Unit - II

Polynomials

* Introduction

  All definitions

* Algebra
* The Algebra of polynomials theorem 5m
* Lagrange Interpolation

  - Derivation of Lagrange's 5m

* Polynomial Ideals

  - Theorem
  - Division Algorithm 10m
  - Taylor's formula 10m

* The prime factorization 10m
* Determinants
* Introduction

  - Commutative rings

* Determinat functions

  - Lemma and theorems 5m

5.9.19 **Linear Algebra:**

Let F be a field. A Linear algebra over the field F is a vector space 'V' over F with an additional operation called Multiplication of vector which associates with each pair of vector α, β in V is a vector $\alpha\beta$ in V called the product of α and β in such a way that,

2

a) Multiplication is associative:

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

b) Multiplication is distributive with respect to addi

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$$

$$(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma.$$

If there is an element 1 in V such that, $1\cdot\alpha = \alpha\cdot 1 = \alpha$ for each α in V. We call V is a linear algebra with identity over F and call 1 the identity of V. The algebra V is called commutative if, $\underline{\alpha\beta = \beta\alpha}$, $\forall \alpha, \beta$ in V.

Polynomial over F:

Let F[x] be the subspace of F^{∞} spanned by the vectors, $1, x, x^2 \ldots$ An element of F[x] is called polynomial over F.

Scalar Polynomial:

If f is non-zero polynomial of degree n it follows that,

$$f = f_0 x^0 + f_1 x^1 + f_2 x^2 + \cdots + f_n x^n, \ f_n \neq 0$$

The scalar $f_0, f_1, \ldots, f_n$ are called co-efficients of f, and we say that f is a polynomial with co-efficient in F and c is a non-zero constant then cf is a scalar polynomial of f.

Monic Polynomial:

A non-zero polynomial f^n such that, $f_n = 1$ is said to be Monic Polynomial.

Vandermonde Matrix:

Let F be a fixed field and that $t_0, t_1, \dots, t_n$ are (n+1) distinct element of F. Let V be a subspace of F[x] consisting of all polynomials of degree $\leq n$. Then the invertiable Matrix is.

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots\cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots\cdots & t_1^n \\ \vdots & & & & \\ 1 & t_n & t_n^2 & \cdots\cdots & t_n^n \end{bmatrix}$$

is called Vandermonde Matrix.

Multiplicity of A Root:

If c is a root of the polynomial f, then the multiplicity of c as a root of f is the largest positive integer r such that, $(x-c)^r$ divides f. $(x-c)^r / f$

Principal Ideal:

Principle Ideal is an ideal Which is generated by non-zero single element of an ideal.

Primary Decomposition of f:

If $P_1, P_2, \dots P_r$ are the distinct Monic Primes occuring in the factorization of f. Then, $f = P_1^{n_1}, P_2^{n_2}, \dots, P_r^{n_r}$

The exponent n_i being the number of times the prime P_i occurs in the factorization.

This unique decomposition is called primary decomposition of f.

6.9.19 Eg:

P.T F^{∞} is linear Algebra with Identity:

Proof:

Let $f = \{f_0, f_1, f_2, \dots\}$ of scalars f_i in F.

and $g = \{g_0, g_1, \dots\}$ g_i in F and $a, b \in F$

then. $af + bg = \{af_0 + bg_0, af_1 + bg_1, \dots\} \rightarrow (1)$

fg is defined by,

$$(fg)_n = \sum_{i=0}^{n} f_i g_{n-i} \rightarrow (2)$$

$$= f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 g_1 + f_2 g_0 + \dots$$

Multiplication is commulative in F^{∞}.

$$\text{consider } (gf)_n = \sum_{i=0}^{n} g_i f_{n-i}$$

$$= \sum_{i=0}^{n} f_i g_{n-i}$$

Thus $(gf)_n = (fg)_n \quad n = 0, 1, \dots$

Multiplication is associative in F^{∞}.

If $h \in F^{\infty}$

$$\text{then } [(fg)h]_n = \sum_{i=0}^{n} (fg)_i h_{n-i}$$

$$= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} f_j g_{i-j} \right) h_{n-i}$$

$$= (f_0 g_0) h_n + (f_0 g_1 + f_1 g_0) h_{n-1} + \dots$$

$$+ (f_0 g_n + f_1 g_{n-1} + \dots + f_n g_0) h_0$$

$= f_0(g_0h_n + g_1h_{n-1} + \cdots + g_nh_0) + f_1(g_0h_{n-1} + \cdots + g_{n-1}h_0) + \cdots$

$= f_0(gh)_n + f_1(gh)_{n-1} + \cdots + f_n(gh)_0$

$= [f(gh)]_n \quad \forall\ n = 0,1,2,\ldots$ So that

Thus.

$(fg)h = f(gh)$

Multiplication is distributive w.r to addition:

consider, $[(f+g)h]_n = \sum_{i=0}^{n} (f+g)_i\, h_{n-i}$

$$[(f+g)h]_n = \sum_{i=0}^{n} (f_i + g_i)\, h_{n-i}$$

$$= \sum_{i=0}^{n} f_i h_{n-i} + \sum_{i=0}^{n} g_i h_{n-i}$$

Scalar Multiplication:

Let $c \in F$ then,

Consider, $c[fg]_n = c \sum_{i=0}^{n} f_i g_{n-i}$

$$= \sum_{i=0}^{n} c f_i g_{n-i}$$

$$= \sum_{i=0}^{n} (cf_i) g_{n-i}$$

$$= [cf \cdot g]_n$$

$$c(fg) = (cf)g$$

Since $i = (1,0,0\ldots)$ is all Identity of F^∞.

Thus F^∞ is commutative Linear algebra over F.

10.9.19

6

Theorem: 1

Let f and g be non-zero polynomial over F.

Then i) fg is non-zero polynomial

ii) deg (fg) = deg f + deg g.

iii) fg is Monic polynomial iff both f and g are Monic polynomial.

iv) fg is a scalar polynomial iff both f and g are scalar polynomial.

v) If $f+g \neq 0$, $\deg(f+g) \leq \max(\deg f, \deg g)$.

proof: Let f and g be non zero polynomial over F.

i) To prove: fg is a non zero polynomial.

Assume that; $f = a$ and $g = b$ where $a \neq 0, b \neq 0$

$\Rightarrow fg = ab \neq 0$ because $a b \in F$.

Hence, fg is a non zero polynomial.

ii) To prove: $\deg(fg) = \deg(f) + \deg(g)$

Let $f = f_0 + f_1 x + \dots + f_m x^m$ with $f_m \neq 0$

$g = g_0 + g_1 x + \dots + g_n x^n$ with $g_n \neq 0$.

$\Rightarrow \deg f = m$; $\deg g = n$ let $fg = h$

$\therefore$ Let $fg = h_0 + h_1 x + \dots + h_{m+n} x^{m+n}$

Where $h_k = \sum_{i=0}^{k} f_i g_{k-i}$, $k = 0, 1, 2, \dots$

consider,

$(fg)_{m+n+k} = h_{(m+n+k)} = \sum_{i=0}^{m+n+k} f_i g_{m+n+k-i}$

clearly, $i = 0, 1, \dots$

$f_i g_{m+n+k-i} \neq 0$

because. $i \le m$. and $m+n+k-i \le n$

Hence, $m+k \le i \le m$

$\Rightarrow k=0$ and $i=m$

$(fg)_{m+n} = f_m g_n$

and $(fg)_{m+n+k} = 0 \quad (k>0)$

$\therefore \deg fg = \deg f + \deg g.$

iii) To prove: fg is a Monic polynomial:

If both f and g are monic polynomial

suppose, $\deg f = m$; $\deg g = n$

$\therefore f_m = 1$; $g_n = 1$

The leading co-efficient is (fg) is $f_m g_n$ which is 1.

$\therefore fg$ is Monic polynomial.

iv) To prove:

Given f and g are non-zero polynomial

consider, $c \deg (fg) = c(\deg f + \deg g)$

$\deg (cfg) = \deg cf + \deg cg$

Since, fg is non zero polynomial and cf and cg are scalar polynomial.

$\therefore c(fg)$ is scalar polynomial of fg.

v) To prove: If $f+g \neq 0$

Then, $\deg (f+g) \le \text{Max} (\deg f, \deg g)$

Let $f = f_0 + f_1 x + \dots + f_m x^m$ with $f_m \neq 0$

$g = g_0 + g_1 x + \dots + g_n x^n$ with $g_n \neq 0$

Then, if $m > n$

$$(f+g) = (f_0+g_0) + (f_1+g_1) + \cdots + (f_n+g_n)x^n + f_{n+1}x^{n+1} + \cdots + f_m x^m$$

$\therefore \deg(f+g) = m$

||^{ly} if $m < n$ Then $\deg(f+g) = n$

$\therefore \deg(f+g) \leq \text{Max}\{\deg f, \deg g\}$

if $m=n$

Theorem : 2

Let F be a field and A be a linear algebra with identity over F. Suppose f and g are polynomials over F, that α is an element of A and that c belongs to F. Then i) $(cf+g)(\alpha) = cf(\alpha) + g(\alpha)$

ii) $(fg)(\alpha) = f(\alpha) \cdot g(\alpha)$

Proof:

Let $\deg f = m$, $\deg g = n$

Then. $f = f_0 + f_1 x + f_2 x \cdots + f_m x^m$; $f_m \neq 0$

ie) $f = \sum_{i=0}^{m} f_i x^i$

$g = g_0 + g_1 x + \cdots + g_n x^n$, $g_n \neq 0$

ie) $g = \sum_{j=0}^{n} g_j x^j$

Then i) $(cf+g) = \sum_{i,j} (cf_i + g_i) x^i$. $j \leq i$

$(cf+g)\alpha = \sum_{i,j} (cf_i + g_i) \alpha^i$

$= \sum_{i=0}^{m} cf_i \alpha^i + \sum_{j=0}^{n} g_j \alpha^j$

$\therefore (cf+g)(\alpha) = cf(\alpha) + g(\alpha)$

ii) $(fg) = \sum_{i,j} (f_i g_j) x^{i+j}$

$(fg)\alpha = \sum_{i,j} (f_i g_j) \alpha^{i+j}$

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$(fg)\alpha = \left(\sum_{i=0}^{m} f_i \alpha^i\right) \left(\sum_{j=0}^{n} g_j \alpha^j\right)$ 9

$\therefore (fg)\alpha = f(\alpha) g(\alpha)$

Lagrange Interpolation:

Statement: Assume that F is fixed field. Let $t_0, t_1, \dots, t_n$ are $(n+1)$ distinct elements of F.

Let V be a subspace of $F[x]$ consisting of all polynomials of degree $\leq n$.

12.9.19

Let the function $L_i : V \to F$ if defined by $L_i(f) = f(t_i)$ $\forall f \in V, 0 \leq i \leq n$.

Clearly, Each L_i is Linear on V.

Proof:
To prove: $\{L_0, L_1, \dots L_n\}$ is a basis for V^*.

We have $L_j(P_i) = P_i(t_j) = \delta_{ij}$ (*)

To polynomial,

$$P_i = \frac{(x-t_0)\dots(x-t_{i-1})(x-t_{i+1})\dots(x-t_n)}{(t_i-t_0)\dots(t_i-t_{i-1})(t_i-t_{i+1})\dots(t_i-t_n)}$$

Dif. fn

$P_i = \prod_{j \neq i} \left(\frac{x-t_j}{t_i-t_j}\right)$ are of degree 'n'.

$\therefore P_i \in V$ $p_i \in V$

Let $f \in V$ then, $f = \sum_i c_i P_i$

for each j, $f(t_j) = \sum_i c_i P_i(t_j)$

$f(t_j) = c_j \underset{(1)}{P_j(t_j)} + \sum_{i \neq j} c_i \underset{(0)}{P_i(t_j)} \Rightarrow c_j = 0$ i, I so

$c_j = f(t_j)$ for $i = 1, 2, \dots n$

$\therefore \{P_0, P_1, \dots P_n\}$ are Linearly Independent.

Since, the polynomial $1, x, \dots x^n$ from a

Hence, dim $V = n+1$

So, $\{p_0, p_1, \dots p_n\}$ must also a basis for V

Since, $L_i(p_j) = \delta_{ij}$ and $L_i \in V^*$

$\therefore \{L_0, L_1, \dots L_n\}$ is a basis for V^*

Let $f \in V$ then $f(t_j) = \sum_{i=0}^{n} c_i p_i(t_j)$

$$= c_i$$

$$\therefore f = \sum_{i=0}^{n} f(t_j)\, p_i \longrightarrow (1)$$

is called Lagrange interpolation for

Let $f = x^j$, $0 \le j \le n$,

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

then (1) becomes,

$$x^j = \sum_{j=0}^{n} (t_i)^j p_i$$

$$x^j = t_0^j p_0 + t_1^j p_1 + \dots + t_n^j p_n$$

for $j=0$, $1 = p_0 + p_1 + \dots + p_n$

$j = 1$, $x = t_0 p_0 + t_1 p_1 + \dots + t_n p_n$

$\vdots$

$j = n$, $x^n = t_0^n p_0 + t_1^n p_1 + \dots + t_n^n p_n$

$$\Rightarrow \begin{vmatrix} 1 & t_0 & t_0^2 \dots & t_0^n \\ 1 & t_1 & t_1^2 \dots & t_1^n \\ \vdots & & & \\ 1 & t_n & t_n^2 \dots & t_n^n \end{vmatrix}$$

$\therefore$ It is invertiable as Vandermonde Matrix.

Isomorphism :

Let F be a field and let a and a~ be Linear algebra over F. The algebra a and a~ are said to Isomorphic. If there is a one to one Mapping $\alpha \to \alpha^{\sim}$ of a onto a~ such that

$$(c\alpha + d\beta)^{\sim} = c\alpha^{\sim} + d\beta^{\sim}$$

$$(\alpha\beta)^{\sim} = \alpha^{\sim}\beta^{\sim} \quad \forall\ \alpha, \beta \in a \text{ and all}$$

scalar c, α in F. The Mapping $\alpha \to \alpha^{\sim}$ is called an isomorphism of a onto a~.

Theorem : 3

If F is a field contining and infinite Number of distinct element the Mapping $f \to f^{\sim}$ is an isomorphism of the algebra of polynomial over F onto the algebra of polynomial function over F.

proof :

by definition,

The Mapping is onto and if $f, g \in F[x]$ it is evident that,

$$(cf + dg)^{\sim} = cf^{\sim} + dg^{\sim}.$$

∀ scalar c and d. Since,

We have already shown that

$$(fg)^{\sim} = f^{\sim} g^{\sim}$$

We need only to show that, the Mapping is one to one for that it is

Proof:

To Prove : $r = 0$ (or) $\deg r < \deg d$.

i) If $f = 0$ (or) $\deg f < \deg d$

Then, $f = 0 \cdot d + f$

ie) $q = 0$, $r = f$

Since, $\deg f < \deg d$

$$\deg r < \deg d. \quad (r = f)$$

ii) If $f \neq 0$ and $\deg f \geq \deg d$

By known lemma,

$\exists$ $g \in F[x]$ such that $f - dg = 0$ (or)

$$d(f - dg) < \deg f$$

If $f - dg \neq 0$ and $\deg(f - dg) \geq \deg f$

choose a polynomial h such that

$$f - dg - dh = 0 \text{ (or)}$$

$$\deg [f - d(g+h)] < \deg (f - dg)$$

continuing this process as long as we get $r = 0$ (or)

$$\deg r < \deg d.$$

To prove: Uniqueness :

Suppose $f = dq_1 + r_1 \rightarrow (1)$

Where, $r_1 = 0$ (or)

$$\deg r_1 < \deg d.$$

Then from given hypothesis,

i) $\Rightarrow dq + r = dq_1 + r_1$

$$d(q - q_1) = r_1 - r \rightarrow (2)$$

If $q - q_1 = 0$ then $r_1 - r = 0$ (15)

$\therefore r = r_1$ & $q = q_1$

If $q - q_1 \neq 0$

Then $d(q - q_1) \neq 0$

$(2) \Rightarrow$

$\deg d + \deg(q - q_1) = \deg(r_1 - r)$

Which is impossible

because, $\deg(r_1 - r) < \deg d$

$q_1 - q = 0$ & $r_1 - r = 0$

$\boxed{q_1 = q \text{ \& } r_1 = r}$

The polynomial satisfying (i) & (ii) are Unique.

3.9.19 Corollary: 1

Let f be a polynomial over the field F and let c be an element of F. Then f is divisible by $x - c$ iff $f(c) = 0$.

Proof:

Let f be a polynomial over F and $c \in F$

Then, by division Algorithm,

$$f = q(x - c) + r$$

put $x = c$

$$f(c) = q(c - c) + r(c)$$

$$f(c) = r(c) \longrightarrow (1)$$

ie) $f(c) = r$

Hence, $r = 0 \Leftrightarrow f(c) = 0$ by (1)

$r = 0 \Leftrightarrow (x - c)$ is a factor of f.

$r = 0 \Leftrightarrow$ f is divisible by $x - c$.

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Corollary: 2

A polynomial f of degree n over a field F has atmost n roots in F.

proof:

To Prove this result by induction on n.

The result is true for $n=0$ & $n=1$.

Assume that the result is true for $n-1$.

If "a" is a root of f.

Then, $\frac{f}{x-a} = q$ (say)

$$f(x) = (x-a)\, q(x) \longrightarrow (1)$$

By induction hypothesis,

$q(x)$ has atmost $(n-1)$ roots.

$\therefore$ $f(x)$ has atmost n roots.

Theorem: 5 Taylor's formula:

Statement:

4m

Let F be a field of characteristic zero, c an element of F; and n a positive integer. If f is a polynomial over F with $\deg f \leq n$, then $f = \sum_{k=0}^{n} \frac{(D^k f)}{k!}(c)(x-c)^k$.

proof:

Taylor's formula is a consequence of and the linearity of the operators $D, D^2, \ldots D^n$ the binomial theorem the binomial theorem, is easily proved by induction and asserts that,

D for derivative

$$(a+b)^m = a^m + mc_1 a^{m-1} b^1 + mc_2 a^{m-2} b^2 + \cdots + b^m.$$

$$= \sum_{k=0}^{m} mc_k\, a^{m-k} b^k.$$

(17)

Where, $mc_k = \binom{m}{k} = \dfrac{m!}{k!(m-k)!}$

$$= \frac{m(m-1)\ldots(m-k+1)(m-k)!}{k!(m-k)!}$$

$$mc_k = \frac{m(m-1)\ldots\ldots(m-k+1)}{1\cdot2\cdot\ldots\ldots k}$$

By the binomial theorem. $(1+x)^n = nc_0 + nc_1x^1 + nc_2x^2 + \ldots + nc_nx^n$

$$x^m = [c+(x-c)]^m$$

$$x^m = \sum_{k=0}^{m} \binom{m}{k} c^{m-k}(x-c)^k \longrightarrow (1)$$

This is the statement of taylor's formula for the case $f = x^m$.

$$\text{If } f = \sum_{m=0}^{n} a_m x^m \longrightarrow (2)$$

Then,

$$D(f(c)) = \sum_{m=0}^{n} a_m (Dx^m)(c)$$

$$D^2(f(c)) = \sum_{m=0}^{n} a_m (D^2x^m)(c)$$

$$\vdots$$

$$D^k(f(c)) = \sum_{m=0}^{n} a_m (D^k x^m)(c)$$

Multiply by $\sum_{k=0}^{n} \dfrac{(x-c)^k}{k!}$

$$\sum_{k=0}^{n} \frac{D^k f(c)(x-c)^k}{k!} = \sum_k \sum_m a_m \frac{(D^k x^m)}{k!}(c)(x-c)^k$$

$$= \sum_m a_m \sum_k \frac{D^k x^m}{k!}(c)(x-c)^k$$

$$= \sum_m a_m x^m \quad \text{by } (1)$$

$$\sum_{k=0}^{n} \frac{(D^k f)}{k!}(c)(x-c)^k = f \quad \text{by } (2).$$

18

Theorem : 6

Let F be a field of characteristic zero and f a polynomial over F with $\deg f \le n$. Then the scalar c is a root of f of Multiplicity r iff $(D^k f)(c) = 0$, $0 \le k \le r-1$ $(D^r f)(c) \ne 0$.

Proof:

Necessary Part:

To Prove that : $(D^k f)(c) \ne 0$

Given c is a root of 'f' of Multiplicity r

ie, $f = (x-c)^r . g \rightarrow (1)$ $\forall g \in F[x]$ with $g(c) \ne 0$

By the taylor's formula,

$$g = \sum_{k=0}^{n-r} \frac{D^k g(c)}{k!} (x-c)^k \rightarrow (2)$$

using (2) in (1) apply;

$$f = (x-c)^r \sum_{k=0}^{n-r} \frac{D^k g(c)}{k!} (x-c)^k$$

$$f = \sum_{k=0}^{n-r} \frac{D^k g(c)}{k!} (x-c)^{r+k} \rightarrow (3)$$

Since, $\deg f = n$

By taylor's formula,

$$f = \sum_{k=0}^{n} \frac{D^k f(c)}{k!} (x-c)^k \rightarrow (4)$$

from (3) & (4)

$$g(c)(x-c)^r + \frac{D[g(c)]}{1!}(x-c)^{r+1} + \cdots$$

$$= f(c) + \frac{D^1[f(c)]}{1!}(x-c) + \frac{D^{11}[f(c)]}{2!}(x-c)^2$$

$$+ \cdots + \frac{D^{r-1} f(c)}{(r-1)!}(x-c)^{r-1}$$

$$+ \frac{D^r f(c)}{r!}(x-c)^r \rightarrow (5)$$

Since, $x-c \nmid g(x) \Rightarrow$ characteristic zero.

$\Rightarrow g(c) \neq 0$

from (5), $\frac{D^r f(c)}{r!} \neq 0.$

$\Rightarrow D^r f(c) \neq 0.$

sufficient part:

To prove that: c is a root of f with multiplicity 'r' given that,

$$D^k f(c) = 0, \text{ for } 0 \le k \le r-1$$

Since, deg $f = n$

By Taylor's formula,

$$f = \sum_{k=0}^{n} \frac{D^k f(c)}{k!} (x-c)^k$$

Since, $D^k f(c) = 0$ for $k = 0, 1, 2 \dots n-1$

$$\therefore f = \sum_{k=r}^{n} \frac{D^k f(c)}{k!} (x-c)^k$$

$\nmid$ does not divise

Hence, $(x-c)^r \mid f$ & $(x-c)^{r+1} \nmid f$.

$\therefore$ c is a root of f with multiplicity 'r'.

Theorem: 7.

If F is a field and M is any non-zero ideal in F[x] there is a unique monic polynomial d in F[x] such that M is the principal ideal generated by d.

$\Rightarrow$ characteristic polynomial

Proof:

Assume that M contains a non-zero polynomial.

Let d = Minimal degree of the polynomial

With out loss of generality.

Assume that d is Monic.

Now, if $f \in M$

Then by division Algorithm

$$f = dq + r \rightarrow (1)$$

Where, $r = 0$ (or) $\deg r < \deg d$.

Since d is Minimal in M.

So, $\deg r < \deg d$ is impossible.

$(1) \Rightarrow \quad \therefore r = 0$

We have $f = dq$.

Hence, $M = d . F[x]$

M is principal ideal generated by d.

To prove: Uniqueness

Suppose, $\exists$ $p, q \in F[x]$

$$d = gp \ \& \ g = dq$$

Thus, $d = dqp$

$$\Rightarrow \deg d = \deg d + \deg q + \deg p$$

$$\deg q + \deg p = 0$$

$$\deg p = 0 = \deg q \rightarrow (2)$$

also d, g are Monic

$$p = q = 1 \rightarrow (3)$$

Thus $d = q = p = 1$

16.9.19

Corollary : 3

If $P_1, P_2, \ldots P_n$ are polynomials over a field not all of which are 0, there is a unique monic polynomial d in $F[x]$ such that,

a) d is in ideal generated by $P_1, P_2, \ldots, P_n$ (21)

b) d divides each of the polynomials P_i.

Any polynomial satisfying (a) & (b) necessarily satisfies.

c) d is divisible by every polynomial which divides each of the polynomials $P_1, P_2, \ldots P_n$.

Proof:

let d be the monic generated of the ideal

$P_1 F[x] + \cdots + P_n F[x]$.

$\Rightarrow$ Every member of the ideal is divisible by d.

$\Rightarrow$ Each of the polynomial P_i is divisible by d.

Now, suppose f is a polynomial which divides each of the polynomials $P_1 \ldots P_n$.

Then $\exists$ $g_1, g_2, \ldots g_n$ such that $P_i = f g_i ; 1 \leq i \leq n$

Since d is ideal.

$P_1 F[x] + \cdots + P_n F[x]$

$\exists$ $q_1, q_2, \ldots q_n$ in $F[x]$ such that $d = P_1 q_1 + \cdots + P_n q_n$

$d = f g_1 q_1 + f g_2 q_2 + \cdots + f g_n q_n$

Thus, $d = f [g_1 q_1 + g_2 q_2 + \cdots + g_n q_n]$

we have show that,

d is monic polynomial satisfying (a), (b), (c)

To Prove: Uniqueness.

If d' is any polynomial satisfying (a), (b) is follows,

from (a), d' is monic polynomial,

$d' = df \rightarrow (1)$

Since d is the monic generator of

$P_1 F[x] + \cdots + P_n F[x]$

$d = d'g \rightarrow (2)$ $\quad g \in F(x)$

From (1) & (2) $\quad d' = d'fg$

$\deg d' = \deg d' + \deg f + \deg g$

$\Rightarrow \deg f + \deg g = 0$

$\deg f = \deg g = 0$

Since $f = g = 1$

$d = d'$

Theorem (8):

Let P, f and g be polynomials over the field F. Suppose that P is a prime polynomial and that P divides the product fg. Then either P divides f or P divides g.

Proof:

Without loss of generality.

Assume that,

P is a Monic Polynomial P/fg.

If P divides f then nothing to prove

If P does not divides f then $(P, f) = 1$

gcd = 1

By the theorem,

$\exists$ $f_0, g_0 \in F[x]$ such that,

$f_0 f + P_0 P = 1$. $\times$ by g.

$g = f_0 fg + P_0 Pg$

$= (fg) f_0 + P(P_0 g)$

Since $P | fg$

$\Rightarrow P/(fg) f_0 \longrightarrow (i)$

Also, $\quad P/P(P_0 g) \longrightarrow (ii)$

$\therefore P/(fg) f_0 + P(P_0 g)$

$\therefore P/g$.

corollary : 7

If p is a prime and divides a product $f_1, f_2, \ldots f_n$ then p divides one of the polynomial $f_1, f_2, \ldots f_n$.

proof:

To Prove : The result by induction on n.

If $n=2$ then the result is true.

Assume that,

The result is true for less then n.

$$p \mid f_1 f_2$$

Now for n,

$$\Rightarrow p \mid f_1 \;\&\; p \mid f_2.$$

$$p \mid f_1, f_2, \ldots f_n$$

$$\Rightarrow p \mid (f_1, f_2, \ldots f_{n-1}) f_n$$

$$\Rightarrow p \mid (f_1, f_2, \ldots, f_{n-1}) \text{ (or) } p \mid f_n$$

If $p \mid f_n$ then by induction hypothesis

$$p \mid (f_1, f_2, \ldots f_{n-1})$$

$$\Rightarrow p \mid f_i \;,\; (1 \le i \le n-1)$$

Theorem: 9

(7) If F is a ~~fixed~~ field, a non scalar monic polynomial in $F[x]$ can be fractored of a product of monic primes in $F[x]$ is one and Expact order for only one way.

Proof:

Let F be a non-scalar monic polynomial over F.

To Prove: The result by Induction on n.

If $n=1$

then $\deg f = 1$ factorise ⇒ reduced

⇒ f is reduced able. not factorise ⇒ reducedable

⇒ The result obsoviely.

Assume that,

The result is true for $< n$.

To prove for n.

If f is irreducedable (not prime)

Then, $f = gh$; Where $g\,h$ are non scalar Monic polynomial degree

Thus by Induction hypothesis.

g and h can be factored as a product of Monic primes in $F[x]$.

$\therefore f$ can be factoried as a product of monic primes in $F[x]$.

To prove: Uniqueness.

Suppose $f = p_1 \ldots p_m = q_1 \ldots q_n$

Where, $p_1, \ldots p_m$ and $q_1 \ldots q_n$ are monic primes in $F[x]$.

Then, $p_m / q_1 \ldots q_n$

$\Rightarrow p_m / q_i$ for some 'i'

Since q_i and p_m are both monic primes

$q_i = p_m \longrightarrow (1)$

$m = n = 1$ if either $m = 1$ (or) $= n = 1$

25

for $\deg f = \sum_{i=1}^{m} \deg p_i = \sum_{j=1}^{n} \deg q_i$

In this case nothing to prove.

So, We may assume $m > 1$ and $n > 1$

by Re arrangeing the q's We can then assume $p_m = q_n$ and that

$$p_1 \ldots p_{m-1}, \cancel{p_m} = q_1 \ldots q_{n-1}. \cancel{p_m}$$

$$p_1 \ldots p_{m-1} = q_1 \ldots q_{n-1}$$

p's and q's has a polynomial of degree $n-1$

By Induction hypothesis so that the sequence $q_1 \ldots q_{n-1}$ is atmost a rearrangement of a sequence $p_1 \ldots, p_{m-1}$.

Hence, f is a product of monic prime is Unique upto the order of the factors.

Theorem :(11).

Let f be a polynomial over the field F with derivative f'. Then f is product of distinct irreducable polynomial over F. iff f and f' are relatively Prime.

Proof:

To prove: f is a product of distinct irreducable Polynomial over F.

Assume that: f and f' are relatively prime $\longrightarrow (*)$

By corollary, (4)

"f is product of Prime Polynomial p is repeated. Then $f = p^2 h$ for some $h \in F[x]$.

Then. $f' = p^2 h' + 2pp'h$ [Two polynomial have gcD is 1 is called Relatively Prime].

and $\Rightarrow p / f'$

Hence, f and f' are not relatively Prime

$\Rightarrow\Leftarrow$ by (*)

Hence, f is a product of distinct Prime polynomial over F.

To prove that f and f' are realively prime.

Assume that $f = p_1 \ldots\ldots p_K$.

Where, $p_1 \ldots p_K$ are distinct non-scalar prime Polynomial over F.

Let $f_j = f / p_j$ then,

$$f' = p_1' f_1 + p_2' f_2 + \cdots + p_K' f_K$$

$$= \sum_{j=1}^{K} p_j' f_j$$

Let p be a prime Polynomial Which divise both f and f'.

Then, $p = p_i$ for some 'i'

Now, p_i / f_j for $j \neq i$

Since p_i / f'

$$\Rightarrow p_i / \sum_{j=1}^{k} p_j' f_j$$

$$\Rightarrow p_i / p_i' f_i$$

$\Rightarrow p_i / p_i'$ (or) p_i / f_i (27)

But $p_i \nmid f_i$ because p_i are distinct

$\Rightarrow p_i / p_i'$

Which is Impossible.

Because $\deg p_i' < \deg p_i$.

Hence, no prime divied f and f'.

So, f and f' are relatively Prime.

Lemma : 2

Let D be an 2- Linear function on $n \times n$ Matrices over K. Suppose D with the property that $D(A)=0$ for all 2×2 Matrices A over K having equal rows then D is Alternative.

proof:

Let $A = [\alpha \beta]$

Where α & β are rows of A.

To prove that, D is alternating.

That is to prove that $D(\alpha, \beta) = -D(\beta, \alpha)$

Since, D is 2- Linear

$$D(\alpha+\beta, \alpha+\beta) = D(\alpha, \alpha) + D(\alpha, \beta) + D(\beta, \alpha) + D(\beta, \beta)$$

By our hypothesis,

$$D(\alpha+\beta, \alpha+\beta) = 0$$

$$D(\alpha, \alpha) = 0$$

$$D(\beta, \beta) = 0$$

$$D(\alpha, \beta) + D(\beta, \alpha) = 0$$

$$D(\alpha, \beta) = -D(\beta, \alpha)$$

Lemma : 3

Let D be an n-Linear function on $n \times n$ Matrices over K. Suppose D has the property that $D(A)=0$. When ever two adjecent rows of A are equal then D is alternative.

Proof:

Show that $D(A)=0$.

When Any two rows of A are equal and

that $D(A') = -D(A)$.

If A' is obtained by interchaning

Some two rows of A.

Let, $A_1 = [\alpha_1, \alpha_2, \dots \alpha_i, \alpha_j, \dots \alpha_n]$

Let B be the Matrix obtained form A by interchanig i^{th} and j^{th} Rows

By known Lemma, Now A_1 obtain from A interchaning i^{th} row $(i+1)^{th}$ row.

Now, A_2 is obtained from A_1 interchaning $A_1 = \{\alpha_1, \alpha_2, \dots \alpha_{i+1}, \alpha_i \dots \alpha_n\}$

i^{th} row with $(i+2)^{th}$ row

$$A_2 = [\alpha_1, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_j, \alpha_i, \alpha_{j+1}, \dots \alpha_n]$$

$$D(A_1) = D(A_2)$$

$$D(A) = (-1)^2 D(A_2)$$

Continue this process.

Upto $k=j-i$ interchanges of adjecent rows.

$$D(A) = (-1)^{j-i} D(A_{j-i})$$

Where,

$$A_{j-1} = [\alpha_1 \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_j, \alpha_i, \alpha_{j+1}, \dots \alpha_n]$$

Similarly,

We Now, Move on α_j to the i^{th} position using $k-1$ interchanges of adjecent row.

Thus obtained B from A by $k+(k-1) = 2k-1$ interchaning on adjecent rows.

Thus, $D(B) = (-1)^{2k-1} D(A)$

$D(B) = -D(A)$

Suppose A is any $n \times n$ - Matrix with two equal rows, say $\alpha_i = \alpha_j$ with $i < j$.

If $j = i+1$

Then, A has two equal & adjacent rows and $D(A) = 0$

If $j > i+1$ then interchaning α_{i+1} and α_j and the resulting Matrix has two equal and adjacent rows.

So $D(B) = 0$

Since, $D(A) = -D(B)$

$D(A) = -(0)$

$D(A) = 0$

$\therefore$ D is alternating.

Theorem: 12:

Let $n \geq 1$ and let D be an alternating $n-1$ linear function on $(n-1) \times (n-1)$ Matrices over K. for each j: $1 \leq j \leq n$, the function E, defined by

$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$

Is an alternating n- linear function on $n \times n$ Matrix A. If D is a determinant function So its each E_j

proof:

To prove that E_j is n-Linear.

Let A is an $n \times n$ Matrix.

Given that $D_{ij}(A) = D(A(i/j))$ is $(n-1)$ linear.
i.j Rows are Interchange

The function D_{ij} is independent of i^{th} rows of A.

$\Rightarrow A_{ij} D_{ij}(A)$ is dependent on i^{th} rows and clearly is Linear.

$\Rightarrow A_{ij} D_{ij}(A)$ is Linear.

W.K.T

A Linear Combination of n - Linear function is n - Linear.

$\Rightarrow \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$ is n - Linear

ie, E_j is n - Linear.

20.9.19 Given that, D is alternative

To prove that, E_j is alternating.

Let A be a Matrix with two adjacent rows are Equal.

Suppose $\alpha_k = \alpha_{k+1}$

$$E_j(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$$

odd-term

$$= (-1)^{j+1} A_{ij} D(A(1/j)) + \cdots\cdots +$$

even

$$(-1)^{j+k} A_{kj} D(A(k/j)) + (-1)^{j+k+1} A_{(k+1)j} D(A(\ldots$$

$$+ \cdots + (-1)^{j+n} A_{nj} D(A(n/j))$$

If $i \neq k$ & $i \neq k+1$

Then the Matrix $A(i/j)$ has two equal rows and $D_{ij}(A) = 0$.

$E_j(A) = (-1)^{k+j} A_{kj} D_{kj}(A) + (-1)^{k+1+j} A_{(k+1)j} D_{(k+1)} A_j$

Since, $\alpha_k = \alpha_{k+1}$

$A_{kj} = A_{(k+1)j}$ and $A(k|j) = A(k+1/j)$

clearly, $E_j(A) = 0$

To prove that, E_j is a determinant function.

Assume that, D is a determinant function.

If I^n is $n \times n$ identity Matrix.

Then,

$I^n(i/j)$ is the $(n-1) \times (n-1)$ Identity Matrix I^{n-1}.

$I_{ij}^{(n)} = \delta_{ij}$ $\quad i=j \Rightarrow 1$, $j \neq i \Rightarrow 0$

by (1)

$$E_j(I^n) = \sum_{j=1}^{n} (-1)^{i+j} \delta_{ij} D_{ij}(I^n)$$

$$= (-1)^{j+j} \delta_{jj} D_{jj}(I^n) + \sum_{i=1}^{n} (-1)^{i+j} \underbrace{\delta_{ij}}_{0} D_{ij}(I^n)$$

$$= (-1)^{2j} D_{ij}(I^n)$$

$$E_j(I^n) = D(I^{n-1})$$

Since $D(I^{n-1}) = 1$

So, $\underline{E_j(I^n) = 1}$

$\therefore$ E_j is a determinant function.

Corollary:

Let K be a commutative Ring with identity and let n be a positive Integer there exists atleast one determinant function on $K^{n \times n}$.

Proof:

To prove the result by induction on n.

If $n=1$ clearly the exists of a determined function on $k^{1\times 1}$.

Assume that,

The result is true of α_n.

Now, for n by known theorem(1)

There exists determinat function on $k^{n\times n}$

9.10.19 2 Mark:

Ring:

A Ring is a set k, together with two operation $(x,y) \to x+y$ and $(x,y) \to xy$

a> k is a commulative group under the operation $(x,y) \to (x+y)$ (k is a commulative group under addition).

b> $(xy)z = x(yz)$ [Multiplication is Associative]

c> $x(y+z) = xy + xz$
$(y+z)x = yx + zx$ {The two distributive Laws is hold}

If $xy = yx$ $\forall x, y \in k$. We say that the k is commulative. If there is an element one in k. Such that $1 \cdot x = x \cdot 1 = x$ for each x, k is Said to be a ring with Identity, and one is called the Identity for k.

n-Linear:

Let k be a commulative ring with Identity and let D be a function n be a positive Integer and let D be a function which assinged to $n\times n$ Matrix A over k a Scalar $P(A)$ in k. We Say that D is n-Linear if for each $1 \leq i \leq n$. D is a Linear function of the i^{th} row when the order $n-1$ rows are held.

1. If $f(x)$ and $g(x)$ are two Polynomial then

2. Given a polynomial $f(x) = a_0 + a_1 x + \cdots + a_m x^m$. Where a's are integer then content of $f(x)$ is

a) gcd are Integer $a_0, \ldots, a_n$

b) Mean of Integer $a_0, \ldots, a_n$

c) Mode of Integers $a_0, \ldots, a_n$

d) none of these.

3. Given the polynomial $P(x) = a_0 + a_1 x + \cdots + a_m x^m$ is degree is m if

a) $a_m = 0$. b) $a_m \neq 0$. c) $a_{m-1} = 0$. d) $a_{m-1} \neq 0$.

4. If $f(x)$ and $g(x)$ are two non zero polynomial of $f(x)$ then.

a) degree of $f(x)\,g(x) = \deg f(x) \cdot \deg g(x)$

b) degree of $f(x) + g(x) = \deg f(x) + \deg g(x)$

c) degree of $f(x) - g(x) = \deg f(x) - \deg g(x)$

d) degree of $f(x) \div g(x) = \deg f(x) \div \deg g(x)$

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Determinant function:

Let K be a commutative ring with identity and let n be a positive Integer. Suppose D is function from n×n Matrix K into K. We Say that D is a Determinant function. If D is n-Linear, alternating, D(I) = 1.

Alternate:

Let D be a n-Linear function we Say D is alternating if the following two condition are Satisfy.

i) D(A) = 0, When ever two rows of A are eq

ii) If A' is a Matrix obtain from A by interchaning two rows of A, then D(A') = -D(A)

Algebric closed:

The field F is called algebric closed. If every prime polynomial over F has degree is one.

Example:

If F is alegebric closed Means every non Scalar Irreducible monic polynomial over F is of the form (x-c)

Relatively Prime:

If $P_1, \ldots, P_n$ are polynomial over a field F not all of which are Zero. The Monic generater d of the ideal. $P_1 F[x] + \cdots + P_n F[x]$ is called the greatest common divisor (g.c.d) of $P_1, P_2, \ldots P_n$. The polynomials $P_1, P_2 \ldots P_n$ are Relatively prime if their greatest common divisor is 1. or equivalently if the ideal they generate is all of $F[x]$.

Irreducible Polynomial:

Let F be a field. A polynomial f in $F[x]$ is said to be reducible over F if there exist polynomials g, h in $F[x]$ of degree ≥ 1 such that $f = gh$ and if not, f is said to be irreducible over F. A non scalar irreducible polynomial over F is called prime polynomial over F and we sometimes say it is a prime in $F[x]$.

UNIT - IV

- Definition

Invariant Subspaces

Simultaneous Triangulation Simultaneous Diagonalization.

Direct Sum Decomposition.

Invariant Direct sums.

The primary decomposition theorem.

Lemma: 1

(8) Q. 5m Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. The minimal polynomial for T_W divides the minimal polynomial for T.

proof:

We have $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ $(A - xI)$

where $A = [T]_B$ and $B = [T]_{B'}$

To prove that: The characteristic polynomial. The characteristic polynomial for T_W divides the characteristic polynomial for T.

$$(A - xI) = \begin{bmatrix} B - xI & C - xI \\ 0 & D - xI \end{bmatrix}$$

$$\det(A-xI) = \det(B-xI)\det(D-xI)$$

$\Rightarrow \det(B-xI) / \det(A-xI)$

$\therefore$ The characteristic polynomial for T_W/the characteristic polynomial for T.

ii) To prove that: The minimal polynomial for T_W divides the minimal polynomial for T.

The k^{th} power of the matrix.

A has the block form

$$A^k = \begin{bmatrix} B^k & C^k \\ 0 & D^k \end{bmatrix}$$

where C^k is some $r \times (n-r)$ matrix

any polynomial which annihilator 'A'

Any polynomial which annihilates B and D also,

The minimal polynomial for B divides the minimal polynomial for A.

$\therefore$ H/P.

Lemma:

If W is an invariant subspace for T, then W is invarient under every polynomial in T. Thus for each α in V the conductor $S(\alpha, W)$ is an ideal in the polynomial algebra $F[x]$.

since $T(W) \subseteq W \rightarrow (1)$

If $\beta \in W$ then $T\beta \in W$

$T^2\beta = T(T\beta) \in TW \rightarrow (2)$

from (1) & (2)

$T^2\beta \in W \quad \forall \beta \in W$

||ly $T^k\beta \in W$

let $f(T) = a_0 + a_1T + \cdots + a_nT^n \cdot a_n \neq 0$

Then $f(T)\beta = a_0\beta + a_1T(\beta) + \cdots + a_nT^n(\beta)$.

$f(T)\beta \in W$ because $T^k\beta \in W$

Thus, $\beta \in W \Rightarrow f(T)\beta \in W$

i.e. $f(T)W \subseteq W$

$\therefore W$ is a invariant under every polynomial in T.

prove that : $S(\alpha, W)$ is an ideal in $F[x]$

$S(\alpha, W) = \{g \in F[x] : g(T)\alpha \in W \ \forall \alpha \in V\}$

since $S(\alpha, W) \subseteq F[x]$

i.e. $S(\alpha, W)$ is a subspace of $F[x]$.

Let $f, g \in S(\alpha, W)$ then $f(T)\alpha, g(T)\alpha \in W$

consider, $((cf+g)(T)(\alpha) = [(cf)T\alpha + (g)T\alpha]$

since, W is a invariant subspace for T.

i.e. $(cf)T\alpha + (g)T\alpha \in W]$

$(cf+g)(T)\alpha \in W$

$\Rightarrow cf + g \in S(\alpha, W)$

Thus $f, g \in S(\alpha, W) \Rightarrow cf + g \in S(\alpha, W)$

So, $S(\alpha, W)$ is a subspace of $F[x]$

ii) $f \in F[x], g \in S(\alpha, W) \Rightarrow fg \in S(\alpha, W)$

Let $g \in S(\alpha, W)$ then $g(T)\alpha \in W$

Since W is a subspace for T $f(T)[g(T)\alpha] \in W$

By definition of $S(\alpha, W)$

we have. $fg \in S(\alpha, W)$

Thus $f \in F[x], g \in S(\alpha, W) \Rightarrow fg \in S(\alpha, W)$

From (i) & (ii).

$S(\alpha, W)$ is an ideal in $F[x]$

H/P.

Lemma: 3

Let V be a finite dimensional v.s over the field F. Let T be a linear operator on V. Such that the minimal polynomial for T is an product of linear factors.

$$p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}, \ c_i \text{ in } F.$$

Let W be a proper $(W \neq V)$ subspace of V. which is invarient under T. There exist a vector α in V such that.

a) α is not in W.

b) $(T-cI)\alpha$ is in W, for some characteristic values c of the operator T.

proof:

let $\beta \in V$ and $\beta \notin W$

let g be the T- conductor of β into W. since p is a minimal

$\therefore g | p$.

since $\beta \notin W$

clearly g is not a scalar polynomial.

$$g = (x-c_1)^{e_1} \cdots (x-c_k)^{e_k}$$

where atleast one integer e_i's positive

choose, j. so that $e_j > 0$.

Then, $(x-c_j) | g$

$$\Rightarrow \frac{g}{x-c_j} = h \text{ (say)}$$

$$\Rightarrow g = h(x-c_j)$$

By definition of g.

The vector $\alpha = h(T)\beta$

consider, $(T-c_jI)\alpha = (T-c_jI)h(T)\beta$

$= g(T)\beta \in W$ $\quad \because g \in S(\beta, W)$

$(T-c_jI)\alpha \in W$

Invarient Subspace:

let V be a vector space and T be a linear operator on V. If W is a subspace of V. we say that W is invarient under T. if for each vector α in W. The vector $T\alpha$ is in W.

i.e. If $T(W)$ is contained in W.

T-conductor of α into W.

Let W be an invarient subspace for T. and let α be a vector in V. The T conductor of α into W if the set $S_T(\alpha, W)$. which consist of all polynomials g (over the scalar field) such that,

$\therefore g(T)\alpha$ in W.

The Invarient so under T.

The subspace W is invarient under $\mathcal{F}$ (family of operator) if W is invarient under each operator in $\mathcal{F}$

Independent Subspace.

let $W_1, W_2 \ldots W_K$ the subspace of the vector space V. we say that

$W_1 \dots W_K$ are independent. If $\alpha_1 + \alpha_2 + \dots \alpha_k = 0$ & $\alpha_i \in W_i$.

$\Rightarrow$ Each α_i is zero.

projunction.

If V is a vector space A projunction of V is a linear operator E on V such that $E^2 = E$.

Nilpotent:

Let N be a linear operator on V.S V. We say that N is nilpotent, if there is some positive integer r s.t

$N^r = 0$.

T- Annihilator.

~~write the definition of T-conductor~~

where $W = 0$ Thus the T-conductor becomes the 'T' annihilator of α.

Direct sum:

Let V be a finite dimensional V.S. Let $W_1 \dots W_K$ be the subspace of V and let $\alpha_1 \dots \alpha_n$ be any basis for V. If W_i is the one dimensional subspace spanned by α_i. Then we say If $W_1 \dots W_K$ are independent and we write it as that sum $W_1 \dots W_K$. Then the sum V is $W_1 \oplus W_2 \oplus \dots \oplus W_K$.

said to be direct sum of $W_1 \dots W_K$.

Thm: 1

Let V be a finite dimensional v.s over the field F and let T be a linear operator on V. Then T is Triangulable iff the minimal polynomial ~~for T is a product~~ for T is a product of linear polynomial over F.

proof:

To proove: T is Triangulable. (Sufficient part)

Suppose that the minimal polynomial factors $p = (x-c_1)^{r_1}(x-c_2)^{r_2} \cdots (x-c_k)^{r_k}$

Let $W = \{0\}$.

clearly W is a proper subspace of V.

$\exists$ $\alpha_1 \in V$ but $\alpha_1 \notin W$ such that,

$(T - a_{11}I)\alpha_1 \in W \quad \forall a_{11} \in F$ (by lemma

$(T - a_{11}I)\alpha_1 = 0$

$T\alpha_1 = a_{11}\alpha_1 \longrightarrow (1)$

Let W_1 be the subspace of V spanned by α_1 and W.

If $W_1 = V$ we nothing to proove

otherwise $W_1 \neq V$.

i.e. W is a proper subspace of V.

we know that,

$\exists\ \alpha_2 \in V$ but not in W,

such that,

$(T - a_{22} I)\alpha_2 \in W = \langle \alpha_1 \rangle\ \forall\ a_{22} \in F$

$\Rightarrow (T - a_{22})\alpha_2 = a_{12}\alpha_1\ \ \forall\ a_{12} \in F$

$\Rightarrow T\alpha_2 - a_{22}\alpha_2 = a_{12}\alpha_1$

$\Rightarrow T\alpha_2 = a_{12}\alpha_1 + a_{22}\alpha_2 \rightarrow (2)$

If $W_2 = \langle \alpha_1, \alpha_2 \rangle = V$. then we have nothing to prove.

otherwise If we continue in the finite stage say n.

$W_n = \langle \alpha_1 \ldots \alpha_n \rangle = V$ because

$\dim V < \alpha$.

$T\alpha_n = a_{1n}\alpha_1 + \ldots + a_{nn}\alpha_n$.

i.e. $\exists$ a basis $\beta = \{\alpha_1 \ldots \alpha_n\}$

such that,

$$T\alpha_j = \sum_{i=1}^{n} a_{ij}\alpha_i \quad , j = 1, \ldots m$$

$$T_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

T is triangular

Necessary part:

we assume that, T is Triangular

To prove that, The minimal polynomial for T is a product of linear polynomial f.

There exist a basis $\mathbb{B} = \{\alpha_1, \dots \alpha_n\}$ such that $[T]_{\mathbb{B}}$ is Triangular.

Let us take,

$$A = [T]_{\mathbb{B}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

The characteristic polynomial f is a product of linear polynomial over F.

Since, The minimal polynomial for g divides in the characteristic polynomial for T.

The minimal polynomial for T is a product of linear polynomial.

$\therefore$ H|P.

Corollary:1

Let F be an algebrically closed fieled eg. The complex number field. every nxn matrix ov F is similar over F to a triangular.

proof: W.K.T

T is Triangular iff the minimal polynomial for T is an product of linear polynomial.

Let A be a nxn matrix over F.

Let f be a characteristic polynomial for A over F.

Since F be an algebraically closed.

$\therefore$ f be a product of linear polynomial over F.

The minimal polynomial for A is a product linear polynomial over F

$\Rightarrow$ A is triangular over F

Every matrix over F is similar to a triangular.

$\therefore$ H/p.

5m U.Q.

Theorem: 5m U.Q.

(14) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then T is diagonalizable iff the minimal polynomial for T has the form

$p = (x-c_1) \dots (x-c_k)$

where $c_1, c_2 \dots c_k$ are distinct element of F.

proof:

Necessary part:

Assume that T is diagonalizable

Then $\exists$ a basis $\mathcal{B} = \{\alpha_1, \dots \alpha_n\}$

Such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} t_1 & \cdots & \\ & \ddots & t_n \end{bmatrix}$$

$\Rightarrow$ characteristic polynomial for T is

$(x-t_1)\cdots(x-t_n)$.

Since minimal polynomial for T divides the characteristic polynomial for T.

So, the minimal polynomial p for T has the form.

$$p=(x-c_1)\cdots(x-c_k)$$

where, $c_1, c_2 \cdots c_k$ are distinct element of F.

Sufficient Part:

To prove T is diagonalizable.

Assume that the minimal polynomial p for T has the form.

$$p=(x-c_1)\cdots(x-c_k).$$

where, $c_1, c_2 \ldots c_k$ are distinct element of F

Let W be the subspace spanned by all of the characteristic vector of T.

If $W \neq V$ then $\exists$ $\alpha \in V$ but $\alpha \notin W$.

Such that,

$$\beta=(T-c_jI)\alpha \in W$$

since $\beta \in W$.

By the definition of W,

$\beta=\beta_1+\cdots+\beta_k$ where, $T\beta_i=c_i\beta_i$

Let h be any polynomial over F.

$h(T)\beta = h(c)\beta$.

$h(T)\beta = h(c_1)\beta_1 + \dots + h(c_k)\beta_k$

is in W, for every polynomial h.

Now, $p = (x - c_j)q$ for some polynomial q.

Also, $q - q(c_j) = (x - c_j)h$.

consider.

$q(T) - q(c_j) = [(T - c_j)h(T)]$

$[q(T) - q(c_j)]\alpha = [(T - c_j)h(T)]\alpha$.

$= h(T)(T - c_j)\alpha$

$= h(T)\beta$

since, $h(T)\beta \in W$

$[q(T) - q(c_j)]\alpha \in W$

since $0 = p(T)\alpha$

$= (T - c_j)q(T)\alpha$

$= q(T)(T - c_j)\alpha$

$= q(T)\alpha \in W$

$\therefore q(c_j)\alpha \in W$. (0, p has multiple roots

$\therefore q(c_j) = 0$.

which $\Rightarrow \Leftarrow$ to factor that p has, distinct roots.

$\therefore$ our assumption $W \neq V$ is wrong

$W = V$.

All the characteristic vector are basis in V. T is diagonalizable.

Simultaneous Triangulation Simultaneous Diagonalization

Lemma: 4

let $\mathcal{F}$ be the commuting family of triangulable linear operator on V. let W be a proper subspace of V which is invariant under F. then exists a vector α in V such that,

a) α is not in W

b) for each T is F the vector $T\alpha$ is in the subspace spanned by α and W.

proof:

without loss of generality.

to assume that,

F contains only a finite number of operator.

let $\{T_1, T_2 \dots T_r\}$ be a maximal linearly independent subset of F

i.e a basis for the subspace spanned by F.

since W is a subspace of V. Then, $\exists$ $\beta_1 \notin W$ and $c_1 \in F$ such that,

$$(T - c_1 I)\beta_1 \in W$$

let $V_1 = \{\beta \in V / (T_1 - c_1 I)\beta \in W\}$

Then V_1 is a subspace of V which is properly larger then W.

since, W is invariant under $\mathcal{F}$.

If $TT_1 = T_1T$ then,

$(T_1 - c_1 I)(T\beta) = T(T_1 - c_1 I)\beta$

If $\beta \in V_1$ then $(T - c_1 I)\beta \in W$.

since W is invarient under each T in $\mathcal{F}$.

we have, $T[T_1 - c_1 I]\beta \in W$

$\Rightarrow [(T_1 - c_1 I)T]\beta \in W$

$\Rightarrow [(T_1 - c_1 I)](T\beta) \in W$

$\Rightarrow (T\beta) \in W \; \forall T \in \mathcal{F} \; \beta \in V_1$ $\quad W \neq V_1$

Now, W is a proper subspace of V_1

Let V_2 be the linear operator on V_1

obtained by restricting.

T_2 to the subspace of V_1.

$\Rightarrow$ The minimal polynomial for U_2 divides

the minimal polynomial for T_2.

since, $W \neq V_1$

If $\beta_2 \in V_1$, but $\beta_2 \notin W$

Such that, $(T_2 - c_2 I)\beta_2 \in W$

Note that:

a) $\beta_2 \notin W$

b) $(T_1 - c_1 I)\beta_2 \in W$

c) $(T_2 - c_2 I)\beta_2 \in W$

Let $V_2 = \{\beta \in V \mid (T_2 - c_2 I)\beta \in W\}$

Then, V_2 is invarient under $\mathcal{F}$

Let U_3 be the restriction of T_3 to V_2.

If we continue in this way, until we reach a vector.

$\alpha = \beta_r$ (not in W) such that, $(T_j - c_j I)\alpha \in W$, $j = 1, \dots r$

since, $\{T_1, \dots T_r\}$ is maximal linearly independent set.

$\exists \, \alpha \notin W$ such that $(T - cI)\alpha \in W$

$\Rightarrow T\alpha - c\alpha \in W \quad \forall \, T \in F; c \in F$

$\Rightarrow T\alpha \in W + c\alpha$

$\Rightarrow T\alpha \in W + \langle \alpha \rangle$

$\therefore T\alpha$ is a subspace spanned by W and α for each T.

$\therefore$ H/P.

Direct sum Decomposition:

Note:

1). For $k = 2$ for definition of independent show that W_1, W_2 are independent subspace of V iff $W_1 \cap W_2 = \{0\}$.

2). If $k > 2$, $W_1, W_2, \dots W_k$ are independent subspace of V then $W_1 \cap W_2 \cap \dots \cap W_k = \{0\}$

i.e. W_j intersects the sum of the other subspace W_i only in the zero vector.

* If $W_1, \ldots W_k$ are linearly independent subspace of V then such vector α in W can be uniquely expressed as a sum $\alpha = \alpha_1 + \ldots + \alpha_k$ α_i in W_i.

proof:

let $W = W_1 + \ldots + W_k$

let $\alpha \in W$ then,

$\alpha = \alpha_1 + \ldots + \alpha_k \rightarrow (1)$ $\alpha_i \in W_i$

To prove : uniqueness

Suppose that, $\alpha = \beta_1 + \ldots + \beta_k$, $\beta_i \in W_i \rightarrow (2)$

(1-2)

$\alpha - \alpha = (\alpha_1 + \ldots + \alpha_k) - (\beta_1 \ldots \beta_k)$

$= (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \ldots + (\alpha_k - \beta_k)$

since, $W_1, W_2 \ldots W_k$ are L.I.

$\alpha_i - \beta_i = 0$ for each i $1 \le i \le k$

$\alpha_i = \beta_i$ for each i, $1 \le i \le k$.

Hence, $\alpha = \alpha_1 + \ldots + \alpha_k$ is unique expression.

Lemma: 5

Let V be a finite - dimensional vector space. Let $W_1 \ldots W_k$ be subspace of V and let $W = W_1 + \ldots + W_k$. The following are equivalent.

a) $W_1 \ldots W_k$ are independent.

b) For each j, $2 \le j \le k$, we have $W_j \cap (W_1 + \ldots + W_{j-1}) = \{0\}$

c). If $\mathcal{B}_i$ is the ordered basis for W_i,

$1 \leq i \leq k$. Then the sequence $\mathbb{B} = \{\mathbb{B}_1 \dots \mathbb{B}_k\}$ is ordered basis for W.

proof:

To prove: (a) $\Rightarrow$ (b)

Assume that $W_1 \dots W_k$ are independent

To prove: For each j, $2 \leq j \leq k$.

we have,

i.e. $\alpha \in W_j \cap (W_1 + \dots + W_{j-1})$

$\Rightarrow \alpha = 0$.

Let, $\alpha \in W_j \cap (W_1 + \dots + W_{j-1})$

Then, $\alpha \in W_j$ and $\alpha \in W_1 + \dots + W_{j-1}$

$\Rightarrow \alpha \in W_j$ and $\alpha = \alpha_1 + \dots + \alpha_{j-1} \cdot \alpha_i \in W_i$.

$\Rightarrow \alpha_1 + \dots + \alpha_{j-1} + (-\alpha) + 0 + 0 \dots = 0$

since, $W_1 \dots W_k$ we are independent.

$\alpha_1 = \alpha_2 \dots \alpha_{j-1} = \alpha = 0$.

Particularly $\alpha = 0$

$W_j \cap (W_1 + \dots W_{j-1}) = \{0\}$

To prove: (b) $\Rightarrow$ (a)

Assume (b),

$W_j \cap (W_1 + \dots + W_{j-1}) = \{0\}$

Assume contradiction of (a)

Suppose that,

$\alpha_1 + \dots + \alpha_k = 0 \quad \alpha_i \in W_i$

Let j be the largest integer i. such that $\alpha_i \neq 0$ then, $0 = \alpha_1 + \dots + \alpha_j$, $\alpha_j \neq 0$.

Thus $\alpha_j' = -\alpha, \dots - \alpha_{j-1}$ is a non-zero vector.

$\Rightarrow W_1, W_2 \dots W_k$ are linearly independent

Now, (a) = (b)

To prove : (a) → (c)

Assume (d) $W_1 \dots W_k$ are independent

let $\mathbb{B}_i$ be a basis for W_i $1 \le i \le k$

and let $\mathbb{B} = (\mathbb{B}_1 \dots \mathbb{B}_k)$

Any linear relation b/w the vectors in $\mathbb{B}$

$\beta_1 \dots + \beta_n = 0$

will have the form $\beta_1 \dots + \beta_n = 0$ where β_i is some linear combination of $\mathbb{B}_i$

Since, $W_1 \dots W_k$ are independent

$\Rightarrow \beta_i = 0$ for each i.

Since, each $\mathbb{B}_i$ is independent

Thus, the sequence

$\mathbb{B} = (\mathbb{B}_1, \dots \mathbb{B}_k)$ is an ordered

Basis for W.

H/P.

Thm : 3

If $V = W_1 \oplus \dots \oplus W_k$, then there exists k linear operator $E_1 \dots E_k$ on V such that.

i) Each E_i is a projection $(E_i^2 = E_i)$;

ii) $E_i \cdot E_j = 0$ if $i \ne j$;

iii) $I = E_1 + \dots + E_k$;

iv) The range of E_j is W_j

Conversely, if $E_1 \dots E_k$ are linear operator on V which statisfies condition i) (ii) & (iii).

and if we let W_i be the range of E_i

then $V = W_1 \oplus \cdots \oplus W_k$.

proof:

Necessary condition:

Given that,

$$V = W_1 \oplus \cdots \oplus W_k$$

Let $\alpha \in V$ then,

$$\alpha = \alpha_1 + \cdots + \alpha_k, \ \alpha_i \in W_i$$

For each j, $1 \le j \le k$.

defined $E_j(\alpha) = \alpha_j \ \forall \alpha \in V$.

here, $E_i E_j$ are linear operator given that definition of range (E_j) is W_j

consider.

i) $E_i^2(\alpha) = E_i(E_i(\alpha))$ $\alpha = \alpha_1 + \cdots + \alpha_k$

$$= E_i(\alpha_i)$$

$$= E_i(0 + 0 + \cdots + \alpha_i + \cdots + 0)$$

$$E_i^2(\alpha) = E_i(\alpha)$$

$$E_i^2 = E_i$$

So, E_i is

ii) To prove that,

$$E_i E_j = 0 \text{ if } i \neq j$$

Consider, $(E_i E_j)(\alpha) = E_i(E_j \alpha)$

$= E_i(\alpha_j)$

$= E_i(0 + \dots + \alpha_j + \dots 0)$

$E_i E_j(\alpha) = \begin{cases} \alpha_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$E_i E_j = 0$ if $i \neq j$

$E_i E_j = 0$ if $i \neq j$.

iii) To prove that. $I = E_1 + \dots + E_k$

let $\alpha \in V$ then,

$\alpha = \alpha_1 + \dots + \alpha_k$

$\Rightarrow I(\alpha) = E_1\alpha + \dots + E_k\alpha \; \forall \alpha \in V.$

$\Rightarrow I = E_1 + E_2 + \dots + E_k$

iv). To prove that. The range of E_i is W_i.

By definition of $E_i E_j(\alpha) = \alpha_j \; \forall \alpha \in V$

$\Rightarrow E_j$ is well defined

E_j is linear.

The range of E_j is W_j.

Sufficient part:

To prove: $V = W_1 \oplus \dots \oplus W_k$

i.e. to prove that.

i) $V = W_1 + W_2 + \dots + W_k$.

ii) $\alpha = \alpha_1 + \dots + \alpha_k$ is unique $\alpha \in V, \alpha_i \in W_i$

$1 \leq i \leq k$.

suppose $E_1, E_2 \dots E_k$ are linear operator on V which satisfy the condition (i)(ii)(iii) given

From (i)(ii)(iii) we have.

$V = W_1 + \cdots + W_k \rightarrow ①$

Next to prove that:

$\alpha \in V, \ \alpha_i \in W_i$

$\alpha = \alpha_1 + \cdots + \alpha_n$ is unique.

By condition (iii)

$$I = E_1 + \cdots + E_k$$

$$\Rightarrow I(\alpha) = (E_1 + \cdots + E_k)\alpha$$

$$\alpha = E_1\alpha + \cdots + E_k\alpha \rightarrow ②$$

if $\alpha = \alpha_1 + \cdots + \alpha_k$ with $\alpha_i \in W_i \rightarrow ③$

since W_i is range of E_i and $\alpha_i \in W_i$

$$\alpha_i = E_i\beta_i \rightarrow ④$$

consider, $E_j \alpha = E_j(\alpha_1 + \cdots + \alpha_n)$

$$= \sum_{i=1}^{k} E_j \alpha_i$$

$$= \sum_{i=1}^{k} E_j(E_i\beta_i) \text{ by } (4)$$

$$= \sum_{i=1}^{k} E_j E_i \beta_i$$

$$= E_j E_j \beta_j + \sum_{i=1}^{k} E_j E_i \beta_i \ (i \neq j)$$

$$= E_j^2 \beta_j + 0$$

$E_j\alpha = E_j\beta_j$ (by i) $E_i^2 = E_i$

$E_j\alpha = \alpha_j \rightarrow ⑤.$

use (5) in (2)

$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$

so, $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k \quad \forall\, \alpha \in V \rightarrow (6)\ \alpha_i \in W_i$ and $1 \le i \le k$

is unique.

From (5) & (6)

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$

Invarient Direct sum :

Thm : A

let T be a linear operator on the space V. and let $W_1 \ldots W_k$ and $E_1 \ldots E_k$ be as in condition . A Then a necessary and sufficient condition that each subspace W_i be invarient under T is that T commutate with each of the projections E_i.

i.e $TE_i = E_i T$, $i = 1, \ldots k$.

proof :

Sufficient part :

suppose T commutates with each E_i

i.e $T.E_i = E_i T$.

Let $\alpha \in W_j$ then $\alpha = E_j \alpha$ (W_j is rang of E_j)

$T\alpha = T(E_j \alpha)$

$= E_j (T\alpha)$

$T\alpha \in$ Range of E_j

$T\alpha \in W_j$

$T(W_j) \subseteq W_j$

Thus, $\alpha \in W_j \Rightarrow T\alpha \in W_j$.

W_j is invariant under T.

Necessary part:

Assume that each W_i is invarient under T.

To prove that: $TE_j = E_jT$.

let $\alpha \in V$.

Then, $\alpha = E_1\alpha + \cdots\cdots + E_k\alpha$.

$$T\alpha = T(E_1\alpha + \cdots\cdots + E_k\alpha)$$

$$T\alpha = TE_1\alpha + \cdots\cdots + TE_k\alpha \rightarrow (1)$$

Since $E_i\alpha \in W_i$ which is invariant under T

$$\Rightarrow T(E_i\alpha) \in W_i$$

Since W_i is the range of E_i

$$T(E_i\alpha) = E_i\beta_i \rightarrow (2) \text{ for some } \beta_i.$$

Then,

$$E_j(TE_i\alpha) = E_j(E_i\beta_i) \quad \text{by } (2)$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ E_j\beta_j & \text{if } i = j \end{cases}$$

consider,

$$E_j(T\alpha) = E_j(TE_1\alpha + \cdots + TE_k\alpha)$$

$$= E_jTE_1\alpha + \cdots + E_jTE_k\alpha$$

$$= E_jE_1\beta_1 + \cdots + E_jE_k\beta_k \quad \text{by } (2)$$

$$= E_jE_j\beta_j \quad (E_j^2 = E_j.$$

$$E_j(T\alpha) = T(E_j\alpha)$$

$$TE_i = TE_i$$

H/P.

Theorem: 5

Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1, c_2 \ldots c_k$ are the distinct characteristic values of T, Then there exists linear operators $E_1 \ldots E_k$ on V such that,

i) $T = c_1E_1 + \ldots + c_kE_k$

ii) $I = E_1 + \ldots + E_k$

iii) $E_iE_j = 0, \quad i \neq j$

iv). $E_i^2 = E_i$ (E_i is a projection).

v). The range of E_i is the characteristic space for T associated with c_i.

Conversely, $\exists$ k distinct scalars $c_1 \ldots c_k$ and k non zero linear operators $E_1 \ldots E_k$ which satisfy condition (i), (ii), (iii). Then T is diagonazable $c_1 \ldots c_k$ are the distinct characteristic values of T and condition (iv) and (v) are satisfied also.

proof:

Necessary part:

Suppose that T is diagonalizable which distinct characteristic values. $c_1, c_2 \ldots c_k$.

Let W_i be the space of characteristic vectors associated with the characteristic values c_i.

ie. $W_i = \{\alpha_i \mid T\alpha_i = c_i\alpha_i\}$.

Then $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$.

Let $\alpha \in V$, then,

$$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k, \; \alpha_i \in W_i$$

Defined $E_j(\alpha) = \alpha_j \quad \forall \alpha \in V$.

To prove (i): $T = c_1 E_1 + \dots + c_k E_k$ for each $\alpha \in V$.

$$\alpha = \alpha_1 + \dots + \alpha_k, \; \alpha_i \in W_i$$

$$T\alpha = T\alpha + \dots + T\alpha_k \qquad \hookrightarrow T\alpha_i = c_i \alpha_i$$

$$= c_1 \alpha_1 + \dots + c_k \alpha_k$$

$$T\alpha = c_1 E_1 \alpha + \dots + c_k E_k \alpha.$$

$$= (c_1 E_1 + \dots + c_k E_k) \alpha$$

$$T = c_1 E_1 + \dots + c_k E_k.$$

To prove (ii): $I = E_1 + \dots + E_k$.

Let $\alpha \in V$ then,

$$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$$

$$\Rightarrow (I)\alpha = E_1 \alpha + E_2 \alpha + \dots + E_k \alpha \; \forall \alpha \in V.$$

$$\Rightarrow I = E_1 + E_2 + \dots + E_k.$$

To prove (iii): $E_i E_j = 0, \; i \neq j$

Consider, $(E_i E_j)\alpha = E_i(E_j \alpha)$

$$= E_i \alpha_j$$

$$E_i E_j(\alpha) = E_i(0 + 0 \dots + \alpha_j + \dots + 0)$$

$$E_i E_j(\alpha) = \begin{cases} \alpha_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$E_i E_j = 0 \text{ if } i \neq j.$$

To poove (iv). $E_i^2 = E_i$

consider,

$$E_i^2(\alpha) = E_i(E_i(\alpha))$$

$$\alpha = \alpha_1 + \dots + \alpha_k$$

$$E_i^2(\alpha) = E_i(E_i(\alpha))$$

$$\alpha = \alpha_1 + \dots + \alpha_k$$

$$E_i^2(\alpha) = E_i(\alpha_i)$$

$$= E_i(0 + \dots \alpha_i + \dots + 0)$$

$$= E_i(\alpha)$$

$$E_i^2 = E_i$$

So, E_j is poojection.

To poove : (v)

$R(E_i)$ = chaoactesistic space of T associated with c_i.

By definition of range,

$$R(E_i) = \{E_i(\alpha); \alpha \in V\}$$

$$= \{\alpha_i ; \alpha_i \in V\}$$

Since $\alpha_i \in W_i$

Hence, $R(E_i) = W_i \rightarrow ①$

chaoactesistic space of T associated with c_i $= \{\alpha \in V;\ T(\alpha) = c_i\alpha$

Since $T(\alpha_i) = c_i\alpha_i$

characteristic space of T associated with,

$$c_i = \{\alpha_i \in V_i ; T(\alpha_i) = c_i \alpha_i\}$$

[by def $W_i = k(E_i)$]

$$= W_i \rightarrow ②$$

$R[E_i]$ = characteristic space of T associated with c_i.

sufficient part:

Given that a linear operator T along with distinct. scalars c_i and non-zero operators E_i which satisfy,

$$\Rightarrow T = c_1 E_1 + \cdots + c_k E_k$$

$$\Rightarrow I = E_1 + \cdots + E_k$$

$$\Rightarrow E_i E_j = 0 \quad ; i \neq j$$

To prove: $E_i^2 = E_i$

[E_i is projection]

by (ii) $I = E_1 + \cdots + E_k$

multiply by E_i on both sides.

$$E_i = (E_1 + \cdots + E_k) E_i$$

$$= E_1 E_i + \cdots + E_i E_i + \cdots + E_k E_i$$

$$= 0 + \cdots + 0 + E_i^2 + \cdots + 0$$

$$E_i = E_i^2$$

To prove that:

T is diagonalizable $c_1, c_2 \ldots c_k$ are distinct characteristic values of T and the condition (v). satisfied also.

by (i).

$$T = c_1E_1 + \cdots + c_kE_k$$

multiply by E_i.

$$TE_i = (c_1E_1 + \cdots + c_kE_k)E_i$$

$$= 0 + c_iE_iE_i + 0 \quad [E_iE_j = 0 \quad i \neq j]$$

$$= c_iE_i^2$$

$$TE_i = c_iE_i \quad [E_i^2 = E_i]$$

$$(T - c_i)E_i = 0.$$

$\Rightarrow$ any vector in the range of $E_i \in$ null space of $(T - c_iI)$.

since, we assumed that

$E_i \neq 0$

$\Rightarrow \exists \; 0 \neq \beta$ in null space of $(T - c_iI)$.

$\Rightarrow \exists \; 0 \neq \beta$ such that $(T - c_iI)\beta = 0$

$\Rightarrow c_i$ is a characteristis values of T.

so c_i are all of the characteristic values of T.

Then $(T - cI) = (c_1E_1 + \cdots + c_kE_k) - c(E_1 + \cdots + E_k)$

$$(T - cI) = (c_1 - c)E_1 + \cdots + (c_k - c)E_k \rightarrow (3)$$

(by (i), (ii)).

I.Q. 10m Thrm : 6

Primary Decomposition theorem.

Statement:

Let T be a linear operator on the finite dimensional vector space V over the field F. Let P be the minimal polynomial for T.

$$P = P_1^{r_1}, \dots P_k^{r_k}$$

where the P_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of

$$P_i(T)^{r_i}, \quad i = 1, 2 \dots k. \text{ Then,}$$

$\Rightarrow V = W_1 \oplus W_2 \oplus \dots \oplus W_k$

$\Rightarrow$ Each W_i is invariant under T

$\Rightarrow$ If T_i is the operator induced by T. then the minimal polynomial T_i is $P_i^{r_i}$.

Proof:

i) To prove that

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_k$$

Let $E_1, E_2 \dots E_k$ be linear operator on V defined by.

$E_i(\alpha) = \alpha_i \quad 1 \le i \le k$

It is enough to show that

i) $E_1 + E_2 + \cdots + E_k = I$

ii) $E_i E_j = 0$ for $i \ne j$

iii) Range of E_i is W_i

iv) $E_i^2 = E_i$ for each i

$$f_i = \frac{p}{p_i^{r_i}} \longrightarrow (1)$$

$$= \prod_{j \ne i} p_j^{r_j}$$

Since $p_1, p_2, \ldots p_k$ are distinct prime polynomial

Then $f_1, f_2, \ldots f_k$ are relatively prime

$\Rightarrow (f_1, f_2, \ldots f_k) = 1$

There exist $g_1, g_2, \ldots g_k \in F[x]$ such that

$$f_1 g_1 + f_2 g_2 + \cdots + f_k g_k = 1$$

ie) $\sum_{i=1}^{k} f_i g_i = 1 \longrightarrow (2)$

Note also that if $i \ne j$

Then $f_i f_j$ is divisible by the polynomial p.

Because $f_i f_j$ contains each $p_m^{r_m}$ as $\rightarrow (3)$ a factor.

Let $E_i = h_i(T) = f_i(T)\, g_i(T)$

from 2 =>

$$p \mid f_i f_j \; ; \; i \neq j$$

$p(x)$

$$\sum_{i=1}^{k} h_i = 1$$

$\Rightarrow h_1 + h_2 + \cdots + h_k = 1$

$\Rightarrow h_1(T) + h_2(T) + \cdots + h_k(T) = I$

$\Rightarrow E_1 + E_2 + \cdots + E_k = I \longrightarrow$ (I)

from (3)

$p \mid f_i f_j \; ; \; i \neq j$

$p(T) \mid f_i(T) f_j(T) \quad i \neq j$

Since $p(T) = 0$

$\Rightarrow f_i(T) \cdot f_j(T) = 0 \quad i \neq j$

$\Rightarrow (f_i(T) \cdot g_i(T)) \cdot (f_j(T) g_j(T)) = 0 \times g_i(T)$

$\Rightarrow h_i(T) \, h_j(T) = 0 \quad i \neq j$

$\Rightarrow E_i E_j = 0 \quad i \neq j \longrightarrow$ (II)

To prove that.

Range of E_i is W_i

let $\alpha \in$ Range of E_i then $\alpha = E_i \alpha$

$\Rightarrow p_i(T)^{r_i}(\alpha) = p_i(T)^{r_i}(E_i \alpha)$

$= p_i(T)^{r_i}\, h_i(T)(\alpha)$

$= p_i(T)^{r_i}\, f_i(T)\, g_i(T)\, \alpha$

$= 0$

beacause $p \mid p_i^{r_i} f_i g_i$

$\Rightarrow \alpha \in$ null space of $p_i(T)^{r_i}$

$\Rightarrow \alpha \in W_i$

Thus $\alpha \in$ Range of E_i

$\Rightarrow \alpha \in W_i$

$\therefore$ Range of $E_i \subseteq W_i$

If $\alpha \in W_i$ then $\alpha \in$ null space of $p_i(T)^{r_i}$

$\Rightarrow (p_i(T)^{r_i})\alpha = 0$

If $i \neq j$ then $p_i^{r_i} / f_i g_i$

since $p_i(T)^{r_i} = 0$

$\Rightarrow f_i(T)\, g_i(T)\, \alpha = 0$

$h_i(T)\, \alpha = 0$

$E_i \alpha = 0 \quad \alpha \in E_i \alpha$

$\Rightarrow \alpha \in$ Range of E_i

Thus $\alpha \in W_i \Rightarrow \alpha \in$ range of E_i

$W_i \subseteq$ range of E_i

Thus $W_i =$ Range of E_i .... (III)

consider,

$$E_i^2\alpha = E_i(E_i\alpha)$$
$$= E_i(\alpha_i)$$
$$= \alpha_i$$
$$E_i^2\alpha = E_i(\alpha)$$
$$E_i^2 = E_i \quad \therefore \quad \text{IV}$$

From I, II, III, IV By thm (5)

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_k$$

ii) To prove that.

Each w_i is invarient under T

let $x_i \in w_i$ then $p_i(T)^{r_i} x_i = 0$

$$\Rightarrow T(p_i(T)^{r_i} x_i) = T(0)$$
$$\Rightarrow p_i(T)^{r_i}\, T(x_i) = 0$$
$$\Rightarrow T(x_i) \in w_i$$

Thus $x \in w_i \Rightarrow T(x_i) \in w_i$

$$T(w_i) \subseteq w_i$$

So, T is invariant under T.

iii) If T_i is the operator induced on w_i by T as, Then prove that.

minimal polynomial for T_i is $p_i^{r_i}$

By hypothesis $p_i(T)^{r_i} = 0$

Because by definition $p_i(T)^{r_i}$ is 0 on w_i

$\Rightarrow$ minimal polynomial for $T_i \mid p_i^{r_i}$

Conversely. g be any polynomial such $g(T_i) = 0$

Then $g(T) f_i(T) = 0$, $g(f_i)T = 0$

minimal polynomial p of T divides $(g f_i)$

i.e. $p_i^{r_i} f_i$ divides $g f_i$

Hence, The minimal for T_i is $p_i^{r_i}$

UNIT-V

7 ⇒ the Rational and Jordon Forms

7.1 ⇒ cyclic subspaces and annihilator

7.2 ⇒ cyclic decomposition and rotational form.

7.3 ⇒ The Jordan form

7.4 ⇒ computation of invariant factor.

cyclic vector (or) cyclic subspace

If α is any vector in V, the T-cyclic subspace generated by α is the subspace $Z(\alpha;T)$ of all vectors of the forms $g(T)\alpha$, g is $F[x]$. If $Z(\alpha;T)=V$, Then α is called a cyclic Vector for T.

Note:

The subspace $Z(\alpha;T)$ is that $Z(\alpha;T)$ is the subspace spanned by the vectors $T^k\alpha$, $k \geq 0$ and thus α is n cyclic vector for T iff vectors span V.

T-annihilator.

If α is any vector in V, the T-annihilator of α is the ideal $M(\alpha;T)$ in $F[x]$ consisting of all polynomial's g over F such that $g(T)\alpha=0$. The unique monic polynomial p_α which generates this ideal will also be called the T-annihilator of α.

complementary:

If W is any subspace of a finite dimensional space V, then there exists a subspace W' such that,

$$V = W \oplus W'$$

we say that W' is complementary to W.

Admissible:

Let T be a linear operator on a vector space V. and let W be a subspace of V. we say that W is T-admisible if

=> W is invariant under T.

=> If $f(T)\beta$ is in W there exist a vector γ in W such that,

$$f(T)\beta = f(T)\gamma.$$

Theorem:1

* Let α be any non-zero vector in V and let p_α be the T-annihilator of α.

i) The degree of p_α is equal to the dimension of the cyclic subspaces $Z(\alpha;T)$

ii) If the degree of p_α is k, then the vectors $\alpha, T\alpha, T^2\alpha, \dots T^{k-1}\alpha$ form a basis for $Z(\alpha; T)$

iii) If U is the linear operator on $Z(\alpha; T)$ induced by T, Then the minimal polynomial for U is p_α.

proof:

Let g be any polynomial over the field F.

since, p_α divides the minimal for T.

$\Rightarrow g$ is divisible by p_α.

Then by division algorithm.

$\exists$ $q, r \in F[x]$ such that,

$$g = p_\alpha q + r$$

where, either $r = 0$ (or) $\deg r < \deg(p_\alpha) = k$.

$$\therefore g(T)\alpha = [p_\alpha(T) q(T)]\alpha + r(T)\alpha \longrightarrow (1)$$

where either $r = 0$ (or) $\deg r < k$.

since, $p_\alpha q$ is in T- annihilator of α.

ie). $p_\alpha(T) q(T)\alpha = 0 \longrightarrow (2)$

from (1) & (2)

$$g(T)\alpha = r(T)\alpha.$$

where either $r = 0$ (or) $\deg(r) < k$.

since, $g(T)\alpha = (r(T)\alpha)$ is linear combination of the vectors $\alpha, T\alpha, \dots T^{k-1}\alpha$.

since $g(T)\alpha \in Z(\alpha, T)$

$Z(\alpha;T)$ generated by $\alpha, T\alpha, \dots T^{k-1}\alpha$.

$\alpha, T\alpha, \dots T^{k-1}\alpha$ spans $Z(\alpha;T)$

$\alpha, T\alpha, \dots T^{k-1}\alpha$ are linearly independent, because any non-trivial linear relation b/w them would given us a.

$g \neq 0$ in $F[x]$ such that,

$g(T)\alpha = 0$ and $\underline{\deg(g) < \deg(p_\alpha)}$

which is absurd.

This process (i) & (ii)

To prove that: (iii)

Let U be the linear operator on $Z(\alpha;T)$ by restricting T to that subspace.

If $g \in F[x]$ then

$$p_\alpha(U)g(T)\alpha = p_\alpha(T)g(T)\alpha$$
$$= g(T)p_\alpha(T)\alpha$$
$$= g(T)0$$
$$= 0$$

Thus the operator $p_\alpha(U)$ send every vector in $Z(\alpha;T)$ into '0'.

$\therefore p_\alpha(U)$ is the zero operator over $Z(\alpha;T)$.

Furthermore if h is a polynomial of degree less than k. We cannot have $h(U) = 0$.

for then $h(V)\alpha = h(T)\alpha = 0$

which is contradiction to the definition of p_α.

$\therefore P_\alpha$ is minimal polynomial for V.

corollary: 1

Q. If T is a linear operator on a finite dimensional vector space, then every T-admissible subspace has a complementary subspace which is also invariant under T.

proof:

Necessary part:

Let W be a T-admissible subspace of V.

If $W = V$ then $W' = (0)$ is a complementary subspace of V and is also invariant.

If $W \neq V$ then W be a proper subspace of V.

Then by "cyclic decomposition theorem" $\exists$ non-zero vectors $\alpha_1 \cdots \alpha_r$ such that

$V = W \oplus Z(\alpha_1; T) \oplus Z(\alpha_2; T) \oplus \cdots \oplus Z(\alpha_r; T)$

let $W' = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$

The clearly W' is T-invariant subspace of V and $V = W \oplus W'$

Thus, w has complementary T-invariant subspace.

~~Sufficient part~~:

Given that W has a complementary T-invariant subspace W'.

Then, $V = W \oplus W'$

Suppose $f(T)\beta \in W$ as $\beta \in V$.

$\therefore \beta$ can be uniquely expressed as

$$\beta = \gamma + \gamma'$$

where, $\gamma \in W$ & $\gamma' \in W'$

Then, $f(T)\beta = f(T)\gamma + f(T)\gamma'$

$$f(T)\gamma' = f(T)\beta - f(T)\gamma \rightarrow (1).$$

$f(T)\beta, f(T)\gamma \in W$

$$\therefore f(T)\gamma' \in W.$$

Since W' is T-invariant subspace and $\gamma' \in W'$

$$\therefore f(T)\gamma' \in W'$$

Thus, $f(T)\gamma' \in W \cap W'$

since, $W \cap W' = (0)$

$$f(T)\gamma' = 0 \rightarrow (2)$$

using (2) in (1)

$$f(T)\beta = f(T)\gamma \rightarrow (3) \text{ where } \gamma \in W$$

Since, W is a T-invariant subspace of $V \rightarrow (4)$

from (3) & (4).

$\therefore W$ is T-Admissible.

Corollary: 2

Let T be a linear operator on a finite-dimensional v.s V. then exists a vector α in V such that the T-annihilator of α is the minimal polynomial for T.

proof:

If $V = \langle 0 \rangle$ then the result is trivially true.

If $V \neq \langle 0 \rangle$. since $W_0 = (0)$ is a proper T-admissible subspace of V.

$\therefore$ By cyclic decomposition theorem.

a non-zero vectors $\alpha_1, \alpha_2 \ldots \alpha_r$ with respect to T-annihilator $p_1, p_2 \ldots p_r$ such that,

$$V = W_0 \oplus Z(\alpha_1; T) \oplus \ldots \oplus Z(\alpha_r; T)$$

$$V = Z(\alpha_1; T) \oplus Z(\alpha_2; T) \oplus \ldots \oplus Z(\alpha_r; T)$$

and $p_{k+1} / p_k, 1 \le k \le r-1$

Now, claim that p_1 is the minimal polynomial for T.

since, $p_{k+1} / p_k, 1 \le k \le r-1$

$\Rightarrow p_k / p_1$ for each k.

Hence, $p_1 = h_k p_k$ for each k $(2 \le k \le r)$

consider any vector $\beta \in V$.

Then β can be uniquely expressed as

$$\beta = f_1(T)\alpha_1 + f_2(T)\alpha_2 + \dots + f_r(T)\alpha_r$$

Then, $p_1(T)\beta = p_1(T)f_1(T)\alpha_1 + p_1(T)f_2(T)\alpha_2 + \dots + p_1(T)f_r(T)\alpha_r$

$p = h_k p_k$

$$= p_1(T)h_1(T)f_1(T)\alpha_1 + h_2(T)p_2(T)f_2(T)\alpha_2 + \dots + h_r(T)p_r(T)f_r(T)\alpha_r$$

$\because p_i(T)\alpha_i = 0 \ \forall i, 1 \le i \le r$

$p_1(T)\beta = 0$

Since $\beta \in V$

$\Rightarrow p_1(T) = 0$ in V.

Now consider any polynomial

$g \in F[x]$ such that $g(T) = 0$ on V.

$\Rightarrow g(T)\alpha_1 = 0$

since, p_1 is the T-annihilator of α_1

$\therefore p_1$ divides g.

Thus p_1 is the minimal polynomial for T.

Hence,

We have shown the existence of a vector $\alpha = (\alpha_1)$ such that the T-annihilator of α is the minimal polynomial for T.

coollary: 3.

T has a cyclic vector iff the characteristic and minimal polynomial for T are identical.

proof:

Necessary part:

If T has a cyclic vector then characteristic polynomial and minimal polynomial for T are identical.

Sufficient part:

Suppose the characteristic polynomial and minimal polynomial are identical.

From coollary (2) $\exists$ a vector α in V.

such that T- annihilator p_α of α is the minimal polynomial for T.

$\therefore$ By hypothesis, p_α is the characteristic polynomial for T.

And hence, $\deg(p_\alpha) = \dim V$.

Then from theorem (1)

It follows that,

$$\dim Z(\alpha; T) = \deg(p_\alpha) = \dim V.$$

since, $Z(\alpha; T) \leq V$

$\therefore Z(\alpha; T)$ and α is the cyclic vector for T.

Them : 4

(7) Generalized cayley - Hamilton theorem.

Statement :

Let T be a linear operator on a finite dimensional vector space V. Let p and f be the minimal and characteristic polynomial for T, respectively.

$\Rightarrow$ p divides f

$\Rightarrow$ p and f have the same prime factor. expect for multiplicities,

$\Rightarrow$ If $p = f_1^{r_1}, \ldots f_k^{r_k}$ is the prime factorial of p than

$f = f_1^{d_1} \ldots f_k^{d_k}$

where, d_i is the nullity of $f_i(T)^n$ divided of f_i.

proof :

i) If $V = \{0\}$ then the result hold trivially.

If $V \neq \{0\}$ since $W_0 = (0)$ be a proper T- admissible subspace of V.

$\therefore$ By cyclic decomposition theorem.

$\exists$ a non-zero vectors $\alpha_1, \alpha_2 \ldots \alpha_n$ with respect T annhilator $P_1, P_2 \ldots P_n$ Such that,

$$V = W_0 \oplus Z(\alpha_1; T) \oplus \ldots \oplus Z(\alpha_r; T).$$

$V = Z(\alpha_1;T) \oplus \cdots \oplus Z(\alpha_r;T)$

and $p_{k+1} | p_k$ $1 \le k \le r-1$ → by corollary : 2.

Also p_i is the minimal polynomial for T.

i.e $p_1 = p$

Let U_i be the linear operators on $Z(\alpha_i;T)$

obtaining by restricting T to $Z(\alpha_i;T)$.

Then clearly (by thm 1)

p_i is both the minimal polynomial and the

characteristic polynomial f for T is the

product.

$f = p_1 p_2 \cdots p_r$.

It is evident from the block matrix form T

with respect to a suitable basis B.

whose each block on the diagonal is the

matrix of U_i corresponding to subspaces

$Z(\alpha_i;T)$.

clearly $(p = p_1)$ divedes f

Thus (i) holds.

ii) P.T: p and f have the same prime

factors except for multiplicities.

since $p | f$.

Any prime factor of p is also a prime factor of f.

Now any prime divison of f is a prime divison for some p_i and hence in turn a prime division of p (as each $p_k | p_1$).

Thus, p and f have the same prime factor's except for multiplicities.

iv. To prove that:

If $p = f_1^{r_1} \cdots f_k^{r_k}$ is the prime factorization of p then $f = f_1^{d_1}, \cdots f_k^{d_k}$.

where d_i is the ~~multiply~~ nullity of $f_i(T)^n$ divided of f_i.

Let $p = f_1^{r_1} \cdots f_k^{r_k}$ be the prime factorization for p.

Using the primary decomposition theorem, If V_i is the null space of $f_i(T)^{r_i}$

Then, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ and $f_i^{r_i}$ is the minimal polynomial for T_i.

where T_i is the operator obtained by restricting T to the T-invariant subspace V_i.

From part (ii) of this theorem.

The characteristic polynomial for T_i and the minimal polynomial for T_i have the same prime factor's.

Since f_i is the only prime division of the minimal polynomial for T_i.

The characteristic polynomial for T_i is of the form $f_i^{d_i}$, where $d_i \geq r_i$

As degree of characteristic polynomial for $T_i = \dim V_i$.

$$d_i = \frac{\dim V_i}{\deg f_i} = \frac{\text{nullity of } f_i(T)^{r_i}}{\deg f_i}$$

Now, since T is the direct sum of the operators $T_1, T_2 \ldots T_K$ the characteristic polynomial f_i of T is the product.

$$f = f_1^{d_1} \ldots f_K^{d_K}.$$

$\therefore$ H/P.

Corollary:

If T is a nilpotent linear operator on a vector space of dimension n, then the characteristic polynomial for T is x^n.

Proof:

Since T is nilpotent operator.

So, $T^k = 0$ for some $k \in N$

Let p be the minimal polynomial for T and f be the characteristic polynomial for T

Then obviously

$$p / x^k.$$

Hence, $p = x^i$ for some $i < n$. by generalized cayley-Hamilton theorem,

p and f have the same prime factors except for multiplicities.

Thus, $f = x^n$

$\therefore$ H/P

Rational form:

An $n \times n$ matrix A is said to be in rational form if there exist non-scalar monic polynomial $p_1, p_2 \ldots p_r$. Such that,

$p_{i+1} | p_i$ for each i $(1 \leq i \leq r-1)$ and A can be expressed as,

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & A_r \end{bmatrix}$$

where $A_1, A_2 \ldots A_r$ are companion matrix of polynomial $p_1, p_2 \ldots p_r$ respectively.

These A is said to be in rational form. If it can be written as a direct sum of companion matrix of the polynomial $p_1 \cdots p_r$. companion matrix of p_α.

Given a polynomial.

$$p_\alpha(x) = c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} + x^k \in F[x]$$

with $c_i \in F$. The matrix.

$$\begin{bmatrix} 0 & 0 & 0 \cdots 0 & -c_0 \\ 1 & 0 & 0 \cdots 0 & -c_1 \\ 0 & 1 & 0 \cdots 0 & -c_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 & -c_{k-1} \end{bmatrix}$$

The matrix is called companion matrix of the monic polynomial p_α.

Thm: 5

Let F be a field and let B be an $n \times n$ matrices over F. Then B is similar over the field F to one and only one matrix which is in rational form.

proof:

Consider the linear operator T on F^n over F such that the matrix of T with respect to standard ordered basis of F^n is B.

Then $\exists$ an ordered basis $\mathcal{B}$ of F^n such that the matrix A of T with r. to the basis $\mathcal{B}$ is in rational form.

i.e. $A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & A_r \end{bmatrix}$

where $A_1, A_2 \cdots A_r$ are companion matrix of polynomial $p_1 \cdots p_r$ respectively,

moreover, polynomials $p_1, p_2 \cdots p_r$ are T-annihilators of some non-zero vectors $\alpha_1, \alpha_2 \cdots \alpha_r$ respectively.

Then obviously B is similar to A.

To prove: uniqueness.

Suppose B is similar to another matrix C (say) which is rational form.

Since C is similar to B and B represented the matrix of T in the standard basis. ∃ an ordered basis D of F^n over F such that,

C is the matrix of T with respect to the ordered basis D.

Now,

Since C is an rotational form ∃ non-scalar monic polynomial $g_1, g_2 \cdots g_K$ such that,

$g_{i+1} | g_i$ for each i $(1 \le i \le k-1)$ and C can be expressed as,

$$C = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ 0 & 0 & C_3 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 & C_K \end{bmatrix}$$

where $C_1, C_2 \ldots C_K$ are companion matrix of polynomial $g_1, g_2 \ldots g_K$ respectively.

Now, for each i $(1 \le i \le K-1)$ as C_i is a companion matrix of g_i.

By the theorem and corollary.

$\exists$ a non-zero vector β_i in V with T-annihilator g_i such that,

$$V = Z(\beta_1; T) \oplus Z(\beta_2; T) \oplus \cdots \oplus Z(\beta_K; T)$$

Then by the uniqueness of the cyclic decomposition theorem,

It follows that,

$r = k$ and $g_i = p_i$ $\forall$ $i = 1, 2, \ldots r$

consequently we have

$\therefore C = A$

H/P.

Equivalent:

The matrix N is equivalent to the matrix M if we can pass from M to N by means of a sequence of operations

$$M = M_0 \to M_1 \to \cdots \to M_k = N$$

Each of which is an elementary row operation or an elementary column operation.

Normal form.

Let N be a matrix in $F[x]^{m \times n}$. We say that N is in (Smith) normal form if,

a) every entry of the main diagonal of N is 0.

b) on the main diagonal of N there appear polynomials $f_1, f_2 \ldots f_l$ such that f_k divides f_{k+1}. $1 \leq k \leq l-1$

The number lies $l = \min(m,n)$

The main diagonal entries are,

$$f_k = N_{kk},\ k = 1, 2 \ldots l.$$

cyclic Decomposition theorem.

statement:

Let T be a linear operator on a finite dimensional vector space V and let W_0 be a proper T-admissiable subspace of V. $\exists$ non-zero vectors $\alpha_1, \alpha_2 \dots \alpha_r$ in V with respectively T annihilator $p_1 p_2 \dots p_r$ such that,

i) $V = W_0 \oplus Z(\alpha_1; T) \oplus \dots \oplus Z(\alpha_r; T)$

ii) p_k divides p_{k-1}; $k = 1, 2 \dots r$

Further more, the integer r and the annihilator $p_1 p_2 \dots p_r$ are uniquely determined by (i), (ii) and the factor that no α_k is zero.

proof:

The proof is uniquely determined into four steps. thought the proof it seems easy to take $W_0 = \{0\}$ and $f(T)\beta$ to $f\beta$.

Step: 1

There exists non-zero vectors $\beta_1, \beta_2 \dots \beta_r$ in V such that.

a) $V = W_0 + Z(\beta_1; T) + \dots + Z(\beta_r; T)$

b) If $1 \le k \le r$ and $W_k = W_0 + Z(\beta_1; T) + \dots + Z(\beta_r; T)$

then the conductor $p_k = S(\beta_k; W_{k-1})$ has maximum degree among along all T-conductors into the subspace of W_{K-1}

ie) for every K.

$$\deg P_K = \max_{\alpha \text{ in } V} \deg S(\alpha; W_{K-1})$$

since, can choos W_0 is an invariant subspace.

If W is a proper T- invarient subspace.

Then $0 < \max_{\alpha} \deg S(\alpha; W) \le \dim V.$

we can choose a vector β so that $\deg S(\beta; W)$ attains that maximum.

The subspace $W + Z(\beta; T)$ is then T invarient and has dimensional large then dim W.

$\therefore \dim W + Z(\beta; T) > \dim W$

Apply this process to $W_1 = W_0$ to obtain β_1. If $W_1 = W_0 + Z(\beta_1; T)$. Then apply the process to W_1 to obtain β_2

continue in same manner

$\dim W > \dim W_0 +$

since, $\dim W_k > \dim W_{k-1}$. We must reach $W_r = V$ and not more than dim r steps.

step : 2

let $\beta_1, \beta_2 \dots \beta_r$ be a non-zero vector which satisfy the conditions (a) and (b) of step (1)

fix k, $1 \le k \le r$. Let β be any vector in V and let $f = s(\beta, W_{k-1})$ if

$$f\beta = \beta_0 + \sum_{1 \le i \le k} g_i \beta_i \ ; \ \beta_i \text{ in } W_i$$

then f divides each polynomial g_i and $\beta_0 = f\gamma_0$. where γ_0 is in W_0.

If $k = 1 \rightarrow W_0$ is T- admissible.

if $k > 1$ apply the division algorithm.

$$g_i = f h_i + r_i \longrightarrow ①$$

and $r_i = 0$ (or) $\deg r_i < \deg f$.

We wish to show that $r_i = 0$ for each i.

Let $$\gamma = \beta - \sum_{i=1}^{k-1} h_i \beta_i \longrightarrow ②$$

since, $\gamma - \beta \in W_{k-1}$

$$s(\gamma ; W_{k-1}) = s(\beta ; W_{k-1}) = f$$

Further more, $$f\gamma = \beta_0 + \sum_{i=1}^{k-1} r_i \beta_i \rightarrow ③$$

γ_i is zero. If γ_i is different from 0

Then we get $\Rightarrow\Leftarrow$

Let j be the largest index i for which $\gamma_j \neq 0$.

Then $f\gamma = \beta_0 + \sum_i \gamma_i \beta_i : \gamma_j \neq 0 \rightarrow (4)$

and $\deg \gamma_i < \deg f$.

Let $p = s(\gamma; W_{j-1})$

Since, W_{K-1} contains W_{j-1} the conductor

$f = s(\gamma; W_{K-1})$ must divide p.

$$p = fg$$

Apply $g(T)$ to the both sides of eqn (4) - 6

$$p\gamma = gf\gamma = g\gamma_j\beta_j + g\beta_0 + \sum_{1 \le i \le j} g\gamma_i\beta_i \rightarrow (6)$$

$$g\gamma_j\beta_j \le W_{j-1}$$

Now, use (2) of step (1)

$$\deg(g\gamma_j) \ge \deg s(\beta_j; W_{j-1})$$

$$= \deg p_j$$

$$\ge \deg s(\gamma; W_{j-1})$$

$$= \deg p$$

$$\deg(g\gamma_j) = \deg(fg).$$

Thus, $\deg \gamma_j \ge \deg f$.

⇒ That conductor to the choice of j.

Now,

f divides each g_i hence that $\beta_0 = f\gamma$

since, W_0 is T-admissible

$$\beta_0 = f\gamma_0 \text{ where } \gamma_0 \in W_0$$

thus each of the subspace $W_1, W_2 \ldots W_r$.

Step: 3 which is satisfy (i), & (ii)

From step (i) the vectors $\beta_1, \beta_2 \ldots \beta_r \in V$

fix k, $1 \le k \le r$.

From step(2) ⇒ $f = p_k$

$$p_k \beta_k = p_k \gamma_0 + \sum_{1 \le i < k} p_k h_i \beta_i \rightarrow (6)$$

where $\gamma_0 \in W_0$ and $h_1, h_2 \ldots h_{k-1}$ are polynomial

$$\therefore \alpha_k = \beta_k - \gamma_0 - \sum_{1 \le i < k} h_i \beta_i \rightarrow (7)$$

$$s(\alpha_k ; W_{k-1}) = s(\beta_k : W_{k-1}) = p_k \rightarrow (8)$$

$$W_{k-1} \cap Z(\alpha_k ; T) = \{0\} \rightarrow (9)$$

satisfy (8) & (9).

$$W_k = W_0 \oplus Z(\alpha_1 ; T) \oplus \ldots \oplus Z(\alpha_k ; T).$$

(ii) condition is verified.

<u>Step: 4</u>:

$$\beta = \beta_0 + f_1 \gamma_1 + \ldots + f_s \gamma_s$$

$$g_i \beta = g_1 \beta_0 + \sum_{i=1}^{s} g_1 f_i \gamma_i$$

$\dim W_0 + \dim Z(\alpha_i ; T) < \dim V$.

The Jordan form.

let N is a nilpotent linear operator on the finite dimensional space V. Let us look at the cyclic decomposition for N which we obtain from cyclic decomposition.

We have a positive integer r and r non-zero vectors $\alpha_1 \ldots \alpha_r$ in V with N-annihilators $p_1 \ldots p_r$. such that.

$$V = Z(\alpha_1 ; N) \oplus \ldots \oplus Z(\alpha_r ; N) \text{ and}$$

p_{i+1} divides p_i for $i = 1, \ldots r-1$, since N is Nilpotent, the minimal polynomial is x^k for some $k \leq n$ Thus each p_i is of the form $p_i = x^{k_i}$. and the divisibility condition simply says that,

$$k_1 \geq k_2 \geq \ldots \geq k_r$$

of course $k_1 = k$ and $k_r \geq 1$.

Annilating polynomial

Suppose T is a linear operator on V. A vector space over the field F. If p is a polynomial over F. Then $p(T)$ is again a linear operator on V. If q is another polynomial over F.

$$(p+q)T = p(T)+q(T)$$

$$(pq)T = p(T)q(T)$$

The collection of polynomial p which annihilate T such that $p(T)=0$ is an ideal in the polynomial algebra $F[x]$.

$\det(xI - A)$.

$(xI - A)_{ij} = \delta_{ij}x - A_{ij}$

Let A be an $n \times n$ matrix over k. Then A is invertiable over k. iff $\det A$ is invertiable in k when A is invertiable the unique inverse for A is,

$$A^{-1} = (\det A)^{-1} \operatorname{adj} A$$

In particular an $n \times n$ matrix over a field is invertiable iff its determinant is different from zero.

proof:

Let A be $n \times n$ matrix over k and given that A is invertiable there is an $n \times n$ matrix A^{-1} with entries in k, such that,

$$AA^{-1} = A^{-1}A = I$$

$\therefore A^{-1}$ is a inverse matrix which exist and its unique

$$1 = \det I = \det(AA^{-1})$$
$$= (\det A)(\det A^{-1})$$
$$= \det(\alpha_1 \ldots \alpha_n) \det(A^{-1})$$
$$= \det(\alpha_1 A^{-1}, \ldots \alpha_n A^{-1})$$

Here each $\alpha_j A^{-1}$ denote $1 \times n$ matrices and det is n-linear.

Here, we wish to mention that this invertiability for matrix with polynomial entries.

If K is the polynomial ring with $F[x]$.

For if f and g are poly.

and $fg = 1$

we have,

$$\deg(fg) = \deg(1)$$

$$\deg f + \deg g = 0$$

$$\deg f = 0 \; ; \; \deg g = 0$$

ie f and g are scalar polynomials

So an $n \times n$ matrix over the polynomial ring $F[x]$

iff its determinant is a non zero scalar polynomial.

UNIT - I

Linear Transformation

Definition :-

Let V and W be the vector spaces over the field F

A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha+\beta) = c\,(T\alpha) + T\beta$$

For all α and β in V are all scalars c in F.

Zero Transformations :-

If V is any vector space the zero transformat 'O' is defined by $O\alpha = 0$ is a linear transformation from V into V

Identity Transformation :-

If V is any vector space the identity transformation 'I' defind by $I\alpha = \alpha$ is a linear transformation from V into V.

Example :-

1) Let F be a field and let V be the vector space of polynomical function f from F into F give by

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_k x^k$$

Let,

$D.f(x) = c_1 + 2c_2x + \dots + kc_kx^{k-1}$

Then D is a linear transformation from V into V the differentiation transformation.

2) Let A be a fixed $m \times n$ matrix with entries in the field F. The function T defined by $T(x) = A(x)$ is linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$

The function U defined $U(\alpha) = \alpha A$ is a linear transformation from F^m into F^n

3) Let R be the field of real number and Let V be the vector space of all functions from R into R, which are continuous def T by.

$$(Tf)x = \int_0^x f(t)\,dt$$

Then T is an linear transformation from V into V.

The function T_f is not only continuous but has continuous first derivative.

The linearity of integration is one of its fundamental property.

Theorem :1

Let V be a finite dimensional vector space over the field F and Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \beta_2, \dots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that

$$T\alpha_j = \beta_j \quad ; \quad j = 1, 2, \ldots, n.$$

Proof:-

Given, α in V.

There is a unique n-tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

For this vector α, we define

$$T\alpha = x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n \longrightarrow (1)$$

Then T is a well defined rule for associating with each vector α in V a vector $T\alpha$ in W

From definition, it clear that $T\alpha_j = \beta_j$ for each j

To prove:-

T is linear

Let $\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n$ be in V and Let c be any scalar

Now,

$$c\alpha + \beta = c(x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n) + (y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n)$$

$$= (cx_1 + y_1)\alpha_1 + (c_2x_2 + y_2)\alpha_2 + \cdots + (cx_n + y_n)\alpha_n$$

and so, By definition.

$$T(c\alpha + \beta) = T(cx_1 + y_1)\alpha_1 + T(cx_2 + y_2)\alpha_2 + \cdots + T(cx_n + y_n)\alpha_n$$

$$= (cx_1 + y_1)\beta_1 + (cx_2 + y_2)\beta_2 + \cdots + (cx_n + y_n)\beta_n$$

$$= c(x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n) + (y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n)$$

$$= cT\alpha + T\beta.$$

To prove:

T is uniquiness

Let U is a linear transformation from V into W with $U\alpha_j = \beta_j$ $(j = 1, 2, \ldots, n)$ Then for the vector $\alpha = \sum_{i=1}^{n} x_n \alpha_i$

We have,

$$U\alpha = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U\alpha_i)$$

$$= \sum_{i=1}^{n} x_i \beta_i$$

$$U\alpha = T\alpha \Rightarrow U = T.$$

Hence T is linear transformation from V into W with $T\alpha_j = \beta_j$ is unique.

Example :1

The vector $\alpha_1 = (1, 2)$, $\alpha_2 = (3, 4)$ are linearly independent and therefore form a basis of R^2 there is the unique linear Transformation from R^2 into R^2 such that,

$$T\alpha_1 = (3, 2, 1)$$

$$T\alpha_2 = (6, 5, 4)$$. Find $T(1, 0)$

Soln:-

Let $\alpha = c_1\alpha_1 + c_2\alpha_2$

$$(1, 0) = c_1(1, 2) + c_2(3, 4)$$

$$(1, 0) = (c_1 + 3c_2, 2c_1 + 4c_2)$$

$$c_1 + 3c_2 = 1 \longrightarrow ①$$

$$2c_1 + 4c_2 = 0 \longrightarrow ②$$

$① \times 2 - ② \Rightarrow \quad 2c_1 + 6c_2 - 2c_1 - 4c_2 = 2 - 0$

$$2c_2 = 2$$

$$c_2 = 1$$

$c_2 = 1$ sub in ① we get

$$c_1 + 3(1) = 1$$

$$c_1 = -2$$

$$(1,0) = -2(1,2) + (3,4)$$

$$T(1,0) = -2T(1,2) + T(3,4)$$

$$= -2(3,2,1) + (6,5,4)$$

$$T(1,0) = (-6+6, -4+5, -2+4)$$

$$T(1,0) = (0,1,2)$$

Example : 2

Let P be a fixed $m \times m$ matrix with entries in the field F and let Q be a fixed $n \times n$ matrix over F. Define a function T from the space, $F^{m \times n}$ into itself by $T(A) = PAQ$. Then T is linear transformation from $F^{m \times n}$ into $F^{m \times n}$.

soln:-

$$T(cA+B) = P(cA+B)Q$$

$$= PcAQ + PBQ$$

$$= c(PAQ) + PBQ$$

$$= cT(A) + T(B)$$

$$T(cA+B) = cT(A) + T(B)$$

$\therefore$ T is linear transformation $F^{m \times n}$ into $F^{m \times n}$

H.P.

Null space:-

Let V and W be a vector space over the field F and T be a linear transformation from V into W

The Null space of T is the set of all vector x in V such that $Tx = 0$.

Rank of T.

If V is finite dimensional. The rank of T is the dimensional of the range of T.

ie) rank $T = \dim \{ v \in U \mid TU \}$

Nullity of T.

If V is finite dimensional. The nullity of T is dimensiona of the null space of T.

Nullity of $T = \dim \{ v \in V \mid Tv = 0 \}$.

Theorem: 2.

Let V and W be a vector space over the field F and Let T be a linear transformation from V into W suppose that V is finite dimensional then,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Proof:-

Let $\dim V = n$ (say)

since, The null space N is subspace of V

Assume, $\dim N = k$ (say)

Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a basis of N.

Then there exists vectors $\{\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n\}$, in V

such that, $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V.

$\Rightarrow \{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)\}$ is range of T.

To prove that,

$\{T(\alpha_{k+1}), T(\alpha_{k+2}), \dots, T(\alpha_n)\}$ is a basis for range of T.

(i) To prove:-

$\{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ span of range of T

Let $\beta \in$ range of T, There exist $\alpha \in V$ such that, $T(\alpha) = \beta$

$T(\alpha) = \beta$

Now, $\alpha \in V$, Then there exists $a_1, a_2, \dots, a_n$ are in F. such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

$$T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$$

$$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_n T(\alpha_n).$$

since, $T(\alpha_j) = 0$ for $j \le k$.

$$\therefore T(\alpha) = a_1 \cdot 0 + \dots + a_k \cdot 0 + a_{k+1} T(\alpha_{k+1}) + \dots + a_n T(\alpha_n)$$

$$\beta = a_{k+1} T(\alpha_{k+1}) + \dots + a_n T(\alpha_n)$$

$\Rightarrow \beta$ is a linear combination of $(T(\alpha_{k+1}, \dots, T(\alpha_n))$

Hence, $\{T(\alpha_{k+1}) \dots T(\alpha_n)\}$ span of range of T. (1)

(ii) To prove:-

$\{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ is linearly Independent

suppose we have scalars c_i.

such that

$$\sum_{i=k+1}^{n} c_i \, T(\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i$ is in null space of T

$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i = \sum_{j=1}^{k} b_j \alpha_j$ with $b_j \in F$

$$\Rightarrow \sum_{j=1}^{k} b_j \alpha_j - \sum_{i=k+1}^{n} c_i \alpha_i = 0$$

Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent

$$\Rightarrow b_1 = b_2 = \ldots = b_k = c_{k+1} = \ldots = c_n = 0.$$

Thus,

$$\sum_{i=k+1}^{n} c_i \, T(\alpha_i) = 0,$$

$$\Rightarrow c_i = 0 \; ; \; i = k+1, \ldots, n$$

$\Rightarrow \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is linear independent for the range of T. ——→ (2)

From (1) and (2)

$\Rightarrow \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ from a basis for range of T

$\Rightarrow$ dim (range of T) $= n-k$

$\Rightarrow$ rank $(T) =$ dim $V -$ dim N

$\Rightarrow$ rank $(T) =$ dim $V -$ nullity T

$\Rightarrow$ rank $(T) +$ nullity $(T) =$ dim V

H/P

Theorem : 3

If A is an $m \times n$ matrix with entries in the field F, Then

row rank (A) = column rank (A)

Proof:-

Let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$ defined by $T(x) = A(x)$

The null space of T is the solution space for the system $AX = 0$

$\rightarrow$ The set of all column matri x. such that $AX = 0$

The range of T is the set of all $m \times 1$ column matrices Y such that $AX = Y$ has a solution of x.

If $A_1, A_2, \dots, A_m$ are the columns of A, Then

$$AX = x_1 A_1 + x_2 A_2 + \dots + x_n A_n.$$

so that the range of T is the subspace spanned by the columns of A.

In other words,

The range of T is column space of A.

$\Rightarrow$ rank T = column rank (A)

If S is the solution space for the system $AX = 0$, Then.

$$\dim S + \text{column rank } (A) = n \longrightarrow ①$$

W.K.T

dimension of the soln space $AX = 0$ = dim A - no. of linear independent rows of A

$\Rightarrow \dim S = n - \text{row rank}(A)$

$\Rightarrow \dim S + \text{row rank}(A) = n \longrightarrow (2)$

From (1) & (2) we get,

$\Rightarrow$ row rank (A) = column rank (A)

H.P

L.T $\Rightarrow T(x_1, x_2) = (x_2, x_1)$

$$T(\alpha x + y) = \alpha T(x) + T(y)$$

$$T(x+y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

$$T(\alpha x + y) = T(\alpha(x_1, x_2) + (y_1, y_2))$$

$$= T((\alpha x_1, x_2) + (y_1, y_2))$$

$$= T((\alpha x_1 + y_1)(\alpha x_2 + y_2))$$

$$= (\alpha x_2 + y_2, \alpha x_1 + y_1)$$

$$= (\alpha x_2, x_1) + (y_2, y_1)$$

$$= \alpha T(x_1, x_2) + T(y_1, y_2)$$

$$= \alpha T(x) + T(y)$$

$\therefore$ This is linear Transformation

The Algebra of Linear Transformation:-

Thm : 4

Let V and W be vector space over the field F. Let T and U be linear transformation from V into W. The function $(T+U)$ defined by $(T+U)(\alpha) = T\alpha + U\alpha$ is a linear transformation from V into W. If c is any element in F, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is linear transformation from V into W.

Proof:-

suppose T and U are linear transformation from V into W and (T+U) defined by

$$(T+U)\alpha = T\alpha + U\alpha$$

Now,

$$(T+U)(c\alpha+\beta) = T(c\alpha+\beta) + U(\alpha+\beta)$$

$$= T(c\alpha) + T(\beta) + U(\alpha) + U\beta$$

$$= c(T\alpha + U\alpha) + (T\beta + U\beta)$$

$$= c(T+U)\alpha + (T+U)\beta$$

Which show that T+U is linear transformation

Next,

$$(cT)(d\alpha+\beta) = c[T(d\alpha+\beta)]$$

$$= c[d(T\alpha) + T\beta]$$

$$= cd(T\alpha) + cT\beta$$

$$= d[c(T\alpha)] + c(T\beta)$$

$$= d[(cT)\alpha] + (cT)\beta$$

Which shows that cT is linear transformation.

Note :-

$L(V, W) \rightarrow$ The space of linear transformation from V into W.

Thm : 5

Let V, W and Z be vector space over the field F. Let T be a linear transformation from V into W and U is a linear transformation from W into Z. Then the composed function UT is defined by $(UT)\alpha = U(T(\alpha))$ is a linear transformation from V into Z.

Proof :-

$$(UT)(c\alpha+\beta) = U[T(c\alpha+\beta)]$$
$$= U[cT\alpha + T\beta]$$
$$= U[cT\alpha] + U[T\beta]$$
$$= c[U(T\alpha)] + U[T(\beta)]$$
$$= c[(UT)(\alpha)] + [(UT)(\beta)]$$

$\therefore$ UT is a linear Transformation from V into Z.

Linear operator :-

If V is a vector space over the field F, a linear operator on V is a linear transformation V into V.

Lemma :-

Let V be a vector space over the field F. Let U, T_1 and T_2 be linear operator on V. Let c be an element of F.

a) $IU = UI = U$

b) $U(T_1+T_2) = UT_1 + UT_2$; $(T_1+T_2)U = T_1U + T_2U$

c) $c(UT_1) = (cU)T_1 = U(cT_1)$

Proof :-

a) Given U be the linear operator on V.

since, I is the identity functions

$\Rightarrow UI = IU = U$

$\therefore$ is obviously true.

b) $U[T_1 + T_2](\alpha) = (T_1 + T_2)(U\alpha)$

$$= T_1(U\alpha) + T_2(U\alpha)$$

$$= (T_1U)(\alpha) + (T_2U)(\alpha)$$

so that,

$$(T_1 + T_2)U = T_1U + T_2U$$

c) $c(UT_1)(\alpha) = cU[T_1(\alpha)]$

$$= (cU)T_1(\alpha)$$

$$= Uc\ T_1(\alpha)$$

$$= U(cT_1)(\alpha)$$

$$c(UT_1) = (cU)T_1 = U(cT_1)$$

H.P.

Thm :-

The set of all linear transformation from V into W together with the addition and scalar multiplication defined by $(T+U)\alpha = T\alpha + U\alpha$ and $(cT)\alpha = c(T\alpha)$ is vector space over the field F.

Proof :-

Let $V(V, W)$ is the set of all linear transformation from V into W.

Defined by.

$$(T+U)\alpha = T\alpha + U\alpha \longrightarrow ①$$

$$(cT)\alpha = c(T\alpha) \longrightarrow ②.$$

To prove :-

$L(V, W)$ is vector space over F.

i) <u>Closure Law</u> :-

Let $T_1, T_2 \in L(V, W)$, $\alpha \in V$

$(T_1 + T_2)\alpha = T_1\alpha + T_2\alpha$

$\therefore$ Closure Law is true

ii) <u>Associative Law</u>

Let $T_1, T_2, T_3 \in L(V, W)$. $\alpha \in V$

$[(T_1+T_2) + T_3]\alpha = (T_1+T_2)\alpha + T_3\alpha$ by ①

$= T_1\alpha + T_2\alpha + T_3\alpha$

$= T_1\alpha + (T_2+T_3)\alpha$

$= [T_1 + (T_2+T_3)]\alpha$

$\therefore (T_1+T_2) + T_3 = T_1 + (T_2+T_3)$.

$\therefore$ Associative Law is true

(iii) <u>Existence Identity</u> :-

Let $0 \in L(V, W)$, $\alpha \in V$

Consider

$(T+0)\alpha = T\alpha + 0\alpha$

$= T\alpha$

$\Rightarrow (T+0) = T$

$\Rightarrow (T+0) = (0+T) = T$

$\therefore$ Identity law is true.

iv) <u>Existence Inverse</u> :-

Let $T \in L(V, W)$, Then there exists $-T \in L(V, W)$

Consider

$[T + (-T)]\alpha = T\alpha + (-T\alpha)$

$= 0\alpha$

$T + (-T) = 0$

$\therefore T + (-T) = (-T) + T = 0$

$\therefore$ Inverse law is true.

(V) Commutative law:

Let $T, U \in L(V, W)$, $\alpha \in V$

consider,

$$(T+U)\alpha = T\alpha + U\alpha$$

$$= U\alpha + T\alpha$$

$$(T+U)\alpha = (U+T)\alpha$$

$$T+U = U+T$$

$\therefore$ commutative law is true.

$\therefore$ $L(V, W)$ is a abelian group.

(vi) $1 \cdot T = T$

consider $(1 \cdot T)\alpha = 1 \cdot T\alpha$ by ②

$$= T\alpha$$

$$1 \cdot T = T$$

(vii) $(c_1 \cdot c_2)T = c_1(c_2 T)$, $c_1, c_2 \in F$

consider

$$[(c_1 \cdot c_2)T](\alpha) = (c_1 c_2)T\alpha \quad \text{by ②}$$

$$= c_1[c_2 T\alpha]$$

$$= c_1[c_2 T]\alpha$$

$$\therefore (c_1 c_2)T = c_1(c_2 T)$$

(viii) $(c_1 + c_2)T = c_1 T + c_2 T$; $c_1, c_2 \in F$

Now,

$$[(c_1 + c_2)T]\alpha = (c_1 + c_2)T\alpha$$

$$= c_1 T\alpha + c_2 T\alpha.$$

$$= (c_1 T)\alpha + (c_2 T)\alpha$$

$$= (c_1 T + c_2 T)\alpha$$

$$\therefore (c_1 + c_2)T = c_1 T + c_2 T$$

ix) $c(T+U) = cT + cU$

Now,

$$[c(T+U)]\alpha = c[(T+U)\alpha] \quad \text{by } ③$$

$$= c[T\alpha + U\alpha] \quad \text{by } ①$$

$$= cT\alpha + cU\alpha$$

$$= (cT)\alpha + (cU)\alpha$$

$$= (cT + cU)\alpha$$

$$\therefore c(T+U) = cT + cU$$

$\therefore$ $L(V, W)$ is vector space over F.

H.P.

Thm :-

Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then the space $L(V, W)$ is finite dimensional and has dimension mn.

Proof:-

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

$B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$

be ordered bases for V and W respectively.

For each pair of integers (p, q) with

$1 \leq p \leq m$ and $1 \leq q \leq n$.

We define a linear transformation $E^{p,q}$ from V into W by

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

$1 \leq p \leq m$
$1 \leq q \leq n$

$$E^{p,q}(\alpha_j) = \delta_{iq}\beta_p$$

i.e.)
$$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \neq q \\ \beta_p & \text{if } i = q. \end{cases}$$

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

According to the theorem,

Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. Let $\beta_1, \beta_2, \dots, \beta_n$ be any vector in W. Then there is precisely one linear transformation from V into W

$T\alpha_j = \beta_j \quad (j = 1, \dots, n)$

such that,

$$T\alpha_j = \beta_j \quad (j = 1, 2, \dots, n)$$

$\therefore$ There is a unique transformation from V into W satisfying these conditions.

Claim :-

The mn transformations $E^{p,q}$ for a basis for $L(V, W)$

Let T be a linear transformation from V into W.

For each j, $1 \le j \le n$.

Let $A_{1j}, \dots, A_{mj}$ be the coordinates of vector $T\alpha_j$ in the ordered bases $\mathcal{B}$

$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$

i.e.) $$T\alpha_j = \sum_{p=1}^{m} A_{pj} \cdot \beta_p \longrightarrow (1)$$

$T\alpha_j = \beta_j$

We wish to show that,

$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}$

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \, E^{p,q} \longrightarrow (2).$$

$T\alpha_j = \sum A_{pj}\beta_p$

$T\alpha_j = \sum_{i=1}^{n} A_{pj}\beta_p$

Let U be the linear transformation in the right hand number of ②

Then for each j

$$Ux_j = \sum_p \sum_q A_{p,q} \, E^{p,q}(x_j)$$

$$= \sum_p \sum_q A_{pq} \, \delta_{jq} \, \beta_p$$

$$= \sum_{p=1}^{m} A_{pj} \, \beta_p$$

$$Ux_j = Tx_j$$

$$U = T$$

To show that

The $E^{p,q}$ span $L(V, W)$

We must prove that,

They are Independent.

But this is clear from the transformation

$U = \sum_p \sum_q A_{pq} E^{p,q}$ is a zero

transformation then $U\alpha_j = 0$ for each j

so, $\sum_{p=1}^{m} A_{pj} \beta_p = 0$

and the independence of the β_p

$\Rightarrow A_{pj} = 0$ for every p and j.

Hence, the space $L(V, W)$ is finite dimensional and has dimension mn.

Invertible :-

The function T from V into W called invertible if there exists a function U from W into V such that UT is the identity function on V and TU is the identity function on W

(ie) $UT = TU = I$

If T is invertible, then function U is denoted by T^{-1}

Note :-

If T is invertible iff

1. T is 1:1
2. T is onto

Theorem : 7

Let V and W be vector space over the field F and let T be a linear transformation from V into W. If T is invertible, then the inverse function T^{-1} is a linear transformation from W into V.

Proof :-

Let β_1 and β_2 be vectors in W and c be a scalar

To show that

$$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2.$$

$$\text{Let } \alpha_i = T^{-1}\beta_i \ , \quad i = 1, 2$$

(ie) Let α_1 and α_2 be the unique vectors in V such that $T\alpha_i = \beta_i$

since T is linear

$$T(c\alpha_1 + \alpha_2) = cT\alpha_1 + T\alpha_2$$

$$= c\beta_1 + \beta_2$$

Thus $c\alpha_1 + \alpha_2$ is unique vector in V which is sent by T into $\alpha\beta_1 + \beta_2$ and so,

$$c\alpha_1 + \alpha_2 = T^{-1}(c\beta_1 + \beta_2)$$

$$\Rightarrow T^{-1}(c\beta_1 + \beta_2) = c(T^{-1}\beta_1) + T^{-1}(\beta_2)$$

$\therefore T^{-1}$ is linear

H.P.

Note :-

1) If T is linear, then $T(\alpha - \beta) = T\alpha - T\beta$

2) Let T be invertible L.T from V onto W and U be invertible L.T from W onto Z, Then

(i) UT is invertible

(ii) $(UT)^{-1} = T^{-1}U^{-1}$

Non - singular :-

A linear transformation T is non singular if $T\gamma = 0$ implies $\gamma = 0$.

(ie) if the null space of T is $\{0\}$.

Note :-

* T is 1:1 iff T is non-singular.

* T is non-singular then T is linear independence.

Theorem : 8

Let T be a linear Transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Proof :-

First suppose T is non-singular

Let S be a linearly independent subset of V.

To prove :-

If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are vectors in S Then the vectors $T\alpha_1, T\alpha_2, \ldots, T\alpha_k$ are linearly independent

If $c_1(T\alpha_1) + c_2(T\alpha_2) + \ldots + c_k(T\alpha_k) = 0$

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k) = 0$

since T is non singular

$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k = 0$

It follows that each $c_i = 0$ because S is an independent set

$\Rightarrow$ The image of S under T is independent

$\Rightarrow$ T carries each linear independent.

Next,

suppose that, T carries independent subset onto independent subsets.

To prove :-

T is non singular

Let α be a non zero vectors in V.

Then the set S consisting of the one vector α is independent.

The image of S is the set consisting of the one vector $T\alpha$ and this set is independent

$\therefore T\alpha \neq 0$, because the set consisting of the zero vector alone is dependent

$\therefore$ The null space of T is the zero subspace

$\therefore$ T is non-singular

H.P.

Theorem: 9

Let V and W be finite dimensional vector space over the field F such that dim V = dim W

If T is linear transformation from V into W, the following are equivalent

(i) T is invertible

(ii) T is non-singular

(iii) T is onto (ie) The range of T is W

(iv) If $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is basis for V, the $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is basis for W.

(v) There is some basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for V. such that $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a basis for W.

Proof :-

(i) → (ii)

Assume that T is invertible

To show that,

T is non-singular.

(ie) $Tv = 0$ if $v = 0$ $\forall$ $v \in V$

W.K.T

T is invertible iff T is 1-1 and onto

Now, $TV = 0$

$TV = T(0)$

Since T is 1-1

$V = 0$.

$\therefore$ T is singular.

(ii) ↔ (iii)

Assume that T is non-singular

To prove :-

T is onto

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is basis for V.

By theorem ⑧, $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a linearly independent in W

Since T is non-singular

$\Rightarrow$ Nullity $(T) = 0$

W.K.T

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

$$\Rightarrow \text{rank}(T) = \dim V$$

since $\dim V = \dim W$

$$\Rightarrow \text{rank}(T) = \dim W$$

Now, Let β be any vector in W

There are scalars $c_1, c_2, \dots, c_n$ such that

$$\beta = c_1(T\alpha_1) + \dots + c_n(T\alpha_n)$$
$$= T(c_1\alpha_1 + \dots + c_n\alpha_n).$$

$\Rightarrow \beta$ is in the range of T

$\therefore T$ is onto

(iii) $\Rightarrow$ (iv)

Assume that T is onto

To prove :-

If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is basis for V then $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is basis for W.

If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is any basis for V the vector $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ span of range of T

since $\dim W = n$.

These n vectors must be linearly independent

$\Rightarrow \{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis for W

(iv) $\Rightarrow$ (v)

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a some basis for V.

from (iv)

$\Rightarrow \{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis of W.

(v) $\rightarrow$ (i)

Assume that there is a some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for V then $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is basis for W.

To prove :-

T is invertible.

It is enough to show that T is one to one and onto.

Since the $T\alpha_i$ span W

It is clear that the range of T is all of W

If $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ is in the null space of T_1 then

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0.$

$\Rightarrow c_1(T\alpha_1) + c_2(T\alpha_2) + \cdots + c_n(T\alpha_n) = 0.$

since the $T\alpha_i$ are independent each $c_i = 0$

Thus $\alpha = 0$, we have

show that the range of T is W and T is non singular

Hence T is invertible

H.P.

Def: Groups :-

A Group consists of the following.

1. A set G

2. A rule (or operation) which associates with each pair of elements xy in G in such a way that (a) $x(yz) = (xy)z$ for x, y and z in G

(b) There is an element e in G.

such that $ex = xe = x$ for every x in G

(c) To each element x in G there corresponds an element x^{-1} in G such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Commutative :-

A group is called commutative if it satisfies the condition $xy = yx$ for each x and y

Field :-

A field can be described as a set with two operations called addition and multiplication

Which is a commutative group under addition and in which the non-zero elements form a commutative group under multiplication with the Distributive law.

$$x(y+z) = xy + xz \text{ holding.}$$

Isomorphism :-

If V and W are vector space over the field F, any one to one linear transformation T of V onto W is called an isomorphism of V onto W.

If there exist an isomorphism of V onto W we say that V is isomorphic to W.

Theorem : 10

Every n-dimensional vector space over the field F is isomorphic of the space F^n.

<u>Proof</u> :-

Let V be an n-dimensional space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.

Let $\alpha \in V$ then $\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for all x_i in F

We define map $T : V \longrightarrow F^n$ by

$T\alpha = \{x_1, x_2, \ldots, x_n\}$ $\ni (\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.)$

Where x_i is the coordinate of α.

$(T\alpha = x_1, x_2, \ldots, x_n$

$\beta = \beta_1 y_1, \beta_2 y_2, \ldots, \beta_n y_n$

$T\beta = y_1, y_2, \ldots, y_n)$

<u>To prove</u> :-

T is linear transformation

Let $\alpha, \beta \in V$ and $c \in F$ (scalar), then

$$\alpha = \sum_{i=1}^{n} \alpha_i x_i \quad \text{and} \quad \beta = \sum_{i=1}^{n} \alpha_i y_i$$

consider $T(\alpha+\beta) = T\left(c \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \alpha_i y_i\right)$

$$= T\left(\sum_{i=1}^{n} c\alpha_i x_i + \sum_{i=1}^{n} \alpha_i y_i\right)$$

$$T(\alpha+\beta) = T\left(\sum_{i=1}^{n} (cx_i + y_i)\alpha_i\right)$$

$$= \{cx_1+y_1, cx_2+y_2, \dots, cx_n+y_n\}$$

$$= \{cx_1, cx_2, \dots, cx_n\} + \{y_1, y_2, \dots, y_n\}$$

$$T(\alpha+\beta) = cT\alpha + T\beta$$

$\therefore$ T is linear transformation

Next

To prove:-

T is one to one.

since every $\alpha \in V$, There is a unique coordinate matrix.

$\therefore$ T is 1-1

Next to prove

T is onto

Let $x = \{x_1, x_2, \dots x_n\} \in F^n$

Then clearly, $\alpha \in V$

$\Rightarrow T\alpha = x$

$\therefore$ T is onto.

$\therefore$ T is isomorphic.

Hence every n-dimensional vector space over F is isomorphic to the space F^n.

H.P.

Representation of Transformation matrices:-

Let V be n-dimensional vector space over the field F and W be m-dimensional vector space over F.

Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ an ordered basis for V and $B' = \{\beta_1, \beta_2, \dots, \beta_n\}$ an ordered basis for W

If T is an linear transformation from V into W

Then T is determined by its action on the vectors α_j.

Each of the n-vectors $T\alpha_j$ is unique expressible as a linear combination.

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i$$

of the β_i, the scalars $A_{1j}, \dots, A_{nj}$ being the coordinates of $T\alpha_j$ in the ordered basis B'.

The $m \times n$ matrix A defined by $A(i,j) - A_{ij}$ is called the matrix of T relative to the pair of ordered basis B and B'.

Theorem: 11

Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. for each pair of ordered bases B, B' for V and W respectively, the function which assigns to be a linear transformation T its matrix relative to $B \cdot B'$ is a isomorphism between the space $L(V, W)$ and the space of all $m \times n$ matrices over the field F.

Proof:-

Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

$B' = \{\beta_1, \beta_2, \dots, \beta_n\}$

Let M be a vector space all $m \times n$ matrices over field F.

Let $\psi : L(V, W) \longrightarrow M$ such that

$$\psi(T) = [T : B : B'] \ \forall \ T \in L(V, W)$$

$$= [a_{ij}]_{m \times n}$$

Let $T_1, T_2 \in L(V, W)$

Let $[T_1, B, B'] = [a_{ij}]_{m \times n}$

$[T_2, B, B'] = [b_{ij}]_{m \times n}$

$$T_1(\alpha_j) = \sum_{i=1}^{m} a_{ij} \beta_j \ , \quad 1 \le j \le n$$

$$T_2(\alpha_j) = \sum_{i=1}^{m} b_{ij} \beta_j \ ; \quad 1 \le j \le n$$

To prove.

ψ is 1-1

Consider $\psi(T_1) = \psi(T_2)$

$\Rightarrow [T_1, B, B'] = [T_2, B, B']$

$\Rightarrow [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$

$\Rightarrow a_{ij} = b_{ij}$

$\Rightarrow \sum_{i=1}^{m} a_{ij} \beta_j = \sum_{i=1}^{m} b_{ij} \beta_j$

$\Rightarrow T_1 \alpha_i = T_2 \alpha_j$

$\Rightarrow T_1 = T_2$

$\therefore \psi$ is 1-1.

To prove:-

ψ is onto

Let $[a_{ij}]_{m \times n} \in M$, Then $\exists$ a linear transformation of T from V into W. such that

$$T\alpha_j = \sum_{i=1}^{m} a_{ij}\, \beta_j \qquad 1 \le j \le n$$

We have,

$$[T, B, B'] = [a_{ij}]_{m \times n}$$

$$\Rightarrow \psi(T) = [a_{ij}]_{m \times n}$$

$\therefore \psi$ is onto.

To prove :-

ψ is linear transformation.

If $a, b \in F$, then

$$\psi(aT_1 + bT_2) = [aT_1 + bT_2, B, B']$$
$$= [aT_1, B, B'] + [bT_2, B, B']$$
$$= a[T_1, B, B'] + b[T_2, B, B']$$
$$= a\psi(T_1) + b\psi(T_2)$$

$\therefore \psi$ is linear transformation

$\therefore L(V, W)$ is isomorphic to M.

Example :1

Let F be a field and let T be the operation on F^2 defined by, $T(x_1, x_2) = (x_1, 0)$. Find matrix of T using standard basis of F.

soln :-

Given

$$T(x_1, x_2) = (x_1, 0)$$

Let $B = \{(1,0), (0,1)\}$

$$T(1,0) = (1,0) = 1.E_1 + 0.E_2$$

$$T(0,1) = (0,0) = 0E_1 + 0.E_2$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example : 2

Let V be the space of all polynomial functional from R into R of the form $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ the space of polynomial functions of degree three or less the differentiation operator D map V into V is defined by $Df(x) = c_1 + 2c_2 x + 3c_3 x^2$ Let B be the ordered basis for V consisting of the four functions f_1, f_2, f_3, f_4 defined by $f_j(x) = x^{j-1}$ find the matrix D in the ordered basis.

soln:-

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$f_j = x^{j-1}$$

$$f_1 = 1, \quad f_2 = x, \quad f_3 = x^2, \quad f_4 = x^3$$

$$D : V \to V$$

$$Df(x) = c_1 + 2c_2 x + 3c_3 x^2.$$

$$D f_1(x) = 0 = 0.f_1 + 0.f_2 + 0.f_3 + 0.f_4$$

$$D f_2(x) = 1 = 1.f_1 + 0.f_2 + 0.f_3 + 0.f_4$$

$$D f_3(x) = 2x = 0.f_1 + 2.f_2 + 0.f_3 + 0.f_4$$

$$D f_4(x) = 3x^2 = 0.f_1 + 0.f_2 + 3f_3 + 0.f_4.$$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem : 13.

Let V, W and Z be the finite dimensional vector space over field F. Let T be a linear transformation from from V into W and U is a linear transformation from W into Z If B, B' and B'' are ordered bases for the space V, W and Z respectively if A is the matrix of T relative to the pair B, B' and B is the matrix of U relative to the pair B', B''.

Then the matrix of the composition UT relative to the pair $\mathcal{B}, \mathcal{B}''$ is the product matrix $C = BA$

Proof:-

~~Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$~~

Let V, W and Z be finite dimensional over F.

$\Rightarrow \dim V = n$, $\dim W = m$ and $\dim Z = p$

Let $T: V \to W$ is linear transformation and $U: W \to Z$ is linear transformation

suppose we have ordered bases

$$\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$\mathcal{B}' = \{\beta_1, \beta_2, \ldots, \beta_n\}$$

and $\mathcal{B}' = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ for the respective space V, W and Z

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times p}$

and $C = [c_{ij}]_{p\times n}$

If α_j is any vector in V.

$$T\alpha_j = \sum_{i=1}^{m} a_{ij}\beta_j \quad 1 \le j \le n$$

If $\beta_j \in W$

$$U\beta_j = \sum_{j=1}^{p} b_{ij}\gamma_j \quad 1 \le j \le m$$

$$UT(\alpha_j) = \sum_{i=1}^{p} c_{ij}\gamma_j \quad 1 \le j \le n$$

If $\alpha \in V$, Then

$$[T\alpha]_{\mathcal{B}} = A[\alpha]_{\mathcal{B}'}$$

$$[U(T\alpha)]_{\mathcal{B}''} = B[T\alpha]_{\mathcal{B}'}$$

and also

$$[(UT)(\alpha)]_{\mathcal{B}''} = BA[\alpha]_{\mathcal{B}'}$$

We have to show that

$$c_{ij} = \sum_{k=1}^{m} B_{ik}A_{kj}.$$

Now,

$$(UT)\alpha_j = U[T\alpha_j]$$

$$\sum_{i=1}^{p} C_{ij}\gamma_i = U\left[\sum_{k=1}^{m} A_{kj}\beta_k\right]$$

$$= \sum_{k=1}^{m} A_{kj}\, U(\beta_k).$$

$$= \sum_{k=1}^{m} A_{kj} \sum_{i=1}^{p} B_{ik}\gamma_i$$

$$\sum_{i=1}^{p} C_{ij}\gamma_i = \sum_{i=1}^{p}\left(\sum_{k=1}^{m} B_{ik}A_{kj}\right)\gamma_i$$

$$\rightarrow C_{ij} = \sum_{k=1}^{m} B_{ik}A_{kj}$$

$$\rightarrow C = BA$$

$\therefore$ The matrix C is UT relative to the pair $\mathcal{B}\cdot\mathcal{B}''$. is the product matrix $C = BA$

H.P.

Theorem : 13

Let V be a finite dimensional vector space over the field F and let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ and $\mathcal{B}' = \{\alpha_1', \alpha_2', \ldots, \alpha_n'\}$ be ordered bases for V. suppose T is a linear operator on V. If $P = [P_1, P_2, \ldots, P_n]$ is the $n\times n$ matrix with columns $P_j = [\alpha_j']_{\mathcal{B}}$, then $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}\cdot P$.

Alternatively, if U is invertible operator on V defined by $U\alpha_j = \alpha_j'$, $j = 1, 2, \ldots n$ then

$$[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}}.$$

Proof :-

Let $\alpha \in V$

$$\alpha = \sum_{j=1}^{n} x_j\alpha_j \longrightarrow ①$$

$$[\alpha]_{\mathcal{B}'} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x \longrightarrow ②$$

since $\beta_j \in V$

$$\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i \longrightarrow ③.$$

$$[\beta_j] = p_j$$

$$① \Rightarrow \alpha = \sum_{j=1}^{n} x_j \left[\sum_{i=1}^{n} p_{ij}\, \alpha_i \right]$$ $\qquad \beta_j = \sum_{i=1}^{n} p_{ij}\, \alpha_i$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_j \cdot p_{ji} \right) \alpha_i$$

$$[\alpha]_B = P_x \longrightarrow ④$$

$$[\alpha]_B = P[\alpha]_{B'} \longrightarrow ⑤$$

We know that

$$[T\alpha]_{B'} = A[\alpha]_B \longrightarrow ⑥$$

If $B = B'$

$$[T\alpha]_B = [T_B][\alpha B] \text{ and}$$

$$[T\alpha]_{B'} = [T_B][\alpha B]$$

$$[T\alpha]_{B'} = [T]_{B'}[\alpha]_{B'}$$

in ④ $\Rightarrow [T\alpha]_{B'} = P[T\alpha]_{B'}$

since T is linear operator.

$$[T_B][\alpha]_B = P[T]_{B'}\ [\alpha]_{B'}$$

$[\alpha]_B = P[\alpha]_{B'}$

$$[T]_B \cdot P[\alpha]_{B'} = P[T]_{B'} \cdot [\alpha]_{B'} \text{ by } ④$$

$$[T]_B P = P[T]_{B'}$$

Pre-multiple by P^{-1} on both sides,

$$P^{-1}[T]_B P = P^{-1} \cdot P[T]_{B'}$$

$$P^{-1}[T]_B P = [T]_{B'} \longrightarrow ⑦$$

Since $\beta_j = \sum_{i=1}^{n} p_{ij}\, \alpha_j$

$$U\alpha_j = \beta_j = \sum_{i=1}^{n} p_{ij}\, \alpha_j \quad (\text{given } \beta_j = U\alpha_j)$$

$$\therefore [U]_B = [\beta_j]_B = P \longrightarrow ⑧$$

$[\text{given } [\beta_j]_B = P]$

sub ⑧ in ⑦ we get.

$$[T]_{B'} = [U]_B^{-1}[T]_B[U]_B$$

Definition : similar

Let A and B be an $n \times n$ matrices over the field F. We say that B is similar to A over F if there is an invertible $n \times n$ matrix P over F such that $\boxed{B = P^{-1} A P}$ then we can say B is similar to A.

Linear Functional :-

If V is a vector space over the field F. a linear transformation f from V into the scalar field F is also called a linear functional on V. If f is a function from V into F. such that

$$f(c\alpha + \beta) = c f(\alpha) + f(\beta)$$

Example :-

Let n be a +ve integer and F be a field. If A is an $n \times n$ matrix with entries in F, the trace of A is the scalar.

$$\operatorname{tr} A = A_{11} + A_{22} + \dots + A_{nn}$$

The trace function is a linear functional on the matrix space $F^{n \times n}$, because

$$\operatorname{tr}(cA + B) = \sum_{i=1}^{n} (cA_{ii} + B_{ii})$$

$$= c \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$$= c \operatorname{tr} A + \operatorname{tr} B.$$

$cA_{ii} = cA_{11} + cA_{12} + \dots + cA_{nn}$

$= c(A_{11} + A_{12} + \dots + A_{nn})$

$= c \sum_{i=1}^{n} A_{ii}$

Dual space :-

If V is a vector space, the collection of all linear functional on V form a vector space in a natural way.

It is the space $L(V, F)$ we denoted by V^* and call it the dual space.

$$V^* = L(V, F)$$

Note :-

$$\dim V = \dim V^*$$

Dual Basis :- (let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be basis for V there is a linear functional f_i on V such that $f_i(\alpha_i) = \delta_{ij}$

If $f_1, f_2, \ldots, f_n$ are n linearly independent functional and V^* has dimension n. then $B^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for V^*. This basis is called the dual basis of B.)

Theorem : 14.

Let V be a finite dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Then there is a unique dual basis $B^* = \{f_1, f_2, \ldots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{ij}$. For each linear functional f on V, we have. $f = \sum_{i=1}^{n} f(\alpha_i) f_i$

and for each vector α in V, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

Proof :-

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.

We know that,

Let V be a finite dimensional vector space over F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over F and let $\beta_1, \ldots, \beta_n$ be any vector in W. Then there is a precisely one linear transformation T from V into W such that

$$T\alpha_j = \beta_j \quad ; \quad j = 1, 2, \ldots, n.$$

$\Rightarrow$ $W = F$ and $\{\beta_1, \beta_2, \ldots, \beta_n\} = \{x_1, \ldots, x_n\}$

$\Rightarrow$ There is a precisely one linear functional f from V into F such that

$$f(\alpha_i) = x_i$$

$\Rightarrow$ There exist a unique linear functional f_1 such that $f_1(\alpha_1) = 1$, $f_1(\alpha_2) = 0$, ..., $f_1(\alpha_n) = 0$.

Where $\{1, 0, 0, \ldots, 0\}$ is a ordered set, of F scalars for each $i = 1, 2, \ldots, n$

There exist unique functional f on V such that

$$f_i(\alpha_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$= \delta_{ij}$$

To prove:-

$B^* = \{f_1, f_2, \ldots, f_n\}$ is basis of V^*.

First prove B^* is a linearly independent.

Consider

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0.$$

$$\Rightarrow (c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)(\alpha) = 0.$$

$$\Rightarrow \sum_{i=1}^{n} c_i f_i(\alpha) = 0 \quad \forall \alpha \in V$$

Put $\alpha = \alpha_j$

$$\sum_{i=1}^{n} c_i f_i(\alpha_j) = 0.$$

$$\sum_{i=1}^{n} c_i \delta_{ij} = 0.$$

$$c_i = 0$$

$\therefore B^* = \{f_1, f_2, \ldots, f_n\}$ are linearly independent

Next to prove.

Every element f in V^* can be expressed as a linear combination of $f_1, f_2, \ldots, f_n$

ie) $f = \sum_{i=1}^{n} a_i f_i$

Let f be any element in V^*

Let $f(\alpha_i) = a_i \quad i = 1, 2, \ldots, n \longrightarrow (1)$

Let $\alpha_j \in B$, Where $j = 1, 2, \ldots, n$

Then $\sum_{i=1}^{n} (a_i f_i)(\alpha_i) = \sum_{i=1}^{n} a_i (f_i(\alpha_i))$

$$= \sum_{i=1}^{n} a_i \delta_{ii}$$

$$= a_i$$

$$\sum_{i=1}^{n} (a_i f_i)(\alpha_i) = f(\alpha_i) \Rightarrow \sum_{i=1}^{n} a_i f_i = f \longrightarrow ②$$

$\therefore V^*$ is a linear span of B^* $\therefore B^*$ is a basis of V^*

To Prove. f is in V^*, then

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

From ② $\Rightarrow$ $f = \sum_{i=1}^{n} a_i f_i$

$$f(\alpha_j) = \sum_{i=1}^{n} a_i f_i(\alpha_j)$$

$$= \sum_{i=1}^{n} a_i \delta_{ij} \quad i=j \longrightarrow ③$$

sub ③ in ② we get.

$$f = \sum_{i=1}^{n} f(\alpha_j) f_i$$

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i \quad (i=j)$$

To prove:-

For each α in V, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha) \alpha_i$$

Now,

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \text{ is a vector in } V \longrightarrow ④.$$

$$f_j(\alpha) = \sum_{i=1}^{n} x_i f_j(\alpha_j)$$

$$= \sum_{i=1}^{n} x_i \delta_{ij}$$

$$f_j(\alpha) = x_j \longrightarrow ⑤$$

sub ⑤ in ④, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)(\alpha_i)$$

H.P.

Hyper space :-

If a vector space of dimension on n, a subspace of dimension $n-1$ is called hyper space.

Annihilator :-

If V is a vector space over the field F and S is subset of V the annihilator of S is the set S^0 of linear functionals f on V such that $f(\alpha) = 0$ for every α in S.

Note :-

If S is the set of consisting of zero vector alone then $S^0 \in V^*$ if $S \in V$, Then S^0 is a zero subspace of V^*

10m

$V^* \to$ dual space.

Theorem : -15

Let V be a finite dimensional vector space over the field F, and let W be a subspace of V. then

$$\dim W + \dim W^0 = \dim V.$$

Proof :-

Let k be the dimension of W and $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ a basis for W

choose vectors $\{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$ in V such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V.

Let $\{f_1, f_2, \ldots, f_n\}$ be a basis of V^* which is dual to this basis of V.

Claim :-

$\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is a basis for the annihilator W^0.

Certainly f_i belongs to W^0 for $i \geq k+1$ because

$$f_i(\alpha_j) = \delta_{ij}$$

and $\delta_{ij} = 0$ if $i \geq k+1$ and $j \leq k$

It follows that, for $i \geq k+1$ $f_i(\alpha) = 0$ whenever α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k$

The functional $f_{k+1}, f_{k+2}, \ldots, f_n$ are linearly independent.

To prove :-

$$W^0 = L(S).$$

(ie) They span W^0.

suppose f is in V^*

Now,

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

so that if f is in W^0, we have

$$f(\alpha_i) = 0 \text{ for } i \leq k$$

and

$$f = \sum_{i=k+1}^{n} f(\alpha_i) f_i$$

Which is show that

$\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is span of W^0

$\therefore$ $\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is basis for the annihilator W^0.

$$\therefore \dim W^0 = n-k$$
$$= \dim V - \dim W$$

$$\Rightarrow \dim W + \dim W^0 = \dim V$$

H.P.

Corollary : 1

If W is a k-dimensional subspace of an n-dimensional vector space V. then W is intersection of $n-k$ hyperspace in V.

Proof :-

First, we proof theorem (15)

$$\dim W + \dim W^0 = \dim V$$

$$\dim W^0 = n-k$$

Let $W^0 = \{f_{k+1}, \dots, f_n\}$

If $k = n-1$, then $W^0 = \{f_n\}$

$$\dim W^0 = 1$$

Let $N f_i = \{\alpha \in V \mid f_n(\alpha) = 0\}$

To prove :- $W = \bigcap_{i=k+1}^{n} N f_i$

Let $\alpha \in W$, $\Rightarrow f_n(\alpha) = 0$

$\Rightarrow \alpha \in N f_n$ contained

$\Rightarrow W \subseteq N f_n$.

$\Rightarrow W \subseteq \cap N f_i \quad i = k+1, \dots, n \longrightarrow (1)$

$\alpha \in \bigcap_{i=k+1}^{n} N f_i$

$\Rightarrow \alpha \in N f_i$

$\Rightarrow f_i(\alpha) = 0, \quad i = k+1, \dots, n.$

$\Rightarrow f_i \in W^0, \quad \alpha \in W$

$\Rightarrow N f_i \subseteq W \longrightarrow (2)$

From (1) & (2) we have

$$W = \bigcap_{i=k+1}^{n} N f_i$$

W is the intersection of $n-k$ hyperspace.

Corollary : 2

If W_1 and W_2 are subspaces of a finite dimensional vector space. then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$

Proof :-

Suppose, $W_1 = W_2$

then $W_1^0 = W_2^0$.

If $W_1 \neq W_2$, then one of the two subspaces contains a vector which is not in the other.

suppose there is a vector α which is in W_2 but not in W_1.

By previous theorem,

There is a linear functional f such that $f(\beta) = 0$ for all β in W_2 But $f(\alpha) \neq 0$.

Then f is in W_1^0 but not in W_2^0

$$\Rightarrow W_1^0 \neq W_2^0$$

H.P.

Double Dual :-

Let V be a vector space over the field F Then, V^* be the dual of V. The dual of V^* is denoted by V^*V^*. It is called the double dual of V.

Note :-

If α is a vector in V. Then α induces a linear functional L_α on V^* defined by $L_\alpha(f) = f(\alpha)$, f in V^*, L_α is linear.

Def of linear operator in V^*

$$L_\alpha(cf+g) = (cf+g)\alpha$$
$$= cf(\alpha) + g(\alpha)$$
$$L_\alpha(cf+g) = cL_\alpha(f) + L_\alpha(g)$$

If V is finite dimensional, $\alpha \neq 0$ Then $L_\alpha \neq 0$

ie, $\exists$ a linear functional f. such that

$$f(\alpha) \neq 0.$$

choose an ordered basis $B = \{\alpha_1, \ldots, \alpha_n\}$ for V such that $\alpha_i = \alpha$ and let f be the linear functional which assign to each vector is V. its functional first coordinates in ordered basis.

Thm : 16

Let V be a finite dimensional vector space over the field F for each vector α in V, define $L_\alpha(f) = f(\alpha)$. f in V^* The mapping $\alpha \to L_\alpha$ is then an isomorphism of V onto V^{**}

Proof:-

We showed that for each α the function L_α is linear.

Suppose α and β are in V and c is in F and Let $\gamma = c\alpha + \beta$. Then for each f in V^*

$$L_\gamma(f) = f(\gamma)$$

$$L_\gamma(f) = f(\gamma)$$

$$= f(c\alpha+\beta)$$

$$= cf(\alpha) + f(\beta)$$

$$= cL_\alpha(f) + L_\beta(f)$$

and so, $L_\gamma = cL_\alpha + L_\beta$

This shows that the mapping $\alpha \to L_\alpha$ is a L.T from V into V^{**}

If V is finite dimensional and $\alpha \neq 0$ then $L_\alpha \neq 0$.

ie, $\exists$ a linear functional "f" such that $f(\alpha) \neq 0$.

According to this transformation is non-singular L.T from V into V^{**}

Since $\dim V = \dim V^* = \dim V^{**}$

By thm, 9

Let V and W be finite dimensional vector space over the field F, S.T $\dim V = \dim W$. If T is a L.T from V into W. the following are equivalent.

i) T is invertible

ii) T is nonsingular

iii) T is onto

therefore $\alpha \to L_\alpha$ is linear and non-singular with

$$\dim V = \dim V^{**} \quad L_\alpha$$

Then the mapping $\alpha \to \underline{L_\alpha}$ is an isomorphism from V onto V^{**}

Corollary:-

Let V be a finite dimensional vector space over F. If 'L' is linear functional over the dual space V^* of V, then there is a unique vector α in V, such that $L(f) = f(\alpha)$ for every f in V^*

soln:-

If $\exists$ a β in V such that, $L(f) = f(\beta)$, then.

$$L(f) = f(\alpha) = f(\beta)$$

$$f(\alpha) = f(\beta)$$

$$f(\alpha) - f(\beta) = 0$$

$$f(\alpha - \beta) = 0$$

$$\alpha - \beta = 0$$

$$\alpha = \beta.$$

Corollary:-

Let V be a finite dimensional vector space over F each basis for V^* is the dual of some basis for V.

Proof:-

Let $B^* = \{f_1, f_2, \dots, f_n\}$ be a basis for V^*

By thm (14).

There is a basis $\{L_1, L_2, \dots, L_n\}$ for V^{**} such that

$$L_i(f_j) = \delta_{ij}$$

and using the above corollary

for each i, there is a vector $\alpha_i \in V$.

such that $L_i(f) = f(\alpha_i)$, for every $f \in V^*$

ie, such that $L_i = L_{\alpha_i}$

It follows that $\{\alpha_1, \dots, \alpha_n\}$ is a basis for V and that B^* is the dual of the basis.

Thm.: 17

If S is any subset of a finite dimensional vector space V then $(S^0)^0$ is the subspace spanned by S.

Proof:-

Let W be the subspace spanned by S. clearly

$W^0 = S^0$

We prove: $W = W^{00}$

By thm,

$$\dim W + \dim W^0 = \dim V$$

$$\dim W^0 + \dim W^{00} = \dim V^*$$

and since $\dim V = \dim V^*$

We have, $\dim W + \dim W^0 = \dim W^0 + \dim W^{00}$

$$\dim W = \dim W^{00}$$

Since W is a subspace of W^{00} we have that

$$W = W^{00}$$

H.P.

Def: Maximal

If V is a vector space a hyperspace in V is a maximal proper subspace of V.

Thm: 18

If f is a non-zero linear functional on a vector space V. then the null space of f is hyperspace in V. Conversely, every hyperspace in V is the null space of a (not unique) non-zero linear functional on V.

Proof:-

Let f be a non-zero linear functions on V and N_f is its null space.

Let α be a vector in V, Which is not in N_f. ie, a vector such that $f(\alpha) \neq 0$ we shall show that every vector in V is in the subspace spanned by N_f and α.

That subspace consists of all vectors $\gamma + c\alpha$, $\gamma \in N_f$, $c \in F$

Let $\beta \in V$, define

$$c = \frac{f(\beta)}{f(\alpha)}$$

Which makes since because $f(\alpha) \neq 0$ Then the vector $\gamma = \beta - c\alpha \in N_f$.

since, $f(\gamma) = f(\beta - (c\alpha))$

$$= f(\beta) - f(c\alpha) = f(\beta) - cf(\alpha)$$

Since β is in the subspace spanned by N_f and α

Now, Let N be a hyper space in V $\exists$ in some vector $\alpha \notin N$ since, N is a maximal proper subspace.

$\therefore$ The subspace spanned by N and α is the entire space V.

$\therefore$ Each vector β in V has the form

$$\beta = \gamma + c\alpha, \quad \gamma \in N, \quad c \in F$$

The vector 'γ' and scalar c are uniquely determined by β.

If we have also,

$$\beta = \gamma' + c'\alpha, \quad \gamma' \in N, \ c' \in F.$$

Then $\quad c(c'-c)\alpha = \gamma - \gamma'$

if $c'-c \neq 0$ then α would be in N Hence

$c' = c$ and $\gamma' = \gamma$

Another way. If β is in V. There is a unique scalar, such that $\beta \to \alpha$ is in "N"

call the scalar $g(\beta)$. It is easy to see that 'g' is a linear functionals on V. and that "N" is the null space of g.

H/P.

Thm : 19

Let $g, f_1, f_2, \dots, f_r$ be linear functions on a vector space V. respective null space $N, N_1, N_2, \dots, N_r$. Then 'g' is a linear combination of $f_1, f_2, \dots, f_r$ iff N contains the intersection of $N_1 \cap N_2 \cap \dots \cap N_r$.

Proof:-

Case (i)

To prove that, N contains the $\cap$ of $N_1 \cap N_2 \cap \dots \cap N_r$

ie, $\bigcap_{i=1}^{r} N_i \subseteq N$

Assume that g is a linear combination of $f_1, f_2, \dots, f_r$

Let, $g = \sum_{i=1}^{r} c_i f_i$, $\forall c_i \in F$

Let, $\alpha \in \bigcap_{i=1}^{r} N_i$

$\Rightarrow \alpha \in N_i$, $i = 1, 2, \dots, r$

$\Rightarrow f_i(\alpha) = 0$

Consider, $g(\alpha) = \left[\sum_{i=1}^{r} c_i f_i\right](\alpha)$

$= \sum_{i=1}^{r} c_i f_i(\alpha) \quad \therefore f_i(\alpha) = 0$

$= \sum_{i=1}^{r} c_i(0) \Rightarrow g(\alpha) = 0.$

$\rightarrow \alpha \in N, \bigcap_{i=1}^{r} N_i \subseteq N$

$N \subseteq N_1 \cap N_2 \cap \ldots \cap N_r$

Case (ii)

To prove that, linear combination of $f_1, f_2, \ldots, f_r$

ie, $g = \sum_{i=1}^{r} c_i f_i$.

Assume $\bigcap_{i=1}^{r} N_i \subseteq N$, Where N_i is null space of "f_i" and N is the null space of "g"

We prove that, part of induction on "r" for $r=1$.

We know this the thm is true.

We assume that, this is proof for upto $r-1$

Now we prove "r"

Let $g', f_1', f_2', \ldots, f_{r-1}'$ be the restriction of $g, f, f_2, \ldots, f_{r-1}$ to the supspace N_r.

Then $g', f_1', f_2', \ldots, f_{r-1}'$ are linear functional on the vector space N_r

If $\alpha \in N$. and $f_i'(\alpha) = 0$, $i = 1, \ldots, r-1$

then $\alpha \in N_1 \cap N_2 \cap \ldots \cap N_r$ and so $g'(\alpha) = 0$

By induction hypothesis,

$$g' = \sum_{i=1}^{r-1} c_i f_i'$$

Let $$h = g - \sum_{i=1}^{r-1} c_i f_i$$

Nearly h is linear functionals on 'r' $\forall \alpha$

$$h(\alpha) - g(\alpha) - \sum_{i=1}^{r-1} c_i f_i(\alpha)$$

$$= 0 \qquad [\because \text{ since } f_i = f_i' \quad g = g' \text{ on } N_r]$$

ie, α belongs to $N_r \Rightarrow h(\alpha) = 0$

By using the next thm,

We get,

'h' is a scalar multiple of f_r

ie, $h = C_r f_r$ on V.

$$g - \sum_{i=1}^{r-1} c_i f_i = C_r f_r$$

$$g = \sum_{i=1}^{r-1} c_i f_i + C_r f_r$$

$$= \sum_{i=1}^{r} c_i f_i$$

$\therefore$ g is a linear combination of $f_1, f_2, \ldots, f_r$.

Lemma :-

If f and g are linear functionals on a vector space V, then g is a scalar multiple of "f" iff the nullspace of "g" contains the nullspace of "f" ie,

Iff $f(\alpha) = 0 \Rightarrow g(\alpha) = 0$.

Proof:-

Case (i)

Assume g is a scalar multiple of "f"

$\Rightarrow g = cf$ for some $c \in F$

consider, $g(\alpha) = c f(\alpha)$

for every $\alpha \in f$

if $f(\alpha) = 0$

$\Rightarrow g(\alpha) = 0$

if $f = g = 0$ then the thm is true.

Case (ii)

If $f \neq 0$ then the null space N_f of f is hyperspace

choose $\alpha \in V$, such that $f(\alpha) \neq 0$

Let $c = \dfrac{g(\alpha)}{f(\alpha)}$

Let $h = g - cf$.

then h is a linear functionals on V

let $\gamma \in N_f$

$$f(\gamma) = 0$$

$$g(\gamma) = 0$$

$h(\gamma) = 0 \quad \forall \ \gamma \in N_f$

if $\alpha \in N_f$

$$\rightarrow f(\alpha) \neq 0$$

$$\Rightarrow h(\alpha) = g(\alpha) - cf(\alpha)$$

$$= g(\alpha) - \frac{g(\alpha)}{f(\alpha)} \cdot f(\alpha)$$

$$= 0$$

$\therefore h(\alpha) = 0$ for every $\alpha \in V$.

$\rightarrow h = 0$ ($h \in$ Null space)

$\rightarrow g - cf = 0$

$\rightarrow g = cf$.

Transpose of Linear transformation:-

Transpose of T:-

(Let V and W be a vector space over the field F for each linear transformation T from V into W. There is a unique linear transformation T^t from W^* into V^* such that,

$$(T^t g)(\alpha) = g(T\alpha)$$

for every g in W^* and α in V. Then this ~~for every g in W^* a~~ transformation. T^t is called as Tranpose of T or adjoint of T)

$T^t(g(x)) = g(Tx)$

Statement.

Above definition.

Proof:-

Let $g, h \in W^*$ and $c \in F$

To prove that T^t is linear.

consider, $[T^t(cg+h)](\alpha) = (cg+h)\,T(\alpha)$

$= cg[T(\alpha)] + h[T(\alpha)]$

$= c[T^t g(\alpha)] + [T^t h](\alpha)]$

$[T^t(c'g+h)]\alpha = [cT^t g + T^t h]\alpha$

$T^t(c'g+h) = cT^t g + T^t h$

$\therefore T^t$ is a linear transformation from W^* into V^*.

To prove : uniqueness :-

Consider that there is another linear transformation U^t from W^* into V^*

such that,

$(U^t g)\alpha = g[T(\alpha)] \ \forall\ g \in W^*\ \alpha \in V.$

since, $(T^t g)\alpha = g(T(\alpha))$

$= (U^t g)\alpha.$

$T^t g = U^t g$

$T^t = U^t$

$\therefore T^t$ is a unique.

Thm : 20.

Statement :-

If V and W be the vector space over the field F and let T be a linear transformation from V into W the null space of T^t is the annihilator of the range T. If V and W are finite. dimensional then (i) rank (T^t) = Rank (T)

(ii) the range of T^t is the annihilator of the null space of T.

Proof :-

First to prove, the annilator of the range of T is equal to the null space of T^t

ie, $[R(T)]^\circ = N(T^t)$

If g is in W^* then by defn

$(T^t g)\,x = g(Tx)$ $\quad T^t(g(x)) = g(Tx)$

let g is in the null space of T^t, which is the subspace of W^*

ie, $g \in N(T^t) \Rightarrow g(Tx)$

Thus the null space of T^t is preceisely the annihilator of the range of T

ie, $N(T^t) = [R(T)]^\circ \longrightarrow ①$

suppose that, V and W are finite dimensional

say, $\dim V = n$, $\dim W = m$

i) let r be the rank of T

ie, $r = \rho(T) = \dim R(T)$

The dimension of the range of T

By thm,

Let V be finite dimensional vector space over the field F, let W be a subspace of V, then

$\dim W + \dim W^\circ = \dim V$.

Now, $R(T)$ is a subspace of W

$\dim R(T) + \dim [R(T)]^\circ = \dim W$

$\dim [R(T)]^\circ = \dim W - \dim R(T)$

$= m - r$

The annihilator of the range of T, that has the dimensional $m-r$.

$\therefore$

By using ① we get,

$\dim N(T^t) = m - r$

But T^t is a linear transformation on an m - dimensional space from W^* into V^*

W.K.T

rank (T) + Nullity $(T) = \dim V$

$\rho(T^t) = \dim W^* -$ Nullity of T^t

$= \dim W -$ Nullity of T^t

$= m - (m - r)$

$= r$

T and T^t have the same rank

ie, $\rho(T) = \rho(T^t)$

ii) Let N be the null space of T. Every functionals is the range of T^t is in the annihilator of N

Let $f \in T^t$, g for some $g \in W^*$

Then $x \in N$,

$f(x) = T^t g(x)$

$= g\,T(x)$

$= g(0) = 0$

Now, the range of T^t is subspace of the space $[N(T)]^0$

ie, $R(T^t) \subseteq [N(T)]^0$

$\dim [N(T)]^0 = n - \dim N(T)$

$= \dim V - \dim N(T)$

$= (\dim R(t) + \dim N(T)) - \dim N(T)$

$= \dim R(t)$

$= \rho(T) = \rho(T^t)$

$= \dim R(T^t)$

$[N[t]]^0 = R(T^t)$

So the range of T^t must be exactly.

$[N(T)]^0$ H/P.

Thm : 22.

Let V, W be finite dimensional vector space over F. Let B an ordered basis for V with dual basis B^* and let B' be an ordered basis for W with dual basis B'^*. Let T be a linear transformation T from V into W. Let A be the matrix of T relative to B, B' and let B be the matrix of T^t relative to B'^*, B^* then $B_{ij} = A_{ji}$

Proof :-

Let $B = \{\alpha_1, \ldots, \alpha_n\}$, $B' = \{\beta_1, \ldots, \beta_m\}$

$B^* = \{f_1, \ldots, f_n\}$, $B'^* = \{g_1, \ldots, g_m\}$

By definition.

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i \quad , j = 1, 2, \ldots, n$$

$$T^t g_j = \sum_{i=1}^{n} B_{ij} f_i \, , \; j = 1, 2, \ldots, m \longrightarrow ①$$

On other hand.

$$\cancel{T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i}$$

$$(T^t g_j)(\alpha_i) = g_j(T\alpha_i)$$

$$= g_j\left(\sum_{k=1}^{m} A_{ki}\beta_k\right)$$

$$= \sum_{k=1}^{m} A_{ki}\, g_j \cdot \beta_k$$

$$= \sum_{k=1}^{m} A_{ki}\,\delta_{jk} = A_{ji}$$

$f_i(\alpha_j) = \delta_{ij}$

$i \neq j \Rightarrow 0$

$i = j \Rightarrow 1$

$C_{ij} = \sum_{i=1}^{m} B_{ik} A_{kj}$

for any linear functions f on V

$$f = \sum_{i=1}^{m} f(\alpha_i) f_i$$

If we apply this formula to the functional $f = T^t g_j$ and the fact that

$(T^t g_j)\alpha_i = A_{ji}$ we have. → matrix term.

Convert to linear transformation

so, $\sum^{n} A_{ji} f_i$.

We have $T^t g_j = \sum_{i=1}^{n} A_{ji} f_i \longrightarrow ③$

from ①

$\sum_{i=1}^{n} B_{ij} f_i = \sum_{i=1}^{n} A_{ji} f_i$

$\sum_{i=1}^{n} B_{ij} f_i - \sum_{i=1}^{n} A_{ji} f_i = 0$

basis element of B^* also, not equal to zero.

$\sum_{i=1}^{n} (B_{ij} - A_{ji}) f_i = 0$

$B_{ij} - A_{ji} = 0$

$\therefore B_{ij} = A_{ji}$

H.P

Transpose of Matrix :- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ Interchange row and column.

If A is an $m \times n$ matrix over the field F. the transpose of A is the $n \times m$ matrix A^t defined by

$A^t_{ij} = A_{ji}$

Thm :-

Statement :-

Let A be any $m \times n$ matrix over F then the row rank of "A" is equal to the column rank of "A"

Proof :- $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \Rightarrow F^n$. B basis of F^n.

Let B be the standard ordered basis for F^n

$B' = \{\beta_1, \ldots, \beta_m\} \Rightarrow F^m$ B' basis of F^m

and B' be the standard ordered basis for F^m

domain as column. $T: F^n \to F^m$. codomain as row

Let "T" be the L.T from F^n into F^m such that the matrix of T relative to the pair B, B' in A

$A = [x_{ij}]_{m \times n}$ row column

ie, $T(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m)$

Where, $y_i = \sum_{j=1}^{n} A_{ij} x_j$

The column rank of A is the rank of the transportation T, because the range T consists of all m-triples

Which are linear combination of the column Vectors of A

Relative to the dual basis B'^* and B^* the transpose mapping T^t is represented by the matrix A^t

Since the column of A^t are the rows of A.

B the same reasons

The row rank (the column rank of A^*) is equal to the rank of T^t

By thm,

$$\text{Rank}(T^t) = \text{rank}(T)$$ we have T and T^t have the same rank

Hence the row rank of A is equal to the column rank of A.

Note :-

If A is $m \times n$ matrix over F and T is the L.T from F^n into F^m defined above then,

$$\text{rank}(T) = \text{row rank}(A)$$

$$= \text{column rank}(A)$$

We say simply the rank of A.

Trace :-

sum of diagonal values is Trace.

Let n be a +ve integer and F be a field If A is an $n \times n$ matrix with entries in F. the trace of A is the scalar.

$$\text{tr}\, A = A_{11} + A_{12} + \cdots + A_{nn}.$$

The trace function is a linear functionals on the matrix space $F^{n \times m}$ because

$$\text{tr}(CA + B) = \sum_{i=1}^{n} (CA_{ii} + B_{ii})$$

$$c \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$$= C\,trA + trB$$

Null space :-

The vector $\alpha_1 = (1,2)$, $\alpha_2 = (3,4)$ are L.I and form a basis for R^2. There is a unique linear transformation from $R^2 \rightarrow R$ such that $T\alpha_1 = (3,2,1)$; $T\alpha_2 = (6,5,4)$ find that $T(1,0)$

Soln:-

If $(1,0) = c_1(1,2) + c_2(3,4)$

$(1,0) = [c_1 + 3c_2, c_1 + 4c_2]$

$c_1 + 3c_2 = 1 \longrightarrow (1)$

$2c_1 + 4c_2 = 0 \longrightarrow (2)$

$(1) \times 2 \rightarrow \quad 2c_1 + 6c_2 = 2$

$2c_1 + 4c_2 = 0$

$2c_2 = 2$

$\boxed{c_2 = 1}$

$(1) \Rightarrow \quad c_1 + 3(1) = 1$

$c_1 = 1 - 3$

$\boxed{c_1 = -2}$

$(1,0) = -2(1,2) + 1(3,4)$

$= -2T(1,2) + T(3,4)$

$= -2(3,2,1) + (6,5,4)$

$= (-6,-4,-2) + (6,5,4)$

$T(1,0) = 0,1,2.$

UNIT - I

Linear Transformation

Definition :-

Let V and W be the vector spaces over the field F

A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha+\beta) = c\,(T\alpha) + T\beta$$

For all α and β in V are all scalars c in F.

Zero Transformations :-

If V is any vector space the zero transformation '0' is defined by $0\alpha = 0$ is a linear transformation from V into V

Identity Transformation :-

If V is any vector space the identity transformation 'I' defind by $I\alpha = \alpha$ is a linear transformation from V into V.

Example :-

1) Let F be a field and let V be the vector space of polynomical function f from F into F give by

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_k x^k$$

Let ,

$D.f(x) = c_1 + 2c_2 x + \ldots + kc_k x^{k-1}$

Then D is a linear transformation from V into V the differentiation transformation.

2) Let A be a fixed $m \times n$ matrix with entries in the field F. The function T defined by $T(x) = A(x)$ is linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$

The function U defined $U(\alpha) = \alpha A$ is a linear transformation from F^m into F^n

3) Let R be the field of real number and Let V be the vector space of all functions from R into R, which are continuous def T by.

$$(Tf)x = \int_0^x f(t)\, dt$$

Then T is an linear transformation from V into V.

The function T_f is not only continuous but has continuous first derivative

The linearity of integration is one of its fundamental property.

Theorem :1

Let V be a finite dimensional vector space over the field F and Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that

$$T\alpha_j = \beta_j \quad ; \; j = 1, 2, \ldots, n.$$

Proof:-

Given, α in V.

There is a unique n-tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

For this vector α, we define

$$T\alpha = x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n \longrightarrow (1)$$

Then T is a well defined rule for associating with each vector α in V a vector $T\alpha$ in W

From definition, it clear that $T\alpha_j = \beta_j$ for each j

To prove:-

T is linear

Let $\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n$ be in V and

Let c be any scalar

Now,

$$c\alpha + \beta = c(x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n) + (y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n)$$

$$= (cx_1 + y_1)\alpha_1 + (c_2x_2 + y_2)\alpha_2 + \cdots + (cx_n + y_n)\alpha_n$$

and so, By definition.

$$T(c\alpha + \beta) = T(cx_1 + y_1)\alpha_1 + T(cx_2 + y_2)\alpha_2 + \cdots + T(cx_n + y_n)\alpha_n$$

$$= (cx_1 + y_1)\beta_1 + (cx_2 + y_2)\beta_2 + \cdots + (cx_n + y_n)\beta_n$$

$$= c(x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n) + (y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n)$$

$$= cT\alpha + T\beta.$$

To prove:

T is uniquiness

Let U is a linear transformation from V into W with $U\alpha_j = B_j$ $(j = 1, 2, \dots, n)$ Then for the vector $\alpha = \sum_{i=1}^{n} x_n \alpha_i$

We have,

$$U_\alpha = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U\alpha_i)$$

$$= \sum_{i=1}^{n} x_i \beta_i$$

$$U_\alpha = T_\alpha \Rightarrow U = T.$$

Hence T is linear transformation from V into W with $T\alpha_j = B_j$ is unique.

Example :1

The vector $\alpha_1 = (1, 2)$, $\alpha_2 = (3, 4)$ are linearly independent and therefore form a basis of R^2 there is the unique linear Transformation from R^2 into R^2 such that,

$$T\alpha_1 = (3, 2, 1)$$

$$T\alpha_2 = (6, 5, 4) \text{. Find } T(1, 0)$$

Soln:-

Let $\alpha = c_1\alpha_1 + c_2\alpha_2$

$$(1, 0) = c_1(1, 2) + c_2(3, 4)$$

$$(1, 0) = (c_1 + 3c_2, 2c_1 + 4c_2)$$

$$c_1 + 3c_2 = 1 \longrightarrow (1)$$

$$2c_1 + 4c_2 = 0 \longrightarrow (2)$$

$① \times 2 - ② \Rightarrow \quad 2c_1 + 6c_2 - 2c_1 - 4c_2 = 2 - 0$

$$2c_2 = 2$$

$$c_2 = 1$$

$c_2 = 1$ sub in ① we get

$$c_1 + 3(1) = 1$$

$$c_1 = -2$$

$$(1,0) = -2(1,2) + (3,4)$$

$$T(1,0) = -2T(1,2) + T(3,4)$$

$$= -2(3,2,1) + (6,5,4)$$

$$T(1,0) = (-6+6, -4+5, -2+4)$$

$$T(1,0) = (0,1,2)$$

Example : 2

Let P be a fixed $m \times m$ matrix with entries in the field F and let Q be a fixed $n \times n$ matrix over F. Define a function T from the space, $F^{m \times n}$ into itself by $T(A) = PAQ$. Then T is linear transformation from $F^{m \times n}$ into $F^{m \times n}$.

soln:-

$$T(cA+B) = P(cA+B)Q$$

$$= PcAQ + PBQ$$

$$= c(PAQ) + PBQ$$

$$= cT(A) + T(B)$$

$$T(cA+B) = cT(A) + T(B)$$

$\therefore$ T is linear transformation $F^{m \times n}$ into $F^{m \times n}$

H.P.

Null space :-

Let V and W be a vector space over the field F and T be a linear transformation from V into W

The Null space of T is the set of all vector x in V. such that $Tx = 0$.

Rank of T.

If V is finite dimensional. The rank of T is the dimensional of the range of T.

ie) $\text{rank } T = \dim \{ v \in U \mid TU \}$

Nullity of T.

If V is finite dimensional. The nullity of T is dimensiona of the null space of T.

$$\text{Nullity of } T = \dim \{ v \in V \mid Tv = 0 \}.$$

Theorem : 2.

Let V and W be a vector space over the field F and Let T be a linear transformation from V into W suppose that V is finite dimensional then,

$$\text{rank } (T) + \text{nullity } (T) = \dim V.$$

Proof:-

Let $\dim V = n$ (say)

since, The null space N is subspace of V

Assume, $\dim N = k$ (say)

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis of N.

Then there exists vectors $\{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$, in V

such that, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V.

$\Rightarrow \{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is range of T.

To prove that,

$\{T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)\}$ is a basis for range of T.

(i) <u>To prove:-</u>

$\{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ span of range of T

Let $\beta \in$ range of T, There exist $\alpha \in V$ such that, $T(\alpha) = \beta$

$T(\alpha) = \beta$

Now, $\alpha \in V$, Then there exists $a_1, a_2, \ldots, a_n$ are in F. such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$$

$$T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n)$$

$$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n).$$

since, $T(\alpha_j) = 0$ for $j \leq k$.

$$\therefore T(\alpha) = a_1 \cdot 0 + \ldots + a_k \cdot 0 + a_{k+1} T(\alpha_{k+1}) + \ldots + a_n T(\alpha_n)$$

$$\beta = a_{k+1} T(\alpha_{k+1}) + \ldots + a_n T(\alpha_n)$$

$\Rightarrow \beta$ is a linear combination of $(T(\alpha_{k+1}, \ldots, T(\alpha_n))$

Hence, $\{T(\alpha_{k+1}) \ldots T(\alpha_n)\}$ span of range of T. (1)

(ii) <u>To prove:-</u>

$\{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is linearly Independent

suppose we have scalars c_i.

such that

$$\sum_{i=k+1}^{n} c_i \, T(\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i$ is in null space of T

$\Rightarrow \sum_{i=k+1}^{n} c_i \alpha_i = \sum_{j=1}^{k} b_j \alpha_j$ with $b_j \in F$

$$\Rightarrow \sum_{j=1}^{k} b_j \alpha_j - \sum_{i=k+1}^{n} c_i \alpha_i = 0$$

Since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a linearly independent

$\Rightarrow b_1 = b_2 = \dots = b_k = c_{k+1} = \dots = c_n = 0.$

Thus,

$$\sum_{i=k+1}^{n} c_i \, T(\alpha_i) = 0,$$

$\Rightarrow c_i = 0 \; ; \; i = k+1, \dots, n$

$\Rightarrow \{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ is linear independent for the range of T. $\longrightarrow$ (2)

From (1) and (2)

$\Rightarrow \{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ from a basis for range of T

$\Rightarrow \dim(\text{range of } T) = n - k$

$\Rightarrow \text{rank}(T) = \dim V - \dim N$

$\Rightarrow \text{rank}(T) = \dim V - \text{nullity } T$

$\Rightarrow \text{rank}(T) + \text{nullity}(T) = \dim V$

H/P

Theorem : 3

If A is an $m \times n$ matrix with entries in the field F, Then

row rank (A) = column rank (A)

Proof:-

Let T be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$ defined by $\boxed{T(x) = A(x)}$

The null space of T is the solution space for the system $AX = 0$

→ The set of all column matri X. such that $AX = 0$

$T(x) = A(x)$

The range of T is the set of all $m \times 1$ column matrices Y such that $AX = Y$ has a solution of X.

If $A_1, A_2, \ldots, A_m$ are the columns of A, Then

$$AX = x_1 A_1 + x_2 A_2 + \ldots + x_n A_n.$$

so that the range of T is the subspace spanned by the columns of A.

In other words,

The range of T is column space of A.

⇒ rank T = column rank (A)

If S is the solution space for the system $AX = 0$, Then.

$$\dim S + \text{column rank } (A) = n \longrightarrow ①$$

W.k.T

dimension of the soln space $AX = 0$ = $\dim A$ - no. of linear independent rows of A

$\Rightarrow \dim S = n - \text{row rank}(A)$

$\Rightarrow \dim S + \text{row rank}(A) = n \longrightarrow (2)$

From (1) & (2) we get,

$\Rightarrow \text{row rank}(A) = \text{column rank}(A)$

H.P

L.T $\Rightarrow T(x_1, x_2) = (x_2, x_1)$

$$T(\alpha x + y) = \alpha T(x) + T(y)$$

$$T(x+y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

$$\begin{aligned}
T(\alpha x + y) &= T(\alpha(x_1, x_2) + (y_1, y_2)) \\
&= T((\alpha x_1, x_2) + (y_1, y_2)) \\
&= T((\alpha x_1 + y_1)(\alpha x_2 + y_2)) \\
&= (\alpha x_2 + y_2, \alpha x_1 + y_1) \\
&= (\alpha x_2, x_1) + (y_2, y_1) \\
&= \alpha T(x_1, x_2) + T(y_1, y_2) \\
&= \alpha T(x) + T(y)
\end{aligned}$$

$\therefore$ This is linear Transformation

The Algebra of Linear Transformation:-

Thm : 4

Let V and W be vector space over the field F. Let T and U be linear transformation from V° into W. The function $(T+U)$ defined by $(T+U)(\alpha) = T\alpha + U\alpha$ is a linear transformation from V into W. If c is any element in F, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is linear transformation from V into W.

Proof:-

suppose T and U are linear transformation from V into W and (T+U) defined by

$$(T+U)\alpha = T\alpha + U\alpha$$

Now,

$$\begin{aligned}(T+U)(c\alpha+\beta) &= T(c\alpha+\beta) + U(\alpha+\beta)\\ &= T(c\alpha) + T(\beta) + U(\alpha) + U\beta\\ &= c(T\alpha+U\alpha) + (T\beta+U\beta)\\ &= c(T+U)\alpha + (T+U)\beta\end{aligned}$$

Which show that T+U is linear transformation

Next,

$$\begin{aligned}(cT)(d\alpha+\beta) &= c[T(d\alpha+\beta)]\\ &= c[d(T\alpha) + T\beta]\\ &= cd(T\alpha) + cT\beta\\ &= d[c(T\alpha)] + c(T\beta)\\ &= d[(cT)\alpha] + (cT)\beta\end{aligned}$$

Which shows that cT is linear transformation.

Note :-

$L(V, W) \rightarrow$ The space of linear transformation from V into W.

Thm : 5

Let V, W and Z be vector space over the field F. Let T be a linear transformation from V into W and U is a linear transformation from W into Z. Then the composed function UT is defined by $(UT)\alpha = U(T(\alpha))$ is a linear transformation from V into Z.

Proof :-

$$(UT)(c\alpha+\beta) = U[T(c\alpha+\beta)]$$
$$= U[cT\alpha + T\beta]$$
$$= U[cT\alpha] + U[T\beta]$$
$$= c[U(T\alpha)] + U[T(\beta)]$$
$$= c[(UT)(\alpha)] + [(UT)(\beta)]$$

$\therefore$ UT is a linear Transformation from V into Z.

Linear operator :-

If V is a vector space over the field F, a linear operator on V is a linear transformation V into V.

Lemma :-

Let V be a vector space over the field F. Let U, T_1 and T_2 be linear operator on V. Let c be an element of F.

a) $IU = UI = U$

b) $U(T_1+T_2) = UT_1 + UT_2$; $(T_1+T_2)U = T_1U + T_2U$

c) $c(UT_1) = (cU)T_1 = U(cT_1)$

Proof :-

a) Given U be the linear operator on V.

since, I is the identity functions

$\Rightarrow UI = IU = U$

$\therefore$ is obviously true.

b) $U[T_1 + T_2](\alpha) = (T_1 + T_2)(U\alpha)$

$= T_1(U\alpha) + T_2(U\alpha)$

$= (T_1U)(\alpha) + (T_2U)(\alpha)$

so that,

$$(T_1 + T_2)U = T_1U + T_2U$$

c) $c(UT_1)(\alpha) = cU[T_1(\alpha)]$

$= (cU)T_1(\alpha)$

$= Uc\ T_1(\alpha)$

$= U(cT_1)(\alpha)$

$$c(UT_1) = (cU)T_1 = U(cT_1)$$

H.P.

Thm :-

The set of all linear transformation from V into W together with the addition and scalar multiplication defined by $(T+U)\alpha = T\alpha + U\alpha$ and $(cT)\alpha = c(T\alpha)$ is vector space over the field F.

Proof :-

Let $V(V, W)$ is the set of all linear transformation from V into W.

Defined by.

$$(T+U)\alpha = T\alpha + U\alpha \longrightarrow (1)$$

$$(cT)\alpha = c(T\alpha) \longrightarrow (2).$$

To prove :-

$L(V, W)$ is vector space over F.

i) <u>Closure Law</u> :-

Let $T_1, T_2 \in L(V, W)$, $\alpha \in V$

$(T_1 + T_2)\alpha = T_1\alpha + T_2\alpha$

$\therefore$ Closure Law is true

ii) <u>Associative Law</u>

Let $T_1, T_2, T_3 \in L(V, W)$. $\alpha \in V$

$$[(T_1+T_2)+T_3]\alpha = (T_1+T_2)\alpha + T_3\alpha \quad \text{by (1)}$$
$$= T_1\alpha + T_2\alpha + T_3\alpha$$
$$= T_1\alpha + (T_2+T_3)\alpha$$
$$= [T_1 + (T_2+T_3)]\alpha$$

$\therefore (T_1+T_2)+T_3 = T_1 + (T_2+T_3)$.

$\therefore$ Associative Law is true

(iii) <u>Existence Identity</u> :-

Let $0 \in L(V, W)$, $\alpha \in V$

Consider

$$(T+0)\alpha = T\alpha + 0\alpha$$
$$= T\alpha$$

$\Rightarrow (T+0) = T$

$\Rightarrow (T+0) = (0+T) = T$

$\therefore$ Identity law is true .

iv) <u>Existence Inverse</u> :-

Let $T \in L(V, W)$, Then there exists $-T \in L(V, W)$

Consider

$$[T+(-T)]\alpha = T\alpha + (-T\alpha)$$
$$= 0\alpha$$

$T+(-T) = 0$

$\therefore T+(-T) = (-T)+T = 0$

$\therefore$ Inverse law is true.

(v) Commutative law:

Let $T, U \in L(V, W)$, $\alpha \in V$

consider,

$$(T+U)\alpha = T\alpha + U\alpha$$

$$= U\alpha + T\alpha$$

$$(T+U)\alpha = (U+T)\alpha$$

$$T+U = U+T$$

$\therefore$ commutative law is true.

$\therefore$ $L(V, W)$ is a abelian group.

(vi) $1.T = T$

consider $(1.T)\alpha = 1.T\alpha$ by ②

$$= T\alpha$$

$$1.T = T$$

(vii) $(c_1 . c_2)T = c_1(c_2T)$, $c_1, c_2 \in F$

consider

$$[(c_1, c_2)T](\alpha) = (c_1c_2)T\alpha \quad \text{by ②}$$

$$= c_1[c_2T\alpha]$$

$$= c_1[c_2T]\alpha$$

$$\therefore (c_1c_2)T = c_1(c_2T)$$

(viii) $(c_1+c_2)T = c_1T + c_2T$; $c_1, c_2 \in F$

Now,

$$[(c_1+c_2)T]\alpha = (c_1+c_2)T\alpha$$

$$= c_1T\alpha + c_2T\alpha.$$

$$= (c_1T)\alpha + (c_2T)\alpha$$

$$= (c_1T + c_2T)\alpha$$

$$\therefore (c_1+c_2)T = c_1T + c_2T$$

ix) $c(T+U) = cT + cU$

Now,

$$[c(T+U)]\alpha = c[(T+U)\alpha] \quad \text{by } ③$$

$$= c[T\alpha + U\alpha] \quad \text{by } ①$$

$$= cT\alpha + cU\alpha$$

$$= (cT)\alpha + (cU)\alpha$$

$$= (cT + cU)\alpha$$

$$\therefore c(T+U) = cT + cU$$

$\therefore$ $L(V, W)$ is vector space over F.

H.P.

Thm :-

Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then the space $L(V, W)$ is finite dimensional and has dimension mn.

Proof:-

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

$B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$

be ordered bases for V and W respectively.

For each pair of integers (p, q) with $1 \le p \le m$ and $1 \le q \le n$.

We define a linear transformation $E^{p,q}$ from V into W by

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

$$E^{p,q}(\alpha_j) = \delta_{iq}\beta_p$$

ie) $$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \ne q \\ \beta_p & \text{if } i = q. \end{cases}$$

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

$E^{p,q}(\alpha_j) = \delta_{jq}\beta_p$

According to the theorem,

Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let $\beta_1, \beta_2, \ldots, \beta_n$ be any vector in W. Then there is precisely one linear transformation from V into W

such that,

$$T\alpha_j = \beta_j \quad (j = 1, 2, \ldots, n)$$

$\therefore$ There is a unique transformation from V into W satisfying these conditions.

Claim :-

The mn transformations $E^{p,q}$ for a basis for $L(V, W)$

Let T be a linear transformation from V into W.

For each j, $1 \leq j \leq n$.

Let $A_{1j}, \ldots, A_{mj}$ be the coordinates of vector $T\alpha_j$ in the ordered bases $\mathcal{B}$.

$$\text{ie) } T\alpha_j = \sum_{p=1}^{m} A_{pj} \cdot \beta_p \longrightarrow (1)$$

We wish to show that,

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \, E^{p,q} \longrightarrow (2).$$

Let U be the linear transformation in the right hand number of (2)

Then for each j

$$Ux_j = \sum_p \sum_q A_{p,q} \, E^{p,q}(x_j)$$

$$= \sum_p \sum_q A_{pq} \, \delta_{jq} \, \beta_p$$

$$= \sum_{p=1}^{m} A_{pj} \, \beta_p$$

$$Ux_j = Tx_j$$

$$U = T$$

To show that

The $E^{p,q}$ span $L(V, W)$

We must prove that,

They are Independent.

But this is clear from the transformation

$U = \sum_p \sum_q A_{pq} E^{p,q}$ is a zero

transformation then $U\alpha_j = 0$ for each j

so, $\sum_{p=1}^{m} A_{pj} \beta_p = 0$

and the independence of the β_p

$\Rightarrow A_{pj} = 0$ for every p and j.

Hence, the space $L(V, W)$ is finite dimensional and has dimension mn.

Invertible :-

The function T from V into W called invertible if there exists a function U from W into V such that UT is the identity function on V and TU is the identity function on W

(ie) $UT = TU = I$

If T is invertible, then function U is denoted by T^{-1}

Note :-

If T is invertible iff

1. T is 1:1
2. T is onto

Theorem : 7

Let V and W be vector space over the field F and let T be a linear transformation from V into W. If T is invertible, then the inverse function T^{-1} is a linear transformation from W into V.

Proof:-

Let β_1 and β_2 be vectors in W and c be a scalar

To show that

$$T^{-1}(c\beta_1+\beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2.$$

$$\text{Let } \alpha_i = T^{-1}\beta_i, \quad i = 1,2$$

(ie) Let α_1 and α_2 be the unique vectors in V such that $T\alpha_i = \beta_i$

since T is linear

$$T(c\alpha_1 + \alpha_2) = cT\alpha_1 + T\alpha_2$$

$$= c\beta_1 + \beta_2$$

Thus $c\alpha_1+\alpha_2$ is unique vector in V which is sent by T into $c\beta_1 + \beta_2$ and so,

$$c\alpha_1 + \alpha_2 = T^{-1}(c\beta_1+\beta_2)$$

$$\Rightarrow T^{-1}(c\beta_1+\beta_2) = c(T^{-1}\beta_1) + T^{-1}(\beta_2)$$

$\therefore$ T^{-1} is linear

H.P.

Note :-

1) If T is linear, then $T(\alpha-\beta) = T\alpha - T\beta$

2) Let T be invertible L.T from V onto W and U be invertible L.T from W onto Z, Then

(i) UT is invertible

(ii) $(UT)^{-1} = T^{-1}U^{-1}$

Non - singular :-

A linear transformation T is non singular if $T\gamma = 0$ implies $\gamma = 0$.

(ie) if the null space of T is $\{0\}$.

Note :-

* T is 1:1 iff T is non-singular.

* T is non-singular then T is linear independence.

Theorem : 8

Let T be a linear Transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Proof :-

First suppose T is non-singular

Let S be a linearly independent subset of V.

To prove :-

If $\alpha_1, \alpha_2, \dots, \alpha_k$ are vectors in S Then the vectors $T\alpha_1, T\alpha_2, \dots, T\alpha_k$ are linearly independent

If $c_1(T\alpha_1) + c_2(T\alpha_2) + \dots + c_k(T\alpha_k) = 0$

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k) = 0$

since T is non singular

$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k = 0$

It follows that each $c_i = 0$ because S is an independent set

$\Rightarrow$ The image of S under T is independent

$\Rightarrow$ T carries each linear independent.

Next,

suppose that, T carries independent subset onto independent subsets.

To prove :-

T is non singular

Let α be a non zero vectors in V.

Then the set S consisting of the one vector α is independent.

The image of S is the set consisting of the one vector $T\alpha$ and this set is independent

$\therefore T\alpha \neq 0$, because the set consisting of the zero vector alone is dependent

$\therefore$ The null space of T is the zero subspace

$\therefore$ T is non-singular

H.P.

Theorem : 9

Let V and W be finite dimensional vector space over the field F such that dim V = dim W. If T is linear transformation from V into W, the following are equivalent

(i) T is invertible

(ii) T is non-singular

(iii) T is onto (ie) The range of T is W

(iv) If $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is basis for V, the $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is basis for W.

(v) There is some basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for V. such that $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a basis for W.

Proof :-

(i) $\rightarrow$ (ii)

Assume that T is invertible

To show that,

T is non-singular.

(ie) $Tv = 0$ if $v = 0$ $\forall$ $v \in V$

W.k.T

T is invertible iff T is 1-1 and onto

Now, $TV = 0$

$TV = T(0)$

Since T is 1-1

$V = 0$.

$\therefore$ T is singular.

(ii) $\leftrightarrow$ (iii)

Assume that T is non-singular

To prove :-

T is onto

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is basis for V.

By theorem ⑧, $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a linearly independent in W

Since T is non-singular

$\Rightarrow$ Nullity $(T) = 0$

W.k.T

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

$$\Rightarrow \text{rank}(T) = \dim V$$

Since $\dim V = \dim W$

$$\Rightarrow \text{rank}(T) = \dim W$$

Now, Let β be any vector in W

There are scalars $c_1, c_2, \dots, c_n$ such that

$$\beta = c_1(T\alpha_1) + \dots + c_n(T\alpha_n)$$
$$= T(c_1\alpha_1 + \dots + c_n\alpha_n).$$

$\Rightarrow \beta$ is in the range of T

$\therefore$ T is onto

(iii) $\rightarrow$ (iv)

Assume that T is onto

To prove :-

If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is basis for V then $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is basis for W.

If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is any basis for V the vector $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ span of range of T

since dim W = n.

These n vectors must be linearly independent

$\Rightarrow \{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis for W

(iv) $\rightarrow$ (v)

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a some basis for V.

from (iv)

$\Rightarrow \{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis of W.

(v) $\rightarrow$ (i)

Assume that there is a some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for V then $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is basis for W.

To prove :-

T is invertible.

It is enough to show that T is one to one and onto.

Since the $T\alpha_i$ span W

It is clear that the range of T is all of W

If $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$ is in the null space of T_1 then

$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0.$

$\Rightarrow c_1(T\alpha_1) + c_2(T\alpha_2) + \cdots + c_n(T\alpha_n) = 0.$

since the $T\alpha_i$ are independent each $c_i = 0$

Thus $\alpha = 0$, we have

show that the range of T is W and T is non singular

Hence T is invertible

H.P.

Def: Groups :-

A Group consists of the following.

1. A set G

2. A rule (or operation) which associates with each pair of elements xy in G in such a way that (a) $x(yz) = (xy)z$ for x, y and z in G

(b) There is an element e in G.

such that $ex = xe = x$ for every x in G

(c) To each element x in G there corresponds an element x^{-1} in G such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Commutative :-

A group is called commutative if it satisfies the condition $xy = yx$ for each x and y

Field :-

A field can be described as a set with two operations called addition and multiplication

Which is a commutative group under addition and in which the non-zero elements form a commutative group under multiplication with the Distributive law.

$$x(y+z) = xy + xz \text{ holding.}$$

Isomorphism :-

If V and W are vector space over the field F, any one to one linear transformation T of V onto W is called an isomorphism of V onto W.

If there exist an isomorphism of V onto W we say that V is isomorphic to W.

Theorem : 10

Every n-dimensional vector space over the field F is isomorphic of the space F^n.

<u>Proof</u>:-

Let V be an n-dimensional space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.

Let $\alpha \in V$ then $\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for all x_i in F

We define map $T : V \longrightarrow F^n$ by

$$T\alpha = \{x_1, x_2, \ldots, x_n\} \quad (\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.)$$

Where x_i is the coordinate of α.

$(T\alpha = x_1, x_2, \ldots, x_n$

$\beta = \beta_1 y_1, \beta_2 y_2, \ldots, \beta_n y_n$

$T\beta = y_1, y_2, \ldots, y_n)$

<u>To prove</u> :-

T is linear transformation

scalar

Let $\alpha, \beta \in V$ and $c \in F$, then

$$\alpha = \sum_{i=1}^{n} \alpha_i x_i \text{ and } \beta = \sum_{i=1}^{n} \alpha_i y_i$$

consider $T(\alpha+\beta) = T\left(c\sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \alpha_i y_i\right)$

$$= T\left(\sum_{i=1}^{n} c\alpha_i x_i + \sum_{i=1}^{n} \alpha_i y_i\right)$$

$$T(\alpha+\beta) = T\left(\sum_{i=1}^{n} (cx_i + y_i)\alpha_i\right)$$

$$= \{cx_1+y_1, cx_2+y_2, \dots, cx_n+y_n\}$$

$$= \{cx_1, cx_2, \dots, cx_n\} + \{y_1, y_2, \dots, y_n\}$$

$$T(\alpha+\beta) = cT\alpha + T\beta$$

$\therefore$ T is linear transformation

Next
To prove :-

T is one to one.

since every $\alpha \in V$, There is a unique coordinate matrix.

$\therefore$ T is 1-1

Next to prove

T is onto

Let $x = \{x_1, x_2, \dots x_n\} \in F^n$

Then clearly, $\alpha \in V$

$\Rightarrow T\alpha = x$

$\therefore$ T is onto.

$\therefore$ T is isomorphic.

$\therefore$ T is isomorphic.

Hence every n-dimensional vector space over F is isomorphic to the space F^n.

H.P.

Representation of Transformation matrices:-

Let V be n-dimensional vector space over the field F and W be m-dimensional vector space over F.

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ an ordered basis for V and $\mathcal{B}' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ an ordered basis for W

If T is an linear transformation from V into W

Then T is determined by its action on the vectors α_j.

Each of the n-vectors $T\alpha_j$ is unique expressible as a linear combination.

$$T\alpha_j = \sum_{i=1}^{m} A_{ij} \beta_i$$

of the β_i, the scalars $A_{1j}, \ldots, A_{nj}$ being the coordinates of $T\alpha_j$ in the ordered basis $\mathcal{B}'$.

The $m \times n$ matrix A defined by $A(i,j) = A_{ij}$ is called the matrix of T relative to the pair of ordered basis $\mathcal{B}$ and $\mathcal{B}'$.

Theorem: 11

Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. for each pair of ordered bases $\mathcal{B}, \mathcal{B}'$ for V and W respectively, the function which assigns to be a linear transformation T its matrix relative to $\mathcal{B}, \mathcal{B}'$ is a isomorphism between the space $L(V, W)$ and the space of all $m \times n$ matrices over the field F.

Proof:-

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

$B' = \{\beta_1, \beta_2, \ldots, \beta_n\}$

Let M be a vector space all $m \times n$ matrices over field F.

Let $\psi : L(V, W) \longrightarrow M$ such that

$$\psi(T) = [T : B : B'] \;\forall\; T \in L(V, W)$$

$$= [a_{ij}]_{m\times n}$$

Let $T_1, T_2 \in L(V, W)$

Let $[T_1 . B . B'] = [a_{ij}]_{m\times n}$

$[T_2, B, B'] = [b_{ij}]_{m\times n}$

$$T_1(\alpha_j) = \sum_{i=1}^{m} a_{ij}\, \beta_j \quad , \quad 1 \le j \le n$$

$$T_2(\alpha_j) = \sum_{i=1}^{m} b_{ij}\, \beta_j \quad ; \quad 1 \le j \le n$$

To prove.

ψ is 1-1

consider $\psi(T_1) = \psi(T_2)$

$$\Rightarrow [T_1, B, B'] = [T_2, B, B']$$

$$\Rightarrow [a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}$$

$$\Rightarrow a_{ij} = b_{ij}$$

$$\Rightarrow \sum_{i=1}^{m} a_{ij}\, \beta_j = \sum_{i=1}^{m} b_{ij}\, \beta_j$$

$$\Rightarrow T_1 \alpha_i = T_2 \alpha_j$$

$$\Rightarrow T_1 = T_2$$

$\therefore \psi$ is 1-1.

To prove:-

ψ is onto

Let $[a_{ij}]_{m\times n} \in M$, Then $\exists$ a linear transformation of T from V into W. such that

$$T\alpha_j = \sum_{i=1}^{m} a_{ij}\, \beta_j \qquad 1 \le j \le n$$

We have,

$$[T, B, B'] = [a_{ij}]_{m\times n}$$

$$\Rightarrow \psi(T) = [a_{ij}]_{m\times n}$$

$\therefore \psi$ is onto.

To prove :-

ψ is linear transformation.

If $a, b \in F$, then

$$\psi(aT_1 + bT_2) = [aT_1 + bT_2, B, B']$$

$$= [aT_1, B, B'] + [bT_2, B, B']$$

$$= a[T_1, B, B'] + b[T_2, B, B']$$

$$= a\psi(T_1) + b\psi(T_2)$$

$\therefore \psi$ is linear transformation

$\therefore L(V, W)$ is isomorphic to M.

Example :1

Let F be a field and let T be the operation on F^2 defined by, $T(x_1, x_2) = (x_1, 0)$. Find matrix of T using standard basis of F.

soln :-

Given

$$T(x_1, x_2) = (x_1, 0)$$

Let $\mathcal{B} = \{(1,0), (0,1)\}$

$$T(1,0) = (1,0) = 1 \cdot E_1 + 0 \cdot E_2$$

$$T(0,1) = (0,0) = 0E_1 + 0 \cdot E_2$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example : 2

Let V be the space of all polynomial functional from R into n of the form $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ the space of polynomial functions of degree three or less the differentiation operator D map V into V is defined by $Df(x) = c_1 + 2c_2 x + 3c_3 x^2$ Let B be the ordered basis for V consisting of the four functions f_1, f_2, f_3, f_4 defined by $f_j(x) = x^{j-1}$ find the matrix D in the ordered basis.

soln:-

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$f_j = x^{j-1}$$

$$f_1 = 1, \quad f_2 = x, \quad f_3 = x^2, \quad f_4 = x^3$$

$$D : V \to V$$

$$D f(x) = c_1 + 2c_2 x + 3c_3 x^2.$$

$$D f_1(x) = 0 = 0.f_1 + 0.f_2 + 0.f_3 + 0.f_4$$

$$D f_2(x) = 1 = 1.f_1 + 0.f_2 + 0.f_3 + 0.f_4$$

$$D f_3(x) = 2x = 0.f_1 + 2.f_2 + 0.f_3 + 0.f_4$$

$$D f_4(x) = 3x^2 = 0.f_1 + 0.f_2 + 3f_3 + 0.f_4.$$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem : 13.

Let V, W and Z be the finite dimensional vector space over field F. Let T be a linear transformation from from V into W and U is a linear transformation from W into Z If B, B' and B'' are ordered bases for the space V, W and Z respectively if A is the matrix of T relative to the pair B, B' and B is the matrix of U relative to the pair B', B''.

Then the matrix of the composition UT relative to the pair $\mathcal{B}, \mathcal{B}''$ is the product matrix $C = BA$

Proof:-

~~Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$~~

Let V, W and Z be finite dimensional over F.

$\Rightarrow$ dim $V = n$, dim $W = m$ and dim $Z = p$

Let $T: V \to W$ is linear transformation and $U: W \to Z$ is linear transformation

suppose we have ordered bases

$\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

$\mathcal{B}' = \{\beta_1, \beta_2, \dots, \beta_n\}$

and $\mathcal{B}'' = \{\gamma_1, \gamma_2, \dots, \gamma_p\}$ for the respective space V, W and Z

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times p}$

and $C = [c_{ij}]_{p\times n}$

If α_j is any vector in V.

$$T\alpha_j = \sum_{i=1}^{m} a_{ij} \beta_j \quad 1 \le j \le n$$

If $\beta_j \in W$

$$U\beta_j = \sum_{j=1}^{p} b_{ij} \gamma_j \quad 1 \le j \le m$$

$$UT(\alpha_j) = \sum_{i=1}^{p} c_{ij} \gamma_j \quad 1 \le j \le n$$

If $\alpha \in V$, Then

$$[T\alpha]_{\mathcal{B}} = A[\alpha]_{\mathcal{B}'}.$$

$$[U(T\alpha)]_{\mathcal{B}''} = B[T\alpha]_{\mathcal{B}'}$$

and also

$$[(UT)(\alpha)]_{\mathcal{B}''} = BA[\alpha]_{\mathcal{B}'}$$

We have to show that

$$c_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}.$$

Now,

$$(UT)\alpha_j = U[T\alpha_j]$$

$$\sum_{i=1}^{p} C_{ij}\gamma_i = U\left[\sum_{k=1}^{m} A_{kj}\beta_k\right]$$

$$= \sum_{k=1}^{m} A_{kj}\, U(\beta_k).$$

$$= \sum_{k=1}^{m} A_{kj} \sum_{i=1}^{p} B_{ik}\gamma_i$$

$$\sum_{i=1}^{p} C_{ij}\gamma_i = \sum_{i=1}^{p}\left(\sum_{k=1}^{m} B_{ik}A_{kj}\right)\gamma_i$$

$$\rightarrow C_{ij} = \sum_{k=1}^{m} B_{ik}A_{kj}$$

$$\rightarrow C = BA$$

∴ The matrix C is UT relative to the pair $\mathcal{B}.\ \mathcal{B}''$. is the product matrix $C = BA$

H.P.

Theorem : 13

Let V be a finite dimensional vector space over the field F and let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ and $\mathcal{B}' = \{\alpha_1', \alpha_2', \ldots, \alpha_n'\}$ be ordered bases for V. suppose T is a linear operator on V. If $P = [P_1, P_2, \ldots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\alpha_j']_{\mathcal{B}}$, then $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} \cdot P$.

Alternatively, if U is invertible operator on V defined by $U\alpha_j = \alpha_j'$, $j = 1, 2, \ldots, n$ then

$$[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1} [T]_{\mathcal{B}} [U]_{\mathcal{B}}.$$

Proof:-

Let $\alpha \in V$

$$\alpha = \sum_{j=1}^{n} x_j \alpha_j \longrightarrow ①$$

$$[\alpha]_{\mathcal{B}'} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x \longrightarrow ②$$

since $\beta_j \in V$

$$\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i \longrightarrow ③.$$

$$[\beta_j] = p_j$$

$$① \Rightarrow \alpha = \sum_{j=1}^{n} x_j \left[\sum_{i=1}^{n} p_{ij}\, \alpha_i\right] \qquad y_j = \sum_{i=1}^{n} p_{ij}\, x_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_j \cdot p_{ji}\right) \alpha_i$$

$$[\alpha]_B = P_x \longrightarrow ④$$

$$[\alpha]_B = P[\alpha]_{B'} \longrightarrow ⑤$$

We know that

$$[T\alpha]_{B'} = A[\alpha]_B \longrightarrow ⑥$$

If $B = B'$

$$[T\alpha]_B = [T_B][\alpha B] \text{ and}$$

$$[T\alpha]_{B'} = [T_B][\alpha B]$$

$$[T\alpha]_{B'} = [T]_{B'}[\alpha]_{B'}$$

in ④ $\Rightarrow [T\alpha]_{B'} = P[T\alpha]_{B'}$

since T is linear operator.

$$[T_B][\alpha]_B = P[T]_{B'}\ [\alpha]_{B'}$$

$[\alpha]_B = P[\alpha]_{B'}$

$$[T]_B \cdot P[\alpha]_{B'} = P[T]_{B'} \cdot [\alpha]_{B'} \quad \text{by } ④$$

$$[T]_B P = P[T]_{B'}$$

Pre-multiple by P^{-1} on both sides,

$$P^{-1}[T]_B P = P^{-1} \cdot P[T]_{B'}$$

$$P^{-1}[T]_B P = [T]_{B'} \longrightarrow ⑦$$

since $\beta_j = \sum_{i=1}^{n} p_{ij}\, \alpha_j$

$$U\alpha_j = \beta_j = \sum_{i=1}^{n} p_{ij}\, \alpha_j \qquad (\text{given } \beta_j = U\alpha_j)$$

$$\therefore [U]_B = [\beta_j]_B = P \longrightarrow ⑧$$

$[\text{given } [\beta_j]_B = P]$

sub ⑧ in ⑦ we get.

$$[T]_{B'} = [U]_B^{-1}\,[T]_B[U]_B$$

Definition : similar

Let A and B be an $n \times n$ matrices over the field F. we say that B is similar to A over F if there is an invertible $n \times n$ matrix P over F such that $\boxed{B = P^{-1}AP.}$ then we can say B is similar to A.

Linear Functional :-

If V is a vector space over the field F. a linear transformation f from V into the scalar field F is also called a linear functional on V. If f is a function from V into F. such that

$$f(c\alpha + \beta) = cf(\alpha) + f(\beta)$$

Example :-

Let n be a +ve integer and F be a field. If A is an $n \times n$ matrix with entries in F, the trace of A is the scalar.

$$tr\ A = A_{11} + A_{22} + \dots + A_{nn}$$

The trace function is a linear functional on the matrix space $F^{n \times n}$, because

$$tr(cA + B) = \sum_{i=1}^{n} (cA_{ii} + B_{ii})$$

$$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$$= c\,trA + tr\,B.$$

$cA_{ii} = cA_{11} + cA_{12} + \dots + cA_{nn}$

$= c(A_{11} + A_{12} + \dots + A_{nn})$

$= c\sum_{i=1}^{n} A_{ii}$

Dual space :-

If V is a vector space, the collection of all linear functional on V form a vector space in a natural way.

It is the space $L(V, F)$ we denoted by V^* and call it the dual space.

$$V^* = L(V, F)$$

Note :-

$\dim V = \dim V^*$

Dual Basis :- (Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be basis for V there is a linear functional f_i on V such that $f_i(\alpha_i) = \delta_{ij}$

If $f_1, f_2, \ldots, f_n$ are n linearly independent functional and V^* has dimension n. then $B^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for V^*. This basis is called the dual basis of B.)

Theorem : 14.

Let V be a finite dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Then there is a unique dual basis $B^* = \{f_1, f_2, \ldots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{ij}$. For each linear functional f on V, we have. $f = \sum_{i=1}^{n} f(\alpha_i) f_i$

and for each vector α in V, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

Proof :-

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.

We know that,

Let V be a finite dimensional vector space over F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over F and let $\beta_1, \ldots, \beta_n$ be any vector in W. Then there is a precisely one linear transformation T from V into W such that

$$T\alpha_j = \beta_j \quad ; \quad j = 1, 2, \ldots, n.$$

$\Rightarrow$ $W = F$ and $\{\beta_1, \beta_2, \ldots, \beta_n\} = \{x_1, \ldots, x_n\}$

$\Rightarrow$ There is a precisely one linear functional f

from V into F such that

$$f(\alpha i) = xi$$

$\Rightarrow$ There exist a unique linear functional f_1 such that $f_1(\alpha_1) = 1$, $f_1(\alpha_2) = 0$, ..., $f_1(\alpha_n) = 0$.

Where $\{1, 0, 0, \ldots, 0\}$ is a ordered set, of F scalars for each $i = 1, 2, \ldots, n$

There exist unique functional f on V such that

$$f_i(\alpha_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$= \delta_{ij}$$

To prove :-

$B^* = \{f_1, f_2, \ldots, f_n\}$ is basis of V^*.

First prove B^* is a linearly independent.

Consider

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0.$$

$$\Rightarrow (c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)(\alpha) = 0.$$

$$\Rightarrow \sum_{i=1}^{n} c_i f_i(\alpha) = 0 \quad \forall \alpha \in V$$

Put $\alpha = \alpha_j$

$$\sum_{i=1}^{n} c_i f_i(\alpha_j) = 0.$$

$$\sum_{i=1}^{n} c_i \delta_{ij} = 0.$$

$$c_i = 0$$

$\therefore B^* = \{f_1, f_2, \ldots, f_n\}$ are linearly independent

Next to prove.

Every element f in V^* can be expressed as a linear combination of $f_1, f_2, \ldots, f_n$

ie) $f = \sum_{i=1}^{n} a_i f_i$

Let f be any element in V^*

Let $f(\alpha_i) = a_i \qquad i = 1, 2, \ldots, n \longrightarrow (1)$

Let $\alpha_j \in B$, Where $j = 1, 2, \ldots, n$

Then $\sum_{i=1}^{n} (a_i f_i)(\alpha_i) = \sum_{i=1}^{n} a_i (f_i(\alpha_i))$

$$= \sum_{i=1}^{n} a_i \delta_{ii}$$

$$= a_i$$

$$\sum_{i=1}^{n} (a_i f_i)(\alpha_i) = f(\alpha_i) \Rightarrow \sum_{i=1}^{n} a_i f_i = f \longrightarrow ②$$

$\therefore V^*$ is a linear span of B^* $\therefore B^*$ is a basis of V^*

To Prove. f is in V^*, then

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

From ② $\Rightarrow$ $f = \sum_{i=1}^{n} a_i f_i$

$$f(\alpha_j) = \sum_{i=1}^{n} a_i f_i(\alpha_j)$$

$$= \sum_{i=1}^{n} a_i \delta_{ij} \qquad i=j \longrightarrow ③$$

sub ③ in ② we get.

$$f = \sum_{i=1}^{n} f(\alpha_j) f_i$$

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i \quad (i=j)$$

To prove :-

For each α in V, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha) \alpha_i$$

Now,

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \text{ is a vector in } V \longrightarrow ④.$$

$$f_j(\alpha) = \sum_{i=1}^{n} x_i f_j(\alpha_j)$$

$$= \sum_{i=1}^{n} x_i \delta_{ij}$$

$$f_j(\alpha) = x_j \longrightarrow ⑤$$

sub ⑤ in ④, we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)(\alpha_i)$$

H.P.

Hyper space :-

If a vector space of dimension on n, a subspace of dimension $n-1$ is called hyper space.

Annihilator :-

If V is a vector space over the field F and S is subset of V the annihilator of S is the set S° of linear functionals f on V such that $f(\alpha) = 0$ for every α in S.

Note :-

If S is the set of consisting of zero vector alone then $S^\circ \in V^*$ if $S \in V$, Then S° is a zero subspace of V^*

$V^* \rightarrow$ dual space 10m

Theorem :- 15

Let V be a finite dimensional vector space over the field F, and let W be a subspace of V. then

$$\dim W + \dim W^\circ = \dim V.$$

Proof :-

Let k be the dimension of W and $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ a basis for W

choose vectors $\{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$ in V such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V.

Let $\{f_1, f_2, \ldots, f_n\}$ be a basis of V^* which is dual to this basis of V.

Claim :-

$\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is a basis for the annihilator W°.

Certainly f_i belongs to W° for $i \geq k+1$ because

$$f_i(\alpha_j) = \delta_{ij}$$

and $\delta_{ij} = 0$ if $i \geq k+1$ and $j \leq k$

It follows that, for $i \geq k+1$ $f_i(\alpha) = 0$ whenever α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k$

The functional $f_{k+1}, f_{k+2}, \ldots, f_n$ are linearly independent.

To prove :-

$$W^0 = L(S).$$

(ie) They span W^0.

suppose f is in V^*

Now,

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

so that if f is in W^0, we have

$$f(\alpha_i) = 0 \text{ for } i \leq k$$

and

$$f = \sum_{i=k+1}^{n} f(\alpha_i) f_i$$

Which is show that

$\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is span of W^0

$\therefore$ $\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is basis for the annihilator W^0.

$$\therefore \dim W^0 = n-k$$
$$= \dim V - \dim W$$

$$\Rightarrow \dim W + \dim W^0 = \dim V$$

H.P.

Corollary : 1

If W is a k-dimensional subspace of an n-dimensional vector space V. then W is intersection of $n-k$ hyperspace in V.

Proof:-

First, we proof theorem (15)

$$\dim W + \dim W^0 = \dim V$$

$$\dim W^0 = n-k$$

Let $W^0 = \{f_{k+1}, \dots, f_n\}$

If $k = n-1$, then $W^0 = \{f_n\}$

$$\dim W^0 = 1$$

Let $N f_i = \{ \alpha \in V \mid f_n(\alpha) = 0\}$

To prove:- $W = \bigcap_{i=k+1}^{n} Nf_i$

Let $\alpha \in W$, $\Rightarrow f_n(\alpha) = 0$

$\Rightarrow \alpha \in Nf_n$ contained

$\Rightarrow W \subseteq Nf_n$.

$\Rightarrow W \subseteq \cap Nf_i \quad i = k+1, \dots, n \longrightarrow (1)$

$\alpha \in \bigcap_{i=k+1}^{n} Nf_i$

$\Rightarrow \alpha \in Nf_i$

$\Rightarrow f_i(\alpha) = 0, \quad i = k+1, \dots, n.$

$\Rightarrow f_i \in W^0, \; \alpha \in W$

$\Rightarrow Nf_i \subseteq W \longrightarrow (2)$

From (1) & (2) we have

$$W = \bigcap_{i=k+1}^{n} Nf_i$$

W is the intersection of n-k hyperspace.

Corollary: 2

If W_1 and W_2 are subspaces of a finite dimensional vector space. then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$

Proof:-

Suppose, $W_1 = W_2$

then $W_1^0 = W_2^0$.

If $W_1 \neq W_2$, then one of the two subspaces contains a vector which is not in the other suppose there is a vector α which is in W_2 but not in W_1.

By previous theorem,

There is a linear functional f such that $f(\beta) = 0$ for all β in W_2 But $f(\alpha) \neq 0$.

Then f is in W_1^0 but not in W_2^0

$$\Rightarrow W_1^0 \neq W_2^0$$

H.P.

Double Dual :-

Let V be a vector space over the field F Then, V^* be the dual of V. The dual of V^* is denoted by V^*V^*. It is called the double dual of V.

Note :-

If α is a vector in V. Then α induces a linear functional L_α on V^* defined by $L_\alpha(f) = f(\alpha)$, f in V^*, L_α is linear.

Def of linear operator in V^*

$$L_\alpha(cf+g) = (cf+g)\alpha$$
$$= cf(\alpha) + g(\alpha)$$
$$L_\alpha(cf+g) = cL_\alpha(f) + L_\alpha(g)$$

If V is finite dimensional, $\alpha \neq 0$ Then $L_\alpha \neq 0$

i.e. $\exists$ a linear functional f. such that

$$f(\alpha) \neq 0.$$

choose an ordered basis $B = \{\alpha_1, \ldots, \alpha_n\}$ for V such that $\alpha_i = \alpha$ and let f be the linear functional which assign to each vector is V. its functional first coordinates in ordered basis.

Thm : 16

Let V be a finite dimensional vector space over the field F for each vector α in V, define $L_\alpha(f) = f(\alpha)$. f in V^* The mapping $\alpha \to L_\alpha$ is then an isomorphism of V onto V^{**}

Proof:-

We showed that for each α the function L_α is linear.

Suppose α and β are in V and c is in F and

Let $\gamma = c\alpha + \beta$. Then for each f in V^*

$$L_\gamma(f) = f(\gamma)$$

$$L\gamma(f) = f(\gamma)$$

$$= f(c\alpha + \beta)$$

$$= cf(\alpha) + f(\beta)$$

$$= cL_\alpha(f) + L_\beta(f)$$

and so, $L\gamma = cL_\alpha + L_\beta$

This shows that the mapping $\alpha \to L_\alpha$ is a L.T from V into V^{**}

If V is finite dimensional and $\alpha \neq 0$ then $L_\alpha \neq 0$.

ie, $\exists$ a linear functional "f" such that $f(\alpha) \neq 0$.

According to this transformation is non-singular L.T from V into V^{**}

Since $\dim V = \dim V^* = \dim V^{**}$

By thm, 9

Let V and W be finite dimensional vector space over the field F, S.T $\dim V = \dim W$. If T is a L.T from V into W. the following are equivalent.

i) T is invertible

ii) T is nonsingular

iii) T is onto

therefore $\alpha \to L_\alpha$ is linear and non-singular with

$$\dim V = \dim V^{**} \quad L_\alpha$$

Then the mapping $\alpha \to \underline{L_\alpha}$ is an isomorphism from V onto V^{**}

Corollary:-

Let V be a finite dimensional vector space over F. If 'L' is linear functional over the dual space V^* of V, then there is a unique vector α in V, such that $L(f) = f(\alpha)$ for every f in V^*

soln:-

If $\exists$ a β in V such that, $L(f) = f(\beta)$, then

$$L(f) = f(\alpha) = f(\beta)$$

$$f(\alpha) = f(\beta)$$

$$f(\alpha) - f(\beta) = 0$$

$$f(\alpha - \beta) = 0$$

$$\alpha - \beta = 0$$

$$\alpha = \beta.$$

Corollary:-

Let V be a finite dimensional vector space over F each basis for V^* is the dual of some basis for V.

Proof:-

Let $B^* = \{f_1, f_2, \ldots, f_n\}$ be a basis for V^*

By thm (14)

There is a basis $\{L_1, L_2, \ldots, L_n\}$ for V^{**} such that

$$L_i(f_i) = \delta_{ij}$$

and using the above corollary

for each i, there is a vector $\alpha_i \in V$.
such that
$$L_i(f) = f(\alpha_i), \text{ for every } f \in V^*$$
ie, such that $L_i = L_{\alpha_i}$

It follows that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V and that B^* is the dual of the basis.

Thm. : 17

If S is any subset of a finite dimensional vector space V then $(S^0)^0$ is the subspace spanned by S.

Proof:-

Let W be the subspace spanned by S. clearly $W^0 = S^0$

We prove : $W = W^{00}$

By thm,
$$\dim W + \dim W^0 = \dim V.$$
$$\dim W^0 + \dim W^{00} = \dim V^*.$$
and since $\dim V = \dim V^*$

We have, $\dim W + \dim W^0 = \dim W^0 + \dim W^{00}$
$$\dim W = \dim W^{00}$$
Since W is a subspace of W^{00} we have that
$$W = W^{00}$$
H.P.

Def : Maximal.

If V is a vector space a hyperspace in V is a maximal proper subspace of V.

Thm : 18.

If f is a non-zero linear functional on a vector space V. then the null space of f is hyperspace in V. Conversely, every hyperspace in V is the null space of a (not unique) non-zero linear functional on V.

Proof :-

Let f be a non-zero linear functions on V and N_f is its null space.

Let α be a vector in V, Which is not in N_f. ie, a vector such that $f(\alpha) \neq 0$ we shall show that every vector in V is in the subspace spanned by N_f and α.

That subspace consists of all vectors $\gamma + c\alpha$, $\gamma \in N_f$, $c \in F$

Let $\beta \in V$, define

$$c = \frac{f(\beta)}{f(\alpha)}$$

Which makes since because $f(\alpha) \neq 0$ Then the vector $\gamma = \beta - c\alpha \in N_f$.

since, $f(\gamma) = f(\beta - (c\alpha))$

$$= f(\beta) - f(c\alpha) = f(\beta) - cf(\alpha)$$

Since β is in the subspace spanned by N_f and α

Now, Let N be a hyper space in V ∃ in some vector $\alpha \notin N$ since, N is a maximal proper subspace.

∴ The subspace spanned by N and α is the entire space V.

∴ Each vector β in V has the form.

$$\beta = \gamma + c\alpha, \quad \gamma \in N, \quad c \in F$$

The vector 'γ' and scalar c are uniquely determined by β.

If we have also,

$$\beta = \gamma' + c'\alpha, \quad \gamma' \in N, \ c' \in F.$$

Then $\varepsilon(c'-c)\alpha = \gamma - \gamma'$

if $c'-c \neq 0$ then α would be in N Hence

$c' = c$ and $\gamma' = \gamma$

Another way. If β is in V. There is a unique scalar, such that $\beta \to \alpha$ is in "N"

Call the scalar $g(\beta)$. It is easy to see that 'g' is a linear functionals on V. and that "N" is the null space of g.

H/P.

Thm : 19

Let $g, f_1, f_2, \ldots, f_r$ be linear functions on a vector space V. respective null space $N, N_1, N_2, \ldots, N_r$. Then 'g' is a linear combination of $f_1, f_2, \ldots, f_r$ iff N contains the intersection of $N_1 \cap N_2 \cap \ldots \cap N_r$.

Proof :-

Case (i)

To prove that, N contains the $\cap$ of $N_1 \cap N_2 \cap \ldots \cap N_r$

ie, $\bigcap_{i=1}^{r} N_i \subseteq N$

Assume that g is a linear combination of $f_1, f_2, \ldots, f_r$

Let, $g = \sum_{i=1}^{r} c_i f_i$, $\forall c_i \in F$

Let, $\alpha \in \bigcap_{i=1}^{r} N_i$

$\Rightarrow \alpha \in N_i$, $i = 1, 2, \ldots, r$

$\Rightarrow f_i(\alpha) = 0$

Consider, $g(\alpha) = \left[\sum_{i=1}^{r} c_i f_i\right](\alpha)$

$= \sum_{i=1}^{r} c_i f_i(\alpha)$ $\quad \therefore f_i(\alpha) = 0$

$= \sum_{i=1}^{r} c_i(0) \Rightarrow g(\alpha) = 0.$

$\Rightarrow \alpha \in N, \bigcap_{i=1}^{r} N_i \subseteq N$

$N \subseteq N_1 \cap N_2 \cap \ldots \cap N_r$

Case (ii)

To prove that, linear combination of $f_1, f_2, \ldots, f_r$

ie, $g = \sum_{i=1}^{r} c_i f_i$.

Assume $\bigcap_{i=1}^{r} N_i \subseteq N$, Where N_i is null space of "f_i"

and N is the null space of "g"

We prove that, part of induction on "r" for $r=1$.

We know this the thm is true.

We assume that, this is proof for upto $r=1$

Now we prove "r"

Let $g', f_1', f_2', \ldots, f_{r-1}'$ be the restriction of

$g, f, f_2, \ldots, f_{r-1}$ to the supspace N_r.

Then $g', f_1', f_2', \ldots, f_{r-1}'$ are linear functional on

the vector space N_r

If $\alpha \in N$. and $f_i'(\alpha) = 0$, $i = 1, \ldots, r-1$

then $\alpha \in N_1 \cap N_2 \cap \ldots \cap N_r$ and so $g'(\alpha) = 0$

By induction hypothesis,

$$g' = \sum_{i=1}^{r-1} c_i f_i'$$

Let $$h = g - \sum_{i=1}^{r-1} c_i f_i$$

Nearly h is linear functionals on 'r' $\forall \alpha$

$$h(\alpha) - g(\alpha) - \sum_{i=1}^{r-1} c_i f_i(\alpha)$$

$$= 0 \qquad [\therefore \text{ since } f_i = f_i' \quad g = g' \text{ on } N_r]$$

ie, α belongs to $N_r \Rightarrow h(\alpha) = 0$

By using the next thm,

We get,

'h' is a scalar multiple of f_r

ie, $h = c_r f_r$ on V.

$$g - \sum_{i=1}^{r-1} c_i f_i = c_r f_r$$

$$g = \sum_{i=1}^{r-1} c_i f_i + c_r f_r$$

$$= \sum_{i=1}^{r} c_i f_i$$

$\therefore$ g is a linear combination of $f_1, f_2, \dots, f_r$.

Lemma :-

If f and g are linear functionals on a vector space V, then g is a scalar multiple of "f" iff the nullspace of "g" contains the nullspace of "f" ie,

Iff $f(\alpha) = 0 \Rightarrow g(\alpha) = 0$.

Proof :-

Case (i)

Assume g is a scalar multiple of "f"

$\Rightarrow g = cf$ for some $c \in F$

consider, $g(\alpha) = c f(\alpha)$

for every $\alpha \in f$

if $f(\alpha) = 0$

$\Rightarrow g(\alpha) = 0$

if $f = g = 0$ then the thm is true.

Case (ii)

If $f \neq 0$ then the null space N_f of f is hyperspace

choose $\alpha \in V$, such that $f(\alpha) \neq 0$

let $c = \dfrac{g(\alpha)}{f(\alpha)}$

let $h = g - cf$.

then h is a linear functionals on V

let $\gamma \in N_f$

$f(\gamma) = 0$

$g(\gamma) = 0$

$h(\gamma) = 0 \ \forall \ \gamma \in N_f$

if $\alpha \in N_f$

$\rightarrow f(\alpha) \neq 0$

$\Rightarrow h(\alpha) = g(\alpha) - cf(\alpha)$

$= g(\alpha) - \dfrac{g(\alpha)}{f(\alpha)} \cdot f(\alpha)$

$= 0$

$\therefore h(\alpha) = 0$ for every $\alpha \in V$.

$\rightarrow h = 0$ ($h \in$ Null space)

$\rightarrow g - cf = 0$

$\rightarrow g = cf$.

Transpose of Linear transformation:-

Transpose of T:-

(Let V and W be a vector space over the field F for each linear transformation T from V into W. There is a unique linear transformation T^t from W^* into V^* such that,

$T^t(g(x)) = g(T\alpha)$

$$(T^t g)(\alpha) = g(T\alpha)$$

for every g in W^* and α in V. Then this ~~for every g in W^*~~ transformation. T^t is called as Tranpose of T or adjoint of T)

Statement.

Above definition.

Proof:-

Let $g, h \in W^*$ and $c \in F$

To prove that T^t is linear.

consider, $[T^t(cg+h)](\alpha) = (cg+h)\,T(\alpha)$

$$= cg[T(\alpha)] + h[T(\alpha)]$$

$$= c[T^t g(\alpha)] + [T^t h](\alpha)$$

$$[T^t(cg+h)]\alpha = [cT^t g + T^t h]\alpha$$

$$T^t(cg+h) = cT^t g + T^t h$$

$\therefore T^t$ is a linear transformation from W^* into V^*.

To prove : uniqueness :-

Consider that there is another linear transformation U^t from W^* into V^*

such that,

$$(U^t g)\alpha = g[T(\alpha)] \;\forall\; g \in W^* \;\alpha \in V.$$

since, $(T^t g)\alpha = g(T(\alpha))$

$$= (U^t g)\alpha.$$

$$T^t g = U^t g$$

$$T^t = U^t$$

$\therefore T^t$ is a unique.

Thm : 20.

Statement :-

If V and W be the vector space over the field F and let T be a linear transformation from V into W the null space of T^t is the annihilator of the range T. If V and W are finite. dimensional then (i) rank (T^t) = Rank (T)

(ii) the range of T^t is the annihilator of the null space of T.

Proof :-

First to prove, the annilator of the range of T is equal to the null space of T^t

ie, $[R(T)]^\circ = N(T^t)$

If g is in W^* then by defn

$(T^t g)\, x = g(Tx)$

$T^t(g(x)) = g(Tx)$

let g is in the null space of T^t, which is the subspace of W^*

ie, $g \in N(T^t) \Rightarrow g(Tx)$

Thus the null space of T^t is preceisely the annihilator of the range of T

ie, $N(T^t) = [R(T)]^\circ \longrightarrow ①$

suppose that, V and W are finite dimensional say, $\dim V = n$, $\dim W = m$

i) let r be the rank of T

ie, $r = \rho(T) = \dim R(T)$

The dimension of the range of T

By thm,

Let V be finite dimensional vector space over the field F, Let W be a subspace of V, then

$\dim W + \dim W^\circ = \dim V$

Now, $R(T)$ is a subspace of W

$\dim R(T) + \dim [R(T)]^\circ = \dim W$

$\dim [R(T)]^\circ = \dim W - \dim R(T)$

$= m - r$

The annihilator of the range of T, that has the dimensional $m-r$.

$\therefore$

By using ① we get,

$$\dim N(T^t) = m - r$$

But T^t is a linear transformation on an, m - dimensional space from W^* into V^*

W.K.T

$$\text{rank}(T) + \text{Nullity}(T) = \dim V$$

$$\rho(T^t) = \dim W^* - \text{Nullity of } T^t$$

$$= \dim W - \text{Nullity of } T^t$$

$$= m - (m - r)$$

$$= r$$

T and T^t have the same rank

ie, $\rho(T) = \rho(T^t)$

ii) Let N be the null space of T. Every functionals is the range of T^t is in the annihilator ~~is thee~~ of N

Let $f \in T^t, g$ for some $g \in W^*$

Then $x \in N$,

$$f(x) = T^t g(x)$$

$$= g\,T(x)$$

$$= g(0) = 0$$

Now, the range of T^t is subspace of the space $[N(T)]^0$

ie, $R(T^t) \subseteq [N(T)]^0$

$$\dim [N(T)]^0 = n - \dim N(T)$$

$$= \dim V - \dim N(T)$$

$$= (\dim R(t) + \dim N(T)) - \dim N(T)$$

$$= \dim R(t)$$

$$= \rho(T) = \rho(T^t)$$

$$= \dim R(T^t)$$

$$[N[t]]^0 = R(T^t)$$

So the range of T^t must be exactly.

$[N(T)]^0$ H/P.

Thm : 22

Let V, W be finite dimensional vector space over F. Let B an ordered basis for V with dual basis B^* and let B' be an ordered basis for W with dual basis B'^*. Let T be a linear transformation T from V into W. Let A be the matrix of T relative to B, B' and let B be the matrix of T^t relative to B'^*, B^* then $B_{ij} = A_{ji}$

Proof :-

$j = 1, 2, \ldots, n$

so $\alpha_j \Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$

Let $B = \{\alpha_1, \ldots, \alpha_n\}$, $B' = \{\beta_1, \ldots, \beta_m\}$

$B^* = \{f_1, \ldots, f_n\}$, $B'^* = \{g_1, \ldots, g_m\}$

By definition.

$T: V \to W$

$T\alpha_j = \sum_{i=1}^{m} A_{ij} \beta_i$, $j = 1, 2, \ldots, n$

$f: W^* \to V^*$

$T^t g_j = \sum_{i=1}^{n} B_{ij} f_i$, $j = 1, 2, \ldots, m \longrightarrow ①$

on other hand.

$B'^* = (g_1, g_2, \ldots, g_n)$

$T\alpha_j \neq \sum_{i=1}^{m} A_{ij} \beta_i$

$(T^t g_j)(\alpha_i) = g_j(T\alpha_i)$

$= g_j\left(\sum_{k=1}^{m} A_{ki} \beta_k\right)$

$f_i(\alpha_j) = \delta_{ij}$

$i \neq j \Rightarrow 0$

$i = j \Rightarrow 1$

Here $B = \delta$

$C_{ij} = \sum_{i=1}^{m} B_{ik} A_{kj}$

$= \sum_{k=1}^{m} A_{ki}\, g_j \cdot \beta_k$

$= \sum_{k=1}^{m} A_{ki}\, \delta_{jk} = A_{ji}$

for any linear functions f on V

$f = \sum_{i=1}^{m} f(\alpha_i) f_i$

If we apply this formula to the fuctional $f = T^t g_j$ and the fact that

$(T^t g_j)\alpha_i = A_{ji}$ we have. matrix term.

Convert to Linear transformation

so, $\sum^{n} A_{ji} f_i$.

We have $T^t g_j = \sum_{i=1}^{n} A_{ji} f_i \longrightarrow (2)$

From (1)

$$\sum_{i=1}^{n} B_{ij} f_i = \sum_{i=1}^{n} A_{ji} f_i$$

$$\sum_{i=1}^{n} B_{ij} f_i - \sum_{i=1}^{n} A_{ji} f_i = 0$$

$$\sum_{i=1}^{n} (B_{ij} - A_{ji}) f_i = 0$$

basis element of B^* so, not equal to zero.

$$B_{ij} - A_{ji} = 0$$

$$\therefore B_{ij} = A_{ji}$$

H.P

Transpose of Matrix :- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ Interchange row and column.

If A is an $m \times n$ matrix over the field F. the transpose of A is the $n \times m$ matrix A^t defined by

$$A^t_{ij} = A_{ji}$$

Thm :-

Statement :-

Let A be any $m \times n$ matrix over F then the row rank of "A" is equal to the column rank of "A"

Proof :-

$B = \{x_1, x_2, \dots, x_n\} \Rightarrow F^n$. B basis of F^n

Let $\underline{B}$ be the standard $\underline{\text{ordered basis}}$ for $\underline{F^n}$

$B' = \{\beta_1, \dots, \beta_m\} \Rightarrow F^m$. B' basis of F^m

and $\underline{B'}$ be the standard ordered basis for F^m

domain as column. $T: F^n \to F^m$. codomain as row

Let "T" be the L.T from F^n into F^m such that the matrix of T relative to the pair B, B' in A

$A = [x_{ij}]_{m \times n}$ (row column)

i.e, $T(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m)$

Where, $y_i = \sum_{j=1}^{n} A_{ij} x_j$

The column rank of A is the rank of the transportation T, because the range T consists of all m-triples

Which are linear combination of the column Vectors of A

Relative to the dual basis B'^{*} and B^{*} the transpose mapping T^t is represented by the matrix A^t

Since the column of A^t are the rows of A.

B the same reasons

The row rank (the column rank of A^*) is equal to the rank of T^t

By thm,

Rank (T^t) = rank (T) we have T and T^t have the same rank

Hence the row rank of A is equal to the column rank of A.

Note :-

If A is $m \times n$ matrix over F and T is the L.T from F^n into F^m defined above then,

$$\text{rank}(T) = \text{row rank}(A)$$
$$= \text{column rank}(A)$$

We say simply the rank of A.

Trace :-

sum of diagonal values ⇒ Trace.

Let n be a +ve integer and F be a field. If A is an $n \times n$ matrix with entries in F. the trace of A is the scalar.

$$\text{tr}\, A = A_{11} + A_{22} + \dots + A_{nn}.$$

The trace function is a linear functionals on the matrix space $F^{n \times m}$ because

$$\text{tr}(cA + B) = \sum_{i=1}^{n} (cA_{ii} + B_{ii})$$

$$c \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$$= C\,trA + trB$$

Null space :-

The vector $\alpha_1 = (1,2)$, $\alpha_2 = (3,4)$ are L.I and form a basis for R^2. There is a unique linear transformation from $R^2 \to R$ such that $T\alpha_1 = (3,2,1)$; $T\alpha_2 = (6,5,4)$ find that $T(1,0)$

Soln:-

If $(1,0) = c_1(1,2) + c_2(3,4)$

$(1,0) = [c_1 + 3c_2, c_1 + 4c_2]$

$c_1 + 3c_2 = 1 \longrightarrow (1)$

$2c_1 + 4c_2 = 0 \longrightarrow (2)$

$(1) \times 2 \rightarrow$

$$\begin{aligned} 2c_1 + 6c_2 &= 2 \\ 2c_1 + 4c_2 &= 0 \\ \hline 2c_2 &= 2 \\ \boxed{c_2 = 1} \end{aligned}$$

$(1) \Rightarrow c_1 + 3(1) = 1$

$c_1 = 1 - 3$

$\boxed{c_1 = -2}$

$(1,0) = -2(1,2) + 1(3,4)$

$= -2T(1,2) + T(3,4)$

$= -2(3,2,1) + (6,5,4)$

$= (-6,-4,-2) + (6,5,4)$

$T(1,0) = 0, 1, 2.$