\forall\; z \in R_1$

$$f(z) = f(K-z)$$

The extension is continuous.

This process is continuous until $f(z)$ is meromorphic function in whole plane which satisfied the extension function

$f(z+\omega_1) = f(z)$ and
$f(z+\omega_2) = f(z)$

hence the proof

Definition

The triangle function of schwartz

The upper of plane is mapped on triangle with angles $\alpha_1\pi, \alpha_2\pi, \alpha_3\pi$
by $F(w) = \int_0^w (w^{\alpha_1-1}(w-1)^{\alpha_2-1})\,dw$

Free boundary arc

Every point of Ω as a neighbourhood whose intersection with the whole boundary $(\partial\Omega)$

Then we say that Ω is the free boundary arc

Elliptic integral.

let Ω be a rectangle we may choose $z_1 = 0, z_2 = 1, z_3 = \rho(>1)$

$$F(w) = \int_0^w \prod_{k=1}^{n-1} (w - z_k)^{\beta_k} dw + c'$$

The function is given by

$$F(w) = \int_0^w \frac{dw}{\sqrt{w(w-1)(w-e)}}$$

Which is an elliptic integral.