Partial Differentional Equation

UNIT-I

Second Order partial differentional Equation

Origin of 2^{nd} order P.D.E — linear differential Eqn with Constant co-efficients — Methods of solving (linear) P.D.E — classification of 2^{nd} order P.D.E — Canonical form — Adjoint opreator — Riemann Method.

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Elliptic Differential equation

Occurence of a laplace and poisson's Equation — Boundary value problems — separation of variables method — Laplace Eqn in cylindrical and Spherical co-ordinates — Dirichlet and Neumann problems for a circle and sphere

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Parabolic Differential Equation

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UNIT: 4

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- cylindrical - spherical Co-ordinates — Duhamel's principle for wave Eqn.

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Integral Transform

Laplace Transform — solution of P.D.E — Diffusion Eqn — wave Equation - Fourier Transforms ~~& their~~ application to P.D.E — Diffusion eqn - wave eqn Laplace equation

chapter: 6 Sec: 6.2 to 6.4

Text Book:-

P.D.E for Engeneering → 2nd Edition.

J.N. Sharma & ~~keh~~

UNIT - I

Second Order P.D.E

Origin of 2nd order partial Differential Eqn:-

Consider the function $z = f(u) + g(v) + w$ —①

Soln: where,

f and g are arbitrary function of u and v respectively and u, v and w are functions of x and y.

we write,

$$p = \frac{\partial z}{\partial x} \; ; \; q = \frac{\partial z}{\partial y} \; ; \; r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} \; , \; t = \frac{\partial^2 z}{\partial y^2}$$

Diff ① w.r.to x and y respectively.

we get.

$$p = f'(u)\, u_x + g'(v)\, v_x + w_x \quad \text{—②}$$

$$q = f'(u)\, u_y + g'(v)\, v_y + w_y \quad \text{—③}$$

$$r = f''(u)(u_x)^2 + f'(u)\, u_{xx} + g''(v)(v_x)^2 + g'(v)\, v_{xx} + w_{xx} \quad \text{—④}$$

$$s = f''(u)\, u_x u_y + g''(v)\, v_x v_y + f'(u)\, u_{xy} + g'(v)\, v_{xy} + w_{xy} \quad \text{—⑤}$$

$$t = f''(u)\, u_y^2 + f'(u)\, u_{yy} + g''(v)\, v_y^2 + g'(v)\, v_{yy} + w_{yy} \quad \text{—⑥}$$

Eliminating the arbitary quantities f', g', f'' and g'' in (2) to (6)

we get

$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s-w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t-w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0$$

If we expand in terms of first column we get

$$\boxed{Rr + Ss + Tt + Pp + Qq = W} \quad \text{---(7)}$$

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where,

R, S, T, P, Q & W are known functions of x and y.

$\therefore$ Equation (1) is solution of eqn (7).

$\therefore$ It is a P.D.E of second order.

PROBLEM:-

If $u = f(x+iy) + g(x-iy)$ where f & g are arbitrary functions S.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Soln:-

Given,

$$u = f(x+iy) + g(x-iy) \quad \text{---(1)}$$

Diff (1) p.w.r.to x & y respectively.

$$\frac{\partial u}{\partial x} = f'(x+iy) + g'(x-iy)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+iy) + g''(x-iy).$$

$$\frac{\partial u}{\partial y} = f'(x+iy)\, i + g'(x-iy)(-i)$$

$(-i)(-i) = +i^2 = -1$

$$\frac{\partial^2 u}{\partial y^2} = -f''(x+iy)(+1) + g''(x-iy)(-1)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

2) If f and g be arbitrary functions of the respectively arguments show that $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ provided $\alpha^2 = 1 - \frac{v^2}{c^2}$

soln:-

Given,

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y) \text{ --- } (1)$$

Diff (1) w.r.to x & y respectively.

$$\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \text{ --- } (2)$$

$$\frac{\partial u}{\partial y} = f'(x - vt + i\alpha y)(i\alpha) + g'(x - vt - i\alpha y)(-i\alpha)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y)(i\alpha)^2 + g''(x - vt - i\alpha y)(-i\alpha)^2$$

$$\frac{\partial^2 u}{\partial y^2} = -f''(x - vt + i\alpha y)\alpha^2 - g''(x - vt - i\alpha y)\alpha^2 \quad \text{--- (3)}$$

Eqn (1) Diff p.w.r.to t

$$\frac{\partial u}{\partial t} = f'(x - vt + i\alpha y)(-v) + g'(x - vt - i\alpha y)(-v)$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x - vt + i\alpha y)v^2 + g''(x - vt - i\alpha y)v^2 \quad \text{--- (4)}$$

(2) + (3) =>

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y)(1-\alpha^2) + g''(x - vt - i\alpha y)(1-\alpha^2)$$

$$= (1-\alpha^2)\left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y)\right]$$

$$= (1-\alpha^2)\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$$= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2}$$

Linear Partial differential equation with Constant Co-efficient

An eqn is of the form

$$F(D, D')z = f(x, y) \quad \text{---} (1)$$

where, $F(D, D')$ is a differential operator of a type

$$\boxed{F(D, D') = \sum_{r} \sum_{s} C_{rs} D^{r} D'^{s}}$$

$2D^3 + 3D'^2 + 5D = x+y$

where,
$D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$, & C_{rs} are Constant

which is called a Linear P.D.E with Constant Co-efficient.

Note:-

i) A solution of $F(D, D')z = 0$ is a ~~solution~~ called a Complementray function and

ii) $F(D, D')z = f(x, y)$ is called a particular integral.

$\therefore$ The general solution of (1) is

$z = C.F + P.I$

NOTE:-

We classify the operator $F(D,D')=0$ into two types.

1) Reducible

2) Irreducible

Reducible:-

The operator $F(D,D')$ is said to be reducible if it can be factrozied into the linear factor of the type $D+aD'+b$. where a and b are Constants.

Ex:-

1) $F(D,D') = D^2 - D'^2$

$= (D+D')(D-D')$

$= (D+D'+0)(D-D'+0)$

2) $F(D,D') = D^2 + D$

$= D(D+1)$

$= (D+0D'+0)(D+0D'+1)$

Irreducible:-

The operator $F(D,D')$ ~~z = f(x,y)~~ is said to be irreducible if it is not reducible.

Ex:-

$F(D,D') = D^2 + D'$

Reducible Equation:-

A P.D.E $F(D, D')z = f(x, y)$ is said to be a reducible, if $F(D, D')$ can be written as a product of linear factor in D & D'.

i.e) $F(D, D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)$

If the equation is not reducible then it is called irreducible equation.

Thm: 2

If $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D, D')$ & $\phi_r(\xi)$ is an arbitrary function of the single variable ξ then

$$U_r = \exp\left(\frac{-\gamma_r}{\alpha_r} x\right) \phi_r(\beta_r x - \alpha_r y), \quad \alpha_r \neq 0$$ is

the solution of the equation $F(D, D')z = 0$

Proof:-

The Given equation is

$F(D, D')z = 0$ —— (1)

we have to prove

$U_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) \phi_r (\beta_r x - \alpha_r y)$ is a solution of equation (1).

It is enough to prove that

$F(D, D') U_r = 0$

<u>Claim:</u>- $(\alpha_r D + \beta_r D' + \gamma_r) U_r = 0$

Let $\alpha_r D(U_r) = \alpha_r \frac{\partial}{\partial x}(U_r)$

$$= \alpha_r \partial/\partial x\left(\exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r (\beta_r x - \alpha_r y)\right)$$

$x^2 - 5x + 6 = (x-2)(x-3)$

$x = 2, 3$

$$= \alpha_r \left[\exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r' (\beta_r x - \alpha_r y) \beta_r + \phi_r (\beta_r x - \alpha_r y) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\left(\frac{-\gamma_r}{\alpha_r}\right)\right]$$

$$\alpha_r D(U_r) = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) \phi_r' (\beta_r x - \alpha_r y) \beta_r \alpha_r - \phi_r (\beta_r x - \alpha_r y) \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \gamma_r \quad (2)$$

$$\beta_r D' U_r = \beta_r \frac{\partial}{\partial y}(U_r)$$

$$= \beta_r \frac{\partial}{\partial y}\left(\exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r (\beta_r x - \alpha_r y)\right)$$

$$= \beta_r \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r' (\beta_r x - \alpha_r y)(-\alpha_r) \quad (3)$$

$$\alpha_r DU_r + \beta_r D' U_r = \phi_r(\beta_r x - \alpha_r y)\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)(-\gamma_r$$

$$= -\phi_r(\beta_r x - \alpha_r y)\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\gamma_r$$

$$= -U_r \gamma_r$$

$$\alpha_r DU_r + \beta_r D'U_r + \gamma_r U_r = 0$$

$$(\alpha_r D + \beta_r D' + \gamma_r) U_r = 0$$

since $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D, D')$

we have write

$$F(D, D') = F_1(D, D')(\alpha_r D + \beta_r D' + \gamma_r)$$

where $F_1(D, D')$ are some other factors of $F(D, D')$.

$$\therefore F(D, D')U_r = F_1(D, D')(\alpha_r D + \beta_r D' + \gamma_r)U_r$$

$$= F_1(D, D')(0) = 0$$

$$F(D, D')U_r = 0$$

$\therefore U_r = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$ is a solution of $F(D, D')z = 0$.

Method of solving linear P.D.E

The solution of reducible Equation

$F(D, D')z = f(x, y)$ —— (1).

Be a given P.D.E

If $f(D, D')z = \prod_{r=1}^{n} [\alpha_r D + \beta_r D' + \gamma_r]z$

If z satisfies

$(\alpha_r D + \beta_r D' + \gamma_r]z = 0$, $r = 0, 1, 2, \ldots n$

$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$

$\alpha_r p + \beta_r q = -\gamma_r z$

which is a lagrangian type of a P.D.E

i.e) $P_p + Q_q = R$

$\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}$ —(2)

Take $\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} \Rightarrow \beta_r dx = \alpha_r dy$

$\int$ing $x \beta_r = y \alpha_r + c_r$

$x \beta_r - y \alpha_r = c_r$ —— (3)

Take

$\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z}$

$$-\frac{\gamma_r}{\alpha_r}\,dx = \frac{dz}{z}$$

Jing $-\frac{\gamma_r}{\alpha_r}\,x = \log z - \log A_r$ [where, A_r Constant]

$$-\frac{\gamma_r x}{\alpha_r} = \log\left(\frac{z}{A_r}\right)$$

$$\exp\left(\frac{-\gamma_r x}{\alpha_r}\right) = \frac{z}{A_r}$$

$$z = A_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

$$z = \phi_r(c_r)\exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

$$z = \phi_r(x\beta_r - y\alpha_r)\exp\left(-\frac{x\gamma_r}{\alpha_r}\right)$$

we assume $\alpha_r \neq 0$

$\therefore$ The Complementary function $\Big\} = z = \sum_{r=1}^{n} \phi_r(x\beta_r - y\alpha_r)\exp\left(\frac{-x\gamma_r}{\alpha_r}\right)$

Particular case:-

If $\alpha_r = 0$ then

$$(2) \Rightarrow \frac{dx}{0} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}$$

1^{st} & 2^{nd} term

$$\frac{dx}{0} = \frac{dy}{\beta_r}$$

$$\beta_r \, dx = 0$$

$$\int \beta_r \, dx = 0$$

$$\boxed{\beta_r x = c_r}$$

2^{nd} & 3^{rd} term

$$\frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}$$

$$\int -\frac{\gamma_r}{\beta_r} dy = \int \frac{dz}{z}$$

$$-\frac{\gamma_r}{\beta_r} y = \log z - \log A_r$$

$$\exp\left(-\frac{\gamma_r}{\beta_r} y\right) = \frac{z}{A_r}$$

$$z = A_r \exp\left(\frac{-\gamma_r y}{\beta_r}\right)$$

$$z = \phi_r(c_r) \exp\left(\frac{-\gamma_r y}{\beta_r}\right)$$

$$z = \phi_r(\beta_r x) \exp\left(-\frac{\gamma_r y}{\beta_r}\right)$$

The above two cases are applicable when there is no repeated factors of the type $\alpha_r D + \beta_r D' + \gamma_r$.

Repeated Factors:-

Let P.D.E. $F(D, D')z = f(x, y)$ have repeated factors.

i.e) suppose $(\alpha_r D + \beta_r D' + \gamma_r)$ is a factor of $F(D, D')$.

Then, we have

$$(\alpha_r D + \beta_r D' + \gamma_r)^2 z = 0$$

Let $(\alpha_r D + \beta_r D' + \gamma_r) z = z_1$ —— (A)

where z_1 is a function of x & y.

Such that

$$(\alpha_r D + \beta_r D' + \gamma_r) z_1 = 0$$

W.K.T

The above eqn has a solution of the form

$$z_1 = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y), \quad \alpha_r \neq 0$$

sub z_1 in equ (A)

$$(\alpha_r D + \beta_r D' + \gamma_r) z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$$

$$\alpha_r Dz + \beta_r D'z + \gamma_r z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$$

$$\alpha_r p + \beta_r q + \gamma_r z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$$

$$\alpha_r p + \beta_r q = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y) - \gamma_r z$$

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{\exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y) - \gamma_r z}$$

Take (1) & (2)

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$$

$\int$ ing $\Rightarrow \beta_r x = \alpha_r y + c_r$

$$c_r = \beta_r x - \alpha_r y$$

Also ① & ③

$$\frac{dx}{\alpha_r} = \frac{dz}{\exp\left(-\frac{\lambda_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y) - \lambda_r z}$$

$$\frac{1}{\alpha_r} \exp\left(-\frac{\lambda_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y) - \frac{\lambda_r z}{\alpha_r} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \frac{\lambda_r z}{\alpha_r} = \frac{1}{\alpha_r} \exp\left(-\frac{\lambda_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$$

This is of the form of L.D.E

$$\frac{dy}{dx} + Py = Q$$

Here, $y = z$, $P = \frac{\lambda_r}{\alpha_r}$ & $Q = \frac{1}{\alpha_r} \exp\left(-\frac{\lambda_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$

$\Rightarrow$ Integrating factor, I.F $= e^{\int P dx}$

$$= e^{\int \frac{\lambda_r}{\alpha_r} dx}$$

$$= \exp\left(\frac{\lambda_r x}{\alpha_r}\right)$$

The solution is

$$y e^{\int P dx} = \int Q \left(e^{\int P dx}\right) dx + c$$

$$z \exp\left(\frac{\lambda_r x}{\alpha_r}\right) = \int \left[\frac{1}{\alpha_r} \exp\left(-\frac{\lambda_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y) \exp\left(\frac{\lambda_r x}{\alpha_r}\right)\right] dx + c$$

$$= \int \frac{1}{\alpha_r} \phi_r(\beta_r x - \alpha_r y)\, dx + c$$

where $C_r = \beta_r x - \alpha_r y$ is a const.

$$= \frac{\phi_r C_r x}{\alpha_r} + \phi_r(C_r)$$

$$\therefore Z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\left(\frac{\phi_r C_r x}{\alpha_r} + \phi_r(C_r)\right)$$

The procedure can be generalized upto any order of repeation factor, adding this to the some of other solution corresponding to the linear factors without repeation.

$\therefore$ we get the required Complementrary function.

5M

(d) Solution of irreducible equation with Constant Co-efficient:-

Let $F(D, D')z = f(x,y)$ —(1) be a irreducible linear P.D.E with Constant Co-efficient.

Let $F(D, D') = F_1(D, D')\, F_2(D, D')$ —(2)

where F_1 is irredusible and F_2 is reducible.

① $\Rightarrow F_1(D, D') F_2(D, D') z = f(x, y)$

Then, the solution of Corresponding linear factor of $F_2(D, D')$ can uptained as

$$z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y), \ \alpha_r \neq 0$$

(or)

$$z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x) \text{ if } \alpha_r = 0.$$

now, to find the soln of irreducible $F_1(D, D')$.

suppose, $z = e^{ax+by}$ is a soln of

$F_1(D, D') z = 0$ —③

$\therefore F_1(D, D') e^{ax+by} = F_1(a, b) e^{ax+by}$

the above equation is must be vanish & we impose the condition that $F_1(a, b) = 0$

$$z = \sum_r C_r e^{a_r x + b_r y}$$ is a Complementary function of equation ③

$$F_1(a_r, b_r) = 0 \quad , r = 1, 2, \ldots$$

Rules for finding complementary function:-

Consider the equation

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y} = 0$$ it can be

written as $(D^2 + a_1 D D' + a_2 D'^2) z = 0$ — ①

where $D = \partial/\partial x$, $D' = \partial/\partial y$

Auxillary equation:-

$$\frac{①}{D'^2} \Rightarrow \frac{D^2}{D'^2} + \frac{a_1 D}{D'} + a_2 = 0$$

$$m^2 + a_1 m + a_2 = 0 \text{ — ②}$$

where $m = D/D'$

which is called auxillary equation

Let m_1 & m_2 be the roots of ②.

case (i):-

when $m_1 \neq m_2$ $\left(\begin{matrix} m - m_1 = 0 \\ m - m_2 = 0 \end{matrix} \right.$

$m = m_1$

$D/D' = m_1$

$① \Rightarrow (D - m_1 D')(D - m_2 D') z = 0$ — ③

Now, the solution of $(D - m_2 D') z = 0$

is also a solution of equation ③

$$(D - m_2 D')z = 0$$

$$Dz - m_2 D'z = 0$$

$$p - m_2 q = 0$$

which is lagrangian form

$$\therefore \frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

1^{st} & 2^{nd} term,

$$\frac{dx}{1} = \frac{dy}{-m_2}$$

$$-m_2 x. = y + c_1$$

Sing

$$c_1 = y + m_2 x.$$

$$0 = \int dz$$

$$z = c_2 = f_2(c_1)$$

$\therefore z = f_2(m_2 x + y)$ is a solution of

$(D - m_2 D')z = 0$. where f_2 is arbitary

function.

iiy

A solution of $(D - m_1 D')z = 0$ is also

solution of ⑤ proceding the same way.

$$z = f_1(y + m_1 x)$$

where f_1 is another arbitrary function.

∴ The C.F of equation (1) is given by

$$z = f_1(y + m_1x) + f_2(y + m_2x).$$

Case (ii):-

$$m_1 = m_2 = m$$

$$(1) \Rightarrow (D - mD')^2 z = 0$$

$$(D - mD')(D - mD')z = 0$$

Let $U = (D - mD')z$

$$(D - mD')U = 0$$

The solution $U = f(y + mx)$ by case (i)

but

$$U = (D - mD')z.$$

$$\therefore f(y + mx) = (D - mD')z$$

$$= p - mq$$

by lagrange's form

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(y + mx)}$$

① & ② term

$$\frac{dx}{1} = \frac{-dy}{m}$$

$\int$ ing

$$x = -\frac{1}{m}y + c$$

$$c_1 = x + y/m = mx + y$$

① & 3^{rd} term

$$\frac{dx}{1} = \frac{dz}{f(y+mx)} \Rightarrow \frac{dx}{1} = \frac{dz}{f(c_1)}$$

$\int$ ing $f(c_1)x + c_2 = z$

$$z = f(y+mx)x$$

$\therefore$ The C.F of equ ① is

$$\boxed{z = F_1(y+mx) + x f_2(y+mx)}.$$

Generalized the result of case (i) and case (ii)

If the ~~roots~~ rules of ~~ax~~ auxilary equation are $m_1, m_2, \ldots$ (distinct).

Then the Complementary function

$$C.F = f_1(y+m_1x) + f_2(y+m_2x) + \cdots$$

If the two roots of auxilary eqn is equal $m_1 = m_2 = m$.

Then, $C.F = f_1(y+mx) + x f_2(y+mx)$.

If three roots of auxilary equation are equal $m_1 = m_2 = m_3 = m$.

$C.F = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx)$

Rules for finding particular integral:-

Consider the equation $\phi(D, D')z = F(x, y)$

where $\phi(D, D') = D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots$

Then

$$P.I = \frac{1}{\phi(D, D')} F(x, y)$$

case(i):-

i) when $F(x, y) = e^{ax+by}$

Then, $P.I = \frac{1}{\phi(a, b)} e^{ax+by}$, $\phi(a, b) \neq 0$

If $\phi(a, b) = 0$, then this case is failure.

ii) when $F(x, y) = \sin(ax+by)$

$$P.I = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by), \quad \phi(-a^2, -ab, -b^2) \neq 0$$

If $\phi(-a^2, -ab, -b^2) = 0$ then this case is failure.

A similar rule for $F(x, y) = \cos(ax+by)$.

iii) $F(x,y) = x^m y^n$, where m & n are $+^{ve}$ integers.

Then $P.I = \frac{1}{\phi(D,D')} x^m y^n$

$$= (\phi(D,D'))^{-1} x^m y^n$$

If m<n we expand as binomiary $[\phi(D,D')]^{-1}$ in power of $\frac{D}{D'}$ and m>n we expand $[\phi(D,D')]^{-1}$ in power of D'/D.

iv) Also $\frac{1}{D} F(x,y) = \int F(x,y)\,dx$, y = constant

$$\frac{1}{D'} F(x,y) = \int F(x,y)\,dy \quad , x = \text{constant}.$$

Case(ii):: General method ~~for~~ to find P.I

The General method is applicable to all cases where $F(x,y)$ is not of form if on above. Now, $F(D,D')$ can be factorized in general into linear factors.

$$P.I = \frac{1}{\phi(D,D')} F(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')\cdots(D-m_nD')} F(x,y).$$

Let us evaluate $\frac{1}{(D-mD')} F(x,y) = Z$

Consider, $F(x,y) = Z(D-mD')$

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{F(x,y)}$$

① & ②

$$\frac{dx}{1} = \frac{dy}{-m}$$

$\int$ ing $\quad -mx + c_1 = y$

$$c_1 = y + mx$$

① & ③rd term

$$dx = \frac{dz}{F(x,y)}$$

$$F(x,y)\,dx = dz$$

$$\int (x, c_1 - mx)\,dx = \int dz$$

$$\frac{1}{(D-mD')} F(x,y) = z = \int F(x, c_1 - mx)\,dx.$$

where c_1 is replaced by $y + mx$

after integration.

Solve the equation $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$.

Soln:-

The given eqn is written as

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)\,z = e^{x+y} \quad —①$$

Here,

$$F(D, D') = D^3 - 2D^2D' - DD'^2 + 2D'^3$$

$$= D^2(D - 2D') - (D'^2[D - 2D'])$$

$$= (D^2 - D'^2)(D - 2D')$$

$$F(D, D') = (D + D')(D - D')(D - 2D') \quad -(2)$$

Now,

$$(D - 2D')z = 0$$

$$p - 2q = 0$$

In lagranges form

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{0}$$

1^{st} & 2^{nd} term

$$\frac{dx}{1} = \frac{dy}{-2}$$

$$\int -2\,dx = \int dy$$

$$-2x + c = y$$

$$c = y + 2x$$

1^{st} and 3^{rd}

$$\frac{dx}{1} = \frac{dz}{0}$$

$$\int 0\,dx = \int 1\,dz$$

$z = \phi(c) = \underline{\phi_1(y + 2x)}$ is a soln of (1)

Now,

$(D+D')z=0$

$p+q=0.$

In lagranges' form

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{0}$$

1st & 2nd term

$$\frac{dx}{1}=\frac{dy}{1}$$

$$\int dx=\int dy$$

$$x+c=y$$

$$c=y-x.$$

1st & 3rd term.

$$\frac{dx}{1}=\frac{dz}{0}$$

$$\int 0\,dx=\int dz.$$

$$z=\phi_2(c)$$

$$z=\phi_2(y-x)$$

$(D-D')z=0$

$p-q=0$

In lagrange's form

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{0}$$

1st & 2nd term

$$\frac{dx}{1}=\frac{dy}{-1}$$

$$\int -dx=\int dy$$

$$-x+c=y$$

$$c=y+x$$

1st & 3rd term

$$\frac{dx}{1}=\frac{dz}{0}$$

$$\int 0\,dx=\int dz$$

$$\phi_3(c)=z$$

$$z=\phi_3(y+x).$$

$$C.F=\phi_1(y+2x)+\phi_2(y-x)+\phi_3(x+y).$$

Particular Integral:-

$$P.I = \frac{1}{\phi(D, D')} F(x, y)$$

$$= \frac{1}{(D+D')(D-D')(D-2D')} e^{x+y}$$

$$= \frac{1}{2(D-D')(1-2)} e^{x+y}$$

$$P.I = -\frac{1}{2(D-D')} e^{x+y} \quad \text{———} (3)$$

Next,

$$w = \frac{1}{D-D'} e^{x+y}$$

$$(D-D')w = e^{x+y}$$

$$p - q = e^{x+y}$$

by lagrange's form

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dw}{e^{x+y}}$$

(1) & (2)

$$\frac{dx}{1} = \frac{dy}{-1}$$

Jing

$$\int -dx = \int dy$$

$$c_1 = x + y.$$

(1) & (3)

$$\frac{dx}{1} = \frac{dw}{e^{x+y}}$$

$$e^{x+y} \cdot dx = dw$$

$$\int e^{c_1} \cdot dx = \int dw$$

$$e^{c_1} \cdot x = w$$

$$w = e^{x+y} \cdot x$$

(3) $\Rightarrow$

$$P.I = -\frac{1}{2} w$$

$$P.I = -\frac{1}{2} \cdot e^{x+y} \cdot x$$

$\therefore$ The general soln is

$$z = C.F + P.I$$

$$z = \phi_1(y+2x) + \phi_2(y-x) + \phi_3(x+y) - \frac{1}{2} x \cdot e^{x+y}$$

2) 15 solve $\dfrac{\partial^4 z}{\partial x^4} + \dfrac{\partial^4 z}{\partial y^4} = 2\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$

10M

Soln:-

Given $D^4 + D'^4 = 2D^2 D'^2$

$$D^4 + D'^4 - 2D^2 D'^2 = 0$$

$$(D^2 - D'^2)^2 = 0$$

$$((D+D')(D-D'))^2 = 0$$

$(D+D')^2 (D-D')^2 = 0$

$(D+D')(D+D')(D-D')(D-D')z = 0$

i) $(D+D')^2 z = 0$ — (2)

Let $(D+D')z = z_1$ — (3)

$\therefore (D+D')z_1 = 0$

$p+q = 0$

by lagranges form

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz_1}{0}$$

(1) & (2)

$\int dx = \int dy$

$x - y = c_1$

(1) & (3) $\frac{dx}{1} = \frac{dz_1}{0}$

$z_1 = c = \phi_1(c_1)$

$z_1 = \phi_1(x-y)$.

(3) $\Rightarrow (D+D')z = z_1 = \phi_1(x-y)$

$p+q = \phi_1(x-y)$.

by lagrange's form,

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\phi_1(x-y)}$$

(1) & (2)

$\int \frac{dx}{1} = \int dy/1$

$x - y = c_2$

(1) & (3)

$$\int \frac{dx}{1} = \int \frac{dz}{\phi_1(x-y)}$$

$$\int \phi_1(c_2)\,dx = \int dz$$

$$x\,\phi_1(x-y) + \phi_2(c_2) = z.$$

$$\therefore z_1 = x\,\phi_1(x-y) + \phi_2(x-y).$$

ii) $(D-D')^2 = 0$

Let $(D-D')z = z_2$

$$\therefore (D-D')z_2 = 0$$

$$p-q = 0$$

by lagrange's form

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{0}$$

(1) & (2)

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\int -dx = \int dy$$

$$-x + c = y$$

$$c_3 = x + y.$$

(1) & (3)

$$\frac{dx}{1} = \frac{dz_2}{0}$$

$$\int 0\,dx = \int dz_2$$

$$z_2 = \phi_3(c_3)$$

$$z_2 = \phi_3(x+y)$$

~~iii) $(D+D')^2 \neq 0$~~ ~~$(D+D')z =$~~

$$(A) \Rightarrow (D-D')z = z_2 = \phi_3(x+y)$$

$$(D-D')z = \phi_3(x+y)$$

$$p-q = \phi_3(x+y)$$

By lagrange form

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{\phi_3(x+y)}$$

(1) & (2)

$$\frac{dx}{1} = \frac{+dy}{-1}$$

$$\int -dx = \int dy$$

$$-x + c_4 = y$$

$$c_4 = x+y$$

1st & 3rd

$$\frac{dx}{1} = \frac{dz_2}{\phi_3(x+y)}$$

$$\int \phi_3(x+y)\, dx = \int dz_2$$

$$x\,\phi_3(x+y) + \phi_4(c_4) = z_2 .$$

$$x \cdot \phi_3(x+y) + \phi_4(x+y) = z_2 .$$

$$z_2 = x\,\phi_3(x+y) + \phi_4(x+y)$$

$$\therefore z = x\,\phi_1(x-y) + \phi_2(x-y) + x\,\phi_3(x+y) + \phi_4(x+y).$$

3) Find the soln of the equation $\nabla_1^2 z = e^{-x}\cos y$

01.07.19

i) which reads to zero as $x \to \infty$

ii) Has the value $\cos y$ when $x=0$.

Soln:-

Given $\nabla_1^2 z = e^{-x}\cos y$.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x}\cos y.$$

$$(D^2+D'^2)z = e^{-x}\cos y \quad -(1)$$

To find C.F:-

Now $(D^2+D'^2)z = 0$

Assume $z = e^{ax+by}$ is a soln of above eqn.

$$(D^2+D'^2)e^{ax+by} = 0$$

$$D^2 e^{ax+by} + D'^2 e^{ax+by} = 0$$

$$a^2 e^{ax+by} + b^2 e^{ax+by} = 0$$

$$(a^2+b^2)e^{ax+by} = 0$$

$\therefore a^2+b^2 = 0 \quad [\because e^{ax+by} \neq 0]$

$\therefore$ The Complementary function is

$$C.F = e^{ax+by}, \text{ where } a^2+b^2=0.$$

To find P.I:-

$$P.I = \frac{1}{(D+D'^2)} e^{-x}\cos y$$

$$= e^{-x}\frac{1}{(D-1)^2+D'^2}\cos y$$

$$= e^{-x}\frac{1}{D^2+1-2D+D'^2}\cos y$$

$\cos(0x+y)$

$a=0,\ b=1$

$$= e^{-x}\frac{1}{0+1-2D-1^2}\cos y$$

$$= -\frac{e^{-x}}{2D}\cos y$$

$$= -\frac{e^{-x}}{2}\int \cos y \, dx$$

$$P.I = -\frac{e^{-x}}{2}\, x\cos y$$

$$\therefore z = C.F + P.I$$

$$\boxed{z = e^{ax+by} - \frac{e^{-x}}{2}\, x\cos y}$$

In general, $e^{-\infty} = 0$

$$z = \sum_r A_r e^{a_r x + b_r y} - \frac{x e^{-x}}{2}\cos y \quad \Big| \; a_r^2 + b_r^2 = 0 \quad \text{---(3)}$$

i) since $z \to 0$ as $x \to \infty$

$\therefore$ It is possible only when a_r is must be negative.

Let $a_r = -\lambda_r$,

but $a_r^2 + b_r^2 = 0$

$\Rightarrow (-\lambda_r)^2 + b_r^2 = 0$

$\underline{b_r^2 = -\lambda_r^2} \Rightarrow b_r^2 = -\lambda_r^2$

$b_r = \sqrt{-\lambda_r^2}$

$b_r = \pm i\lambda_r$

$$(3) \Rightarrow z = \sum_{r=0}^{\infty} A_r e^{-\lambda_r x \pm i\lambda_r y} - \frac{x}{2} e^{-x}\cos y \quad \text{---(4)}$$

Now,

$B_r \cos(\lambda_r y + \varepsilon_r) = B_r[\cos \lambda_r y \cos \varepsilon_r - \sin \lambda_r y \sin \varepsilon_r]$

$= A_r \cos \lambda_r y \pm i A_r \sin \lambda_r y$

where $A_r = B_r \cos \varepsilon_r$

$\pm i A_r = -B_r \sin \varepsilon_r$

$= A_r (\cos \lambda_r y \pm i \sin \lambda_r y)$

$B_r \cos(\lambda_r y + \varepsilon_r) = A_r e^{\pm i \lambda_r y}$.

$(4) \Rightarrow z = \sum_{r=0}^{\infty} B_r e^{-\lambda_r x} \cos(\lambda_r y + \varepsilon_r) - \frac{x}{2} e^{-x} \cos y.$

ii) since $z = \cos y$, where $x = 0$.

$$z = \sum_{r=0}^{\infty} B_r e^{-\lambda_r x} \cos(\lambda_r y + \varepsilon_r) - \frac{x}{2} e^{-x} \cos y$$

$$\cos y = \sum_{r=0}^{\infty} B_r \cos(\lambda_r y + \varepsilon_r)$$

$\Rightarrow \varepsilon_r = 0 \ \forall r$

and

$$\lambda_r = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$

$$\Rightarrow B_r = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$

$$z = e^{-x} \cos y - \frac{x e^{-x}}{2} \cos y.$$

$$\boxed{z = e^{-x} \cos y \left(1 - \frac{x}{2}\right)}$$

1) show that the soln of $\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}$ possesses

Solution

of the form

$\sum_{r=0}^{\infty} C_r e^{-kt} \cos(\alpha_r x + \xi_r) \cos(\omega_r t + \delta_r)$ where

$C_r, \alpha_r, \delta_r, \xi_r$ are constant and $\omega_r^2 = \alpha_r^2 c^2 - k^2$.

$z - y$
$x - t$
$y - x$
$\frac{\partial z}{\partial x} - D = \frac{\partial y}{\partial t}$

Soln:- $(D^2 + 2kD - c^2 D'^2) y = 0$ —(1)

Let $y = e^{ax+bt}$ be the soln of eqn (1)

$(D^2 + 2kD - c^2 D'^2) e^{ax+bt} = 0.$

$b^2 e^{ax+bt} + 2kb\, e^{ax+bt} - c^2 a^2 e^{ax+bt} = 0$

$\Rightarrow b^2 + 2kb - c^2 a^2 = 0 \quad \{\because e^{ax+bt} \neq 0$

$ax^2 + bx + c$

$b = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2k \pm \sqrt{4k^2 - 4(1)(-a^2c^2)}}{2(1)}$

$= \frac{-2k \pm \sqrt{4k^2 + 4a^2c^2}}{2}$

$= \frac{-2k \pm 2\sqrt{k^2 + a^2c^2}}{2}$

$b = -k \pm \sqrt{k^2 + a^2c^2}$

In general,

$b_r = -k \pm \sqrt{k^2 + a_r^2 c^2}$

Let us take $a_r^2 = -\alpha_r^2$

$$a_r = \pm i \alpha_r$$

then $b_r = -k \pm \sqrt{k^2 - \alpha_r^2 c^2}$

$$= -k \pm i \sqrt{\alpha_r^2 c^2 - k^2}$$

$$= -k \pm i \sqrt{\omega_r^2}$$

$b_r = -k \pm i\omega_r$, where $\omega_r^2 = \alpha_r^2 c^2 - k^2$

$$\therefore y = e^{a_r x + b_r t}$$

$$= e^{a_r x} \cdot e^{b_r t}$$

$$y = e^{\pm i\alpha_r x} \cdot e^{-kt} \cdot e^{\pm i\omega_r t}$$

In general,

$$y = \sum_{r=0}^{\infty} C_r e^{-kt} e^{\pm i\alpha_r x} e^{\pm i\omega_r t}$$

$e^{0 \pm i\alpha_r x}$

$e^{0x}[A\cos \beta x \ldots$

$$y = \sum_{r=0}^{\infty} C_r e^{-kt} \cos(\alpha_r x + \xi_r) \cos(\omega_r t + \delta_r)$$

Classification of 2^{nd} order P.D.E

2M. Def:- Quasi linear P.D.E of 2^{nd} Order.

A second order P.D.E, which is linear w.r.to second order partial derivatives r, s, t is said to be quasi linear P.D.E of 2^{nd} order

Ex:-

The equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ — (1)

where $f(x, y, z, p, q)$ need not be linear, is a quasi linear P.D.E.

Here the co-efficients are R, S, T may be function of x and y, eqn (1) is said to be

i) Elliptic if $S^2 - 4RT < 0$

ii) parabolic if $S^2 - 4RT = 0$

iii) hyperbolic $S^2 - 4RT > 0$, at a point (x_0, y_0)

Canonical form of PDE:-

Consider the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{—} (1)$$

Let us the transformation

$$\xi = \xi(x, y), \quad \eta = \eta(x, y), \quad z = z(\xi, \eta)$$

where ξ & η are continuous differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)}$$

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix}$$

$$J = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}(\xi, \eta)$$

$$= \left(\frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right)$$

$$p = z_\xi \xi_x + z_\eta \eta_x$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}(\xi, \eta)$$

$$q = z_\xi \xi_y + z_\eta \cdot \eta_y$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(z_\xi \xi_x + z_\eta \eta_x)$$

$$= z_\xi \xi_{xx} + \xi_x \frac{\partial z_\xi}{\partial x} + z_\eta \eta_{xx} + \eta_x \frac{\partial z_\eta}{\partial x}$$

$$= z_\xi \xi_{xx} + \xi_x \left[\frac{\partial z_\xi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z_\xi}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right]$$

$$+ z_\eta \eta_{xx} + \eta_x \left[\frac{\partial z_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right]$$

$$= z_\xi \xi_{xx} + \xi_x \left[z_{\xi\xi} \xi_x + z_{\xi\eta} \eta_x\right] + z_\eta \eta_{xx} +$$

$$\eta_x \left[z_{\eta\xi} \xi_x + z_{\eta\eta} \eta_x\right]$$

$$r = z_\xi \xi_{xx} + z_{\xi\xi} \xi_x^2 + z_{\eta\eta} \eta_x^2 + 2 \eta_x \xi_x z_{\eta\xi} + z_\eta \eta_{xx}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(z_\xi \xi_y + z_\eta \eta_y)$$

$$= z_\xi \xi_{yy} + \xi_y \frac{\partial z_\xi}{\partial y} + z_\eta \eta_{yy} + \eta_y \frac{\partial z_\eta}{\partial y}$$

$$= z_\xi \xi_{yy} + \xi_y \left[\frac{\partial z_\xi}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z_\eta}{\partial \eta} \frac{\partial \eta}{\partial y} \right] \quad (2)$$

$$= z_\xi \xi_{yy} + \xi_y \left[z_{\xi\xi} \xi_y + z_{\xi\eta} \eta_y \right] + z_\eta \eta_{yy} + \eta_y$$

$$\left[z_{\eta\xi} \xi_y + z_{\eta\eta} \eta_y \right]$$

$$t = z_\xi \xi_{yy} + z_{\xi\xi} \xi_y^2 + z_{\eta\eta} \eta_y^2 + 2 \eta_y \xi_y z_{\eta\xi} + z_\eta \eta_{yy}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left[z_\xi \xi_x + z_\eta \eta_x \right]$$

$$= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

$$= \frac{\partial}{\partial y} \left(z_\xi \xi_x \right) + \frac{\partial}{\partial y} \left(z_\eta \cdot \eta_x \right)$$

$$= \frac{\partial z_\xi}{\partial y} \xi_x + z_\xi \xi_x \xi_y + \frac{\partial z_\eta}{\partial y} \eta_x +$$

$$z_\eta \eta_x \cdot \eta_y$$

$$= \left(\frac{\partial z_\xi}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z_\xi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) \xi_x +$$

$$z_\xi \xi_x \xi_y + \left(\frac{\partial z_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) \eta_x$$

$$+ z_\eta \cdot \eta_x \eta_y$$

$$= \left(z_{\xi\xi} \xi_y + z_{\xi\eta} \eta_y \right) \xi_x + z_\xi \cdot \xi_x \xi_y +$$

$$\left(z_{\eta\xi} \xi_y + z_{\eta\eta} \eta_y \right) \eta_x + z_\eta \eta_x \eta_y$$

$$s = z_{\xi\xi} \cdot \xi_x \xi_y + z_\xi \xi_x \xi_y + z_{\xi\eta} \xi_x \eta_y +$$

$$z_{\eta\eta} \eta_x \eta_y + z_\eta \cdot \eta_x \eta_y + z_{\eta\xi} \eta_x \xi_y$$

sub this value eqn (1)

$$R\left[Z_{\xi\xi}\xi_x^2 + Z_\xi \xi_{xx} + Z_{\eta\eta}\eta_x^2 + Z_\eta \eta_{xx} + 2Z_{\xi\eta}\xi_x\eta_x\right] +$$

$$T\left[Z_\xi \xi_{yy} + Z_{\xi\xi}\xi_y^2 + 2Z_{\xi\eta}\xi_y\eta_y + Z_{\eta\eta}\eta_y^2 + \eta_{yy}Z_\eta\right] +$$

$$S\left[Z_{\xi\xi}\xi_x\xi_y + Z_\xi \xi_x\xi_y + Z_{\xi\eta}\xi_x\eta_y + Z_{\eta\eta}\eta_x\eta_y + Z_\eta \eta_x\eta_y + Z_{\eta\xi}\eta_x\xi_y\right] + f(x, y, z, p, q) = 0$$

$$\Rightarrow Z_{\xi\xi}\left(R\xi_x^2 + T\xi_y^2 + S\xi_x\xi_y\right) + Z_{\eta\eta}\left(R\eta_x^2 + T\eta_y^2 + S\eta_x\eta_y\right)$$

$$+ Z_{\eta\xi}\left(2R\xi_x\eta_x + 2T\xi_y\eta_y + S(\xi_x\eta_y + \eta_x\xi_y)\right) + f(\xi, \eta, z, Z_\xi, Z_\eta) = 0$$

$Z_{\eta\xi}$

$$\therefore A(\xi_x, \xi_y)Z_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y) + A(\eta_x, \eta_y)Z_{\eta\eta} = -f(\xi, \eta, z, Z_\xi, Z_\eta) \quad (2)$$

where $A(\xi_x, \xi_y) = (R\xi_x^2 + T\xi_y^2 + S\xi_x\xi_y)$.

$$A(\eta_x, \eta_y) = (R\eta_x^2 + T\eta_y^2 + S\eta_x\eta_y)$$

$$2B(\xi_x, \xi_y, \eta_x, \eta_y) = (2R\xi_x\eta_x + 2T\xi_y\eta_y + S(\xi_x\eta_y + \eta_x\xi_y)$$

Put $A = A(\xi_x, \xi_y)$

$B = 2B(\xi_x, \xi_y, \eta_x, \eta_y)$

$C = A(\eta_x, \eta_y)$

Let $B^2 = 4\left[B(\xi_x, \xi_y, \eta_x, \eta_y)\right]^2$.

from the above eqn we can show that

$$B^2 - 4AC = (S^2 - 4RT) J$$

$$4\left[B(\xi_x, \xi_y, \eta_x, \eta_y)\right]^2 - 4A(\xi_x, \xi_y)\, A(\eta_x, \eta_y)$$

$$= (S^2 - 4RT) J \quad \text{--- } (3)$$

where $J = \xi_x \eta_y - \eta_x \xi_y$.

Find the canonical form of

i) Hyperbolic. $(S^2 - 4RT) > 0$

ii) parabolic $(S^2 - 4RT) = 0$

iii) Elliptic $(S^2 - 4RT) < 0$

proof :-

case(i) :- Hyperbolic

$(S^2 - 4RT) > 0$

consider the quadratic equation

$$R\lambda^2 + S\lambda + T = 0$$

has real and distinct roots.

Let λ_1 & λ_2 be the real & distinct

roots of $R\lambda^2 + S\lambda + T = 0$.

choose. ξ and η

such that

$$\xi_x = \lambda_1 \xi_y \quad \& \quad \eta_x = \lambda_2 \eta_y \quad \text{--- } (1)$$

$$\lambda_1 = \frac{\xi_x}{\xi_y} \quad \& \quad \lambda_2 = \frac{\eta_x}{\eta_y}$$

Consider,

$$\xi_x - \lambda_1 \xi_y = 0$$

by lagranges form

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

(1) & (2) term

$$\frac{dx}{1} = \frac{dy}{-\lambda_1}$$

$$-\lambda_1 \, dx = dy$$

$$-\lambda_1 = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \lambda_1 = 0$$

$\Rightarrow$

$$p(x, y) = c_1$$

(1) & (3)rd terms

$$\frac{dx}{1} = \frac{d\xi}{0}$$

$$d\xi = 0$$

Jing $\xi = \phi_1(c_1)$

$$= \phi_1(p(x, y))$$

$$\xi = f_1(x, y)$$

$|||^{ly}$ we have

$$\frac{dy}{dx} + \lambda_2 = 0 \quad \text{and} \quad \eta = f_2(x, y)$$

W.K.T

$$A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$$

$$= \xi_y^2\left[R\left(\frac{\xi_x}{\xi_y}\right)^2 + S\frac{\xi_x}{\xi_y} + T\right]$$

$$= \xi_y^2\left[R\lambda^2 + S\lambda + T\right]$$

$$A(\xi_x, \xi_y) = \xi_y^2(0) = 0 \quad \left[\because \lambda \text{ is a root } R\lambda^2 + S\lambda + T = 0\right.$$

$$\Rightarrow A = 0$$

$||^{ly}$ $A(\eta_x, \eta_y) = 0 \quad \Rightarrow C = 0$

Also, $B^2 - 4AC = S^2 - 4RT > 0$

$$\therefore B^2 - 4AC > 0$$

$$B^2 - 0 > 0$$

$$B > 0$$

(2) in canonical form becomes,

$$A(\xi_x, \xi_y) z_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y) z_{\eta\xi} + A(\eta_x, \eta_y) z_{\eta\eta} + f(\xi, \eta, z, z_\xi, z_\eta) = 0$$

$$2B(\xi_x, \xi_y, \eta_x, \eta_y) z_{\eta\xi} = -f(\xi, \eta, z, z_\xi, z_\eta)$$

$$z_{\eta\xi} = \frac{-f(\xi, \eta, z, z_\xi, z_\eta)}{2B(\xi_x, \xi_y, \eta_x, \eta_y)}$$

$$z_{\eta\xi} = g(\xi, \eta, z, z_\xi, z_\eta)$$

which is canonical form of Hyperbolic

case(ii):- $(S^2 - 4RT) = 0$ parabolic

Consider, $R\lambda^2 + S\lambda + T = 0$

has equal & real roots. such that $\lambda_1 = \lambda_2 = \lambda$

choose ξ

such that,

$$\boxed{\xi_x = \lambda \xi_y}$$

$$\xi = f_1(x, y) \quad (\text{by case (i)})$$

now $A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$

$$= \xi_y^2 \left[R\left(\frac{\xi_x}{\xi_y}\right)^2 + S\left(\frac{\xi_x}{\xi_y}\right) + T\right]$$

$$= \xi_y^2 \left(R\lambda_1^2 + S\lambda_1 + T\right)$$

$$A(\xi_x, \xi_y) = 0$$

Also $B^2 - 4AC = S^2 - 4RT = 0$

$$B^2 - A(0) = 0$$

$$B^2 = 0$$

$$B = 0$$

eqn (3) becomes,

$$0(z_{\xi\xi}) + 0\, z_{\eta\xi} + A z_{\eta\eta} = -f(\xi, \eta, z, z_\xi, z_\eta)$$

$$z_{\eta\eta} = g(\xi, \eta, z, z_\xi, z_\eta)$$

which is Canonical form of parabolic

P.D.E.

Case (iii): $S^2 - 4RT < 0$ Elliptic.

Consider

$R\lambda^2 + S\lambda + T = 0$ has two imaginary roots λ_1, λ_2

Consider the transformation

$\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$, where α & β are real.

$\alpha = \frac{1}{2}(\xi + \eta)$; $\beta = -i/2 (\xi - \eta)$

$\xi - \eta = 2i\beta$

$\beta = \frac{1}{2i}(\xi - \eta)$

$\beta = -\frac{i}{2}(\xi - \eta)$

$\alpha_\xi = 1/2 (1) = 1/2$; $\beta_\xi = -i/2 (1) = -i/2$

$\alpha_\eta = 1/2 (0+1) = 1/2$; $\beta_\eta = -i/2(-1) = i/2$

$$Z_\xi = Z_\alpha \alpha_\xi + Z_\beta \beta_\xi$$

$$= Z_\alpha 1/2 + Z_\beta (-i/2)$$

$$Z_\xi = 1/2 (Z_\alpha - i Z_\beta)$$

$$Z_{\xi\eta} = 1/2 \frac{d}{d\eta} (Z_\alpha - i Z_\beta)$$

$$= 1/2 \left[Z_{\alpha\alpha} \alpha_\eta + Z_{\alpha\beta} \beta_\eta\right] - i/2 \left[Z_{\beta\alpha} \alpha_\eta + Z_{\beta\beta} \beta_\eta\right]$$

$$= 1/2 \left[Z_{\alpha\alpha} \alpha_\eta + Z_{\alpha\beta} \beta_\eta - i Z_{\beta\alpha} \alpha_\eta - i Z_{\beta\beta} \beta_\eta\right]$$

$$= \frac{1}{2}\left[Z_{\alpha\alpha} (1/2) + Z_{\alpha\beta} (i/2) - i Z_{\beta\alpha} (1/2) - i Z_{\beta\beta} (i/2)\right]$$

$$Z_{\xi\eta} = 1/2 \left[Z_{\alpha\alpha}(1/2) + Z_{\beta\beta}(1/2)\right] = \frac{1}{4}\left(Z_{\alpha\alpha} + Z_{\beta\beta}\right)$$

proceeding in the similar lines as in case (i):

$$z_{\eta\xi} = g(\xi, \eta, z, z_\xi, z_\eta) \quad -(3)$$

$$\frac{1}{4}\left[z_{\alpha\alpha} + z_{\beta\beta}\right] = g(\xi, \eta, z, z_\xi, z_\eta)$$

$$\left[z_{\alpha\alpha} + z_{\beta\beta}\right] = 4\left[g(\xi, \eta, z, z_\xi, z_\eta)\right]$$

$$z_{\alpha\alpha} + z_{\beta\beta} = \phi(\alpha, \beta, z, z_\alpha, z_\beta)$$

which is canonical form of Elliptic P.D.E.

Reduce the P.D.E $y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$ into canonical form and find the solution.

Soln:-

The given equation is

$$y^2 r - 2xys + x^2 t = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y} \quad (1)$$

Here $R = y^2$, $S = -2xy$, $T = x^2$

$$S^2 - 4RT = (-2xy)^2 - 4y^2x^2$$

$$S^2 - 4RT = 0$$

$\therefore$ eqn (1) is parabolic.

Now, $R\lambda^2 + S\lambda + T = 0$

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

$$(y\lambda - x)^2 = 0$$

$$(y\lambda - x)(y\lambda - x) = 0$$

$$\lambda = \frac{x}{y}, \frac{x}{y}.$$

choose ξ and η such that $\xi_x = \lambda \xi_y$.

$$\xi_x - \frac{x}{y}\xi_y = 0$$

by lagrange's form

$$\frac{dx}{1} = \frac{dy}{-x/y} = \frac{d\xi}{0}$$

(1) & (2)nd term

$$\frac{dx}{1} = \frac{dy}{-x/y}.$$

$$x\,dx = -y\,dy$$

$\int$ing $\frac{x^2}{2} = -y^2/2 + c_1$

$$x^2 + y^2 = c_1$$

from (1) & (3)

$$\frac{dx}{1} = \frac{d\xi_1}{0}$$

$$d\xi = 0$$

$\int$ing

$$\xi_1 = c_1 = x^2 + y^2$$

$|||^{ly}$ again choose η which is independent of ξ

i.e) $\eta = x^2 - y^2$ and $\xi = x^2 + y^2$

$\eta_x = 2x$ $\qquad\qquad$ $\xi_x = 2x$

$\eta_y = -2y$ $\qquad\qquad$ $\xi_y = 2y$

Also, $z = z(\xi, \eta)$

$$\frac{\partial z}{\partial x} = z_\xi \cdot \xi_x + z_\eta \cdot \eta_x$$

$$= z_\xi \, 2x + z_\eta \cdot 2x$$

$$\frac{\partial z}{\partial x} = 2x \left(z_\xi + z_\eta \right)$$

$$\frac{\partial z}{\partial y} = z_\xi \xi_y + z_\eta \eta_y$$

$$= z_\xi \cdot 2y + z_\eta (-2y)$$

$$\frac{\partial z}{\partial y} = 2y \left(z_\xi - z_\eta \right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(2x\left(z_\xi + z_\eta\right)\right)$$

$$= 2\left[(z_\xi + z_\eta)\cdot 1 + x\left(z_{\xi\xi}\xi_x + z_{\xi\eta}\eta_x + z_{\eta\xi}\xi_x + z_{\eta\eta}\eta_x\right)\right]$$

$$= 2\left[z_\xi + z_\eta + x\left(z_{\xi\xi}\, 2x + 2\, z_{\xi\eta}\, 2x + z_{\eta\eta}(2x)\right)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = 2\left[z_\xi + z_\eta + 2x^2\left(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}\right)\right]$$

III^{ly}

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(2y\left(z_\xi - z_\eta\right)\right)$$

$$\frac{\partial^2 z}{\partial y^2} = 2\left(z_\xi - z_\eta + 2y^2\left(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}\right)\right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y\left(z_\xi - z_\eta\right)\right)$$

$$= 2y\left(z_{\xi\xi}\xi_x + z_{\xi\eta}\eta_x - z_{\eta\xi}\xi_x - z_{\eta\eta}\eta_x\right]$$

$$= 2y\left(z_{\xi\xi}\cdot 2x + \cancel{z_{\xi\eta}(2x)} - \cancel{z_{\eta\xi}\, 2x} - z_{\eta\eta}(2x)\right]$$

$$= 2y\left(z_{\xi\xi}\cdot 2x - z_{\eta\eta}\cdot 2x\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xy\left(z_{\xi\xi} - z_{\eta\eta}\right)$$

sub above values in (1)

$$y^2\left[2\left(z_\xi + z_\eta + 2x^2\left(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}\right)\right] - 2xy\left(4xy\left(z_{\xi\xi} - z_{\eta\eta}\right)\right] + x^2 2\left[z_\xi - z_\eta + 2y^2\left(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}\right)\right]$$

$$= \frac{y^2}{x}\left(2x\left(z_\xi + z_\eta\right) + \frac{x^2}{y}\left(2y\left(z_\xi - z_\eta\right)\right)\right.$$

$$\Rightarrow 2y^2 z_\xi + 2y^2 z_\eta + 4x^2y^2 z_{\xi\xi} + \underline{4x^2y^2 z_{\eta\eta}} - 8x^2y^2 z_{\xi\eta} - 8x^2y^2 z_{\xi\xi} + \underline{8x^2y^2 z_{\eta\eta}} + 2x^2 z_\xi - 2x^2 z_\eta + 4x^2y^2 z_{\xi\xi} - 8x^2y^2 z_{\xi\eta} + \underline{4x^2y^2 z_{\eta\eta}} = \cancel{2y^2 z_\xi} + \cancel{2y^2 z_\eta} + \cancel{2x^2 z_\xi} - 2x^2 z_\eta$$

$$16x^2y^2z_{\eta\eta} = 0$$

we can find $\frac{\partial^2 z}{\partial \eta^2} = 0$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \eta}\right) = 0$$

$$\frac{\partial z}{\partial \eta} = A$$

$$\partial z = A\,\partial\eta$$

$$z = A\eta + B$$

where A & B are arbitrary function of ξ

$$\therefore z = \eta A(\xi) + B(\xi)$$

$$z = (x^2-y^2)A(x^2+y^2) + B(x^2+y^2)$$

10m Q2 : Reduce the equation

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n}\frac{\partial^2 z}{\partial y^2} = ny^{2n-1}\frac{\partial z}{\partial y}$$ to form canonical form & its general solution.

Soln:-

The given equation is

$$(n-1)^2 r - y^{2n} t = ny^{2n-1}\frac{\partial z}{\partial y} \quad \text{---} (1)$$

Here,

$$R = (n-1)^2, \quad S = 0, \quad T = -y^{2n}$$

$$S^2 - 4RT = -4(n-1)^2(-y^{2n})$$

$$= \cancel{-4[n^2+1-2n](-y^{2n})}$$

$= \cancel{+4n^2 y^{2n} + 4y^{2n} - 8ny^{2n}}$

$= +4(n-1)^2 (y)^{2n}$ $\qquad 4ny^{2n}\left[n + \frac{1}{n} - 2\right]$

$= 2^2 (n-1)^2 (y)^{2n}$

$= (2(n-1)y^n)^2 > 0$

$\therefore$ eqn (1) is hyperbolic.

Now,

$$R\lambda^2 + S\lambda + T = 0$$

$$(n-1)^2 \lambda^2 - y^{2n} = 0$$

$$(n-1)^2 \lambda^2 = y^{2n}$$

$$\lambda^2 = \frac{y^{2n}}{(n-1)^2}$$

$$\lambda = \pm \frac{y^{2n}}{(n-1)^2}$$

$$\lambda = \pm \frac{y^n}{(n-1)}$$

$$\lambda_1 = \frac{y^n}{(n-1)}, \quad \lambda_2 = -\frac{y^n}{(n-1)}$$

choose ξ and η

such that,

$$\xi_x = \lambda_1 \xi_y \quad \text{and} \quad \eta_x = \lambda_2 \eta_y$$

$$\xi_x = \frac{y^n}{(n-1)} \xi_y \quad \& \quad \eta_x = -\frac{y^n}{(n-1)} \eta_y$$

$$\xi_x - \frac{y^n}{(n-1)}\xi_y = 0 \quad \& \quad \eta_x + \frac{y^n}{(n-1)}\eta_y = 0.$$

By lagrange's form

$$\frac{dx}{1} = \frac{dy}{-y^n/(n-1)} = \frac{d\xi}{0}$$

(1) & (2)nd term,

$$\frac{dx}{1} = \frac{dy}{-y^n/(n-1)}$$

$$dx = -(n-1)\, y^{-n} dy$$

Jing

$$x = (\cancel{1-n})\frac{y^{-n+1}}{\cancel{-n+1}} + c$$

$$x = y^{1-n} + c. \quad \boxed{x - y^{-n} = c.}$$

By lagrange's form.
(1) & (3)rd term,

$$\frac{dx}{1} = \frac{d\xi}{0}$$

$$\int 0\, dx = \int d\xi$$

Jing

$$\xi = c = x - y^{1-n}$$

$$\eta = x + y^{1-n}$$

i.e.

$$\xi_x = 1 \quad ; \quad \xi_y = -(1-n)\, y^{-n}$$

$$\eta_x = 1 \quad ; \quad \eta_y = (1-n)\, y^{-n}$$

$z_{\xi\eta} = 0$

Also,

$$z = z(\xi, \eta)$$

$$\frac{\partial z}{\partial x} = z_\xi \cdot \xi_x + z_\eta \cdot \eta_x$$

$$= z_\xi \cdot 1 + z_\eta \cdot 1$$

$$\frac{\partial z}{\partial x} = z_\xi + z_\eta$$

$$\frac{\partial z}{\partial y} = z_\xi\, \xi_y + z_\eta \cdot \eta_y$$

$$= z_\xi\left[-(1-n)\, y^{-n}\right] + z_\eta \cdot (1-n)\, y^{-n}$$

$$\frac{\partial z}{\partial y} = (1-n)\, y^{-n}\left(z_\eta - z_\xi\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\partial x\,(z_\xi + z_\eta)\right)$$

$$= z_{\xi\xi}\, \xi_x + z_{\xi\eta}\, \eta_x + z_{\eta x}\, \eta_x + z_{\eta\eta}\, \eta_x$$

$$= z_{\xi\xi}(1) + z_{\xi\eta}(1) + z_{\eta x}(1) + z_{\eta\eta}(1)$$

$$= z_{\xi\xi} + z_{\xi\eta} + z_{\xi\eta} + z_{\eta\eta}$$

$$\frac{\partial^2 z}{\partial x^2} = z_{\xi\xi} + 2 z_{\xi\eta} + z_{\eta\eta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[(1-n)\, y^{-n}(z_\eta - z_\xi)\right]$$

$$= (1-n)\left[(z_\eta - z_\xi)(-n)\, y^{-n+1} + y^{-n}\left(z_{\eta\xi}\,\xi_y + z_{\eta\eta}\,\eta_y - z_{\xi\eta}\,\eta_y + z_{\xi\xi}\,\xi_y\right]\right.$$

$$= (1-n)\left[(z_\eta - z_\xi)(-n)\, y^{-n+1} + y^{-n}\right.$$

$$\left[z_{\eta\xi}(1-n)\, y^{-n} + z_{\eta\eta}(1-n)\, y^{-n} - z_{\xi\eta}(1-n)\, y^{-n} - z_{\xi\xi}(-)(1-n)\, y^{-n}\right]$$

$$= (1-n)\left[(-n)\, y^{-n+1}(z_\eta - z_\xi)\right] + y^{-n}$$

$$\left[z_{\eta\eta}(1-n)\, y^{-n} - 2 z_{\eta\xi}(1-n)\, y^{-n} - z_{\xi\xi}(1-n)\, y^{-n}\right]$$

$$\frac{\partial^2 z}{\partial y^2} = (1-n)\left[(-n)\, y^{-n+1}(z_\eta - z_\xi)\right] + y^{-2n}(1-n)\left[z_{\eta\eta} - 2 z_{\eta\xi} - z_{\xi\xi}\right].$$

Sub values of eqn (1)

$$(n-1)^2\left[z_{\xi\xi} + 2 z_{\xi\eta} + z_{\eta\eta}\right] - y^{2n}\left\{(1-n)\left[-n\, y^{-(n+1)}\right.\right.$$

$$\left.\left.(z_\eta - z_\xi) + y^{-2n}(1-n)\,(z_{\eta\eta} - 2 z_{\xi\eta} + z_{\xi\xi})\right]\right\} =$$

$$n\, y^{2n-1}(1-n)\, y^{-n}\left[z_\eta - z_\xi\right]$$

$$(n-1)^2\left[z_{\xi\xi}+2z_{\xi\eta}+z_{\eta\eta}\right]+y^{2n}(1-n)y^{-(n+1)}(z_\eta-z_\xi)$$

$$-(1-n)^2(z_{\eta\eta}-2z_{\xi\eta}+z_{\xi\xi})=ny^{2n-1}(1-n)y^{-n}(z_\eta-z_\xi)$$

$$(n-1)^2\left[z_{\xi\xi}+2z_{\xi\xi}+z_{\eta\eta}\right)-\left[(1-n)^2(z_{\eta\eta}-2z_{\xi\eta}+z_{\xi\xi})\right]=$$

$$(n-1)^2\left[z_{\xi\xi}+2z_{\xi\xi}+z_{\eta\eta}\right)-\left[(n-1)^2(-1)^2(z_{\eta\eta}-2z_{\xi\eta}-z_{\xi\xi}\right]=$$

$$(n-1)^2\left[\cancel{z_{\xi\xi}}+2z_{\xi\eta}+\cancel{z_{\eta\eta}}-\cancel{z_{\eta\eta}}+2z_{\xi\eta}-\cancel{z_{\xi\xi}}\right]=0$$

$$(n-1)^2\,4\,z_{\xi\eta}=0$$

$$z_{\xi\eta}=0$$

$$\frac{\partial^2 z}{\partial\xi\,\partial\eta}=0$$

$$\frac{\partial}{\partial\xi}\left(\frac{\partial z}{\partial\eta}\right)=0$$

$$z=f_1(\xi)\eta+f_2(\eta)$$

$$z=f_1(x-y^{1-n})(x+y^{1-n})+f_2(x+y^{-n})$$

Which is a required solution.

ADJOINT OPERATOR:-

Let $LU = \phi$, where L is a differential operator is given by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_n(x)$$

The adjoint operator L^* associated with L is to form the product vLu

i.e) $$\int_A^B vLu\,dx = [vu]_A^B - \int_A^B uL^* v\,dx$$

which is obtain after repeated integration by parts.

Here, L^* is called the adjoint operator of L.

$R w^* \quad R w^* \quad w x^*$
$R x w^* \quad R x^*$
$R x^*$

SELF Adjoint Operator:-

If the operator $L = L^*$ then L is called self adjoint operator.

Problem:-

If L is a operator $R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2}$ $P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z$ and M is a adjoint operator defined by $MW = \frac{\partial^2}{\partial x^2}(RW) + \frac{\partial^2}{\partial x \partial y}(SW) +$

$\frac{\partial^2}{\partial y^2}(TW) - \frac{\partial}{\partial x}(PW) - \frac{\partial}{\partial y}(QW) + ZW$

Then show that 11.00 11.00 3.00 12.00 5.00 3, 6.15 8 9.00

$$\iint_S (wLz - zMw)\,dx\,dy = \int_C [U\cos(n,x) + V\cos(n,y)]\,ds$$

where c is closed curve enclosing on a surface S. where $U = Rw\frac{\partial z}{\partial x} - z\frac{\partial}{\partial x}(Rw) - z\frac{\partial}{\partial y}(Sw) + Pzw$ where $V = Sw\frac{\partial z}{\partial x} + Tw\frac{\partial z}{\partial y} - z\frac{\partial}{\partial y}(Tw) + Qzw$ If $R_x + \frac{1}{2}S_y = P$ and $\frac{1}{2}S_x + T_y = Q$ show that the operator L is self adjoint

Soln:-

$$wLz = wR\frac{\partial^2 z}{\partial x^2} + wS\frac{\partial^2 z}{\partial x \partial y} + wT\frac{\partial^2 z}{\partial y^2} + wP\frac{\partial z}{\partial x} + wQ\frac{\partial z}{\partial y} + wZz$$

$$zMw = z\frac{\partial^2}{\partial x^2}(Rw) + z\frac{\partial^2}{\partial x \partial y}(Sw) + z\frac{\partial^2}{\partial y^2}(Tw) - z\frac{\partial}{\partial x}(Pw) - z\frac{\partial}{\partial y}(Qw) + zZw$$

$$wLz - zMw = \left(wR\frac{\partial^2 z}{\partial x^2} - z\frac{\partial^2}{\partial x^2}(Rw)\right) + \left(wS\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 (Sw)}{\partial x \partial y}\right) + \left(wT\frac{\partial^2 z}{\partial y^2} - z\frac{\partial^2 (wT)}{\partial y^2}\right) + \left(wP\frac{\partial z}{\partial x} + z\frac{\partial (Pw)}{\partial x}\right) + \left(wQ\frac{\partial z}{\partial y} + z\frac{\partial (Qw)}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(wR\frac{\partial z}{\partial x} - z\frac{\partial}{\partial x}Rw - z\frac{\partial (Sw)}{\partial y} + zPw\right)$$

$$+ \frac{\partial}{\partial y}\left(wS\frac{\partial z}{\partial x} + wT\frac{\partial z}{\partial y} - z\frac{\partial (wT)}{\partial y} + Qwz\right)$$

$$\left[\; z\frac{\partial (Pw)}{\partial x} + Pw\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(zPw)\right.$$

$$wLz - zMw = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$$

$$\iint_S (wLz - zMw)\,dx\,dy = \iint \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)dx\,dy$$

$$= \int_C (lU + mV)\,ds$$

$$= \int_C \{U\cos(n,x) + V(\cos(n,y))\}$$

Also, $zMw = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x\partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) -$

$$\frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw$$

$$= R_{xx}w + R_x w_x + R_x w_x + R w_{xx} + S_y w_x +$$

$$S_x w_y + S w_{xy} + w S_{xy} + T_{yy} w + T_y w_y +$$

$$T_y w_y + T w_{yy} - P_x w - P w_x - Q_y w - Q w_y + Zw$$

$\left[\because R_x + \tfrac{1}{2}S_y = P\right.$

$2R_x + S_y = 2P$

$\left.S_x + 2T_y = 2Q\right]$

$$= R w_{xx} + w_x(2R_x + S_y) + S w_{xy} + w_y(2T_y + S_x)$$

$$+ T w_{yy} + w(R_{xx} + S_{xy} + T_{yy} - P_x - Q_y + Z) - P w_x$$

$$- Q w_y$$

$$= R w_{xx} + 2 w_x P + S w_{xy} + 2 w_y Q + T w_{yy} + w Z_1 -$$

$$P w_x - Q w_y$$

where $Z_1 = R_{xx} + S_{xy} + T_{yy} - P_x - Q_y + Z$

$\therefore MW = R\frac{\partial^2 w}{\partial x^2} + S\frac{\partial^2 w}{\partial x \partial y} + T\frac{\partial^2 w}{\partial y^2} + P\frac{\partial w}{\partial x} + Q\frac{\partial w}{\partial y}$

$+ wz,$

$M(w) = L(w)$

$\therefore M = L$

$\therefore \frac{L}{M}$ is self adjoint.

Q

2) Construct an operator adjoint to the laplace operator and prove that it is self adjoint.

(10)

$L(u) = U_{xx} + U_{yy}$

Proof:-

W.K.T

The laplace operator L is defined by

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Here,

$R = 1$, $S = 0$, $T = 1$, $P = 0$, $Q = 0$ and $z = 0$

The adjoint operator is defined by

$$L^* w = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + zw$$

$$L^*(w) = \frac{\partial^2}{\partial x^2}(w) + \frac{\partial^2}{\partial y^2}(w)$$

$$L^*(w) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$L^*(w) = L(w)$$

$\therefore$ L is a self adjoint.

3) Determine the adjoint operator of $L(u)$

$L(u) = c^2 u_{xx} - u_{tt}$ then prove that the Operator L is self adjoint.

Proof:-

The adjoint operator of

$$c^2 \frac{\partial^2}{\partial x^2} - 1 \frac{\partial^2}{\partial y^2}$$

$$L(u) = c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$$

Here,

$$R = c^2,\ S = 0,\ T = -1,\ P = 0,\ Q = 0,\ Z = 0$$

$$L^*(u) = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw$$

$$= \frac{\partial^2}{\partial x^2}(c^2 w) + \frac{\partial^2}{\partial y^2}(-w)$$

$$= \frac{\partial^2}{\partial x^2} c^2 w - \frac{\partial^2}{\partial y^2} w.$$

$$L^*(w) = \frac{\partial^2 w}{\partial x^2} c^2 - \frac{\partial^2 w}{\partial y^2}$$

$\therefore$ L is self adjoint.

Riemann Method:-

Consider the linear hyperbolic PDE

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = f(x,y) \quad \text{—(1)}$$

where a, b, c are function of x & y.

Let $L = \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c$

eqn (1) becomes

$L(z) = f(x,y)$

Let M be the adjoint operator of L which is given by

$$M(w) = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial}{\partial x}(aw) - \frac{\partial}{\partial y}(bw) + cw$$

Now,

$$wL(z) - zM(w) = \left(w\frac{\partial^2 z}{\partial x \partial y} + aw\frac{\partial z}{\partial x} + bw\frac{\partial z}{\partial y} + \cancel{wcz}\right)$$

$$- \left(z\frac{\partial^2 w}{\partial x \partial y} - z\frac{\partial}{\partial x}(aw) - z\frac{\partial}{\partial y}(bw) + \cancel{zcw}\right)$$

$$= \left(w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} + \underbrace{aw\frac{\partial z}{\partial x} + z\frac{\partial (aw)}{\partial x}}_{\frac{\partial (awz)}{\partial x}}\right.$$

$$\left. + \underbrace{bw\frac{\partial z}{\partial y} + z\frac{\partial (bw)}{\partial y}}_{\frac{\partial (bwz)}{\partial y}}\right)$$

$$= \frac{\partial}{\partial x}\left(waz - z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(wbz + w\frac{\partial z}{\partial x}\right)$$

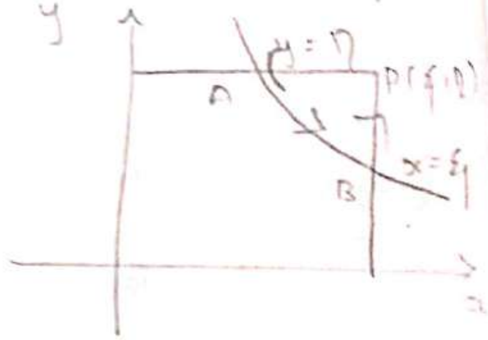
$$wLz - zMw = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \quad -(2)$$

where $u = waz - z\frac{\partial w}{\partial y}$

$V = wbz + w\frac{\partial z}{\partial x}$ — (*)

Consider the arc of a curve, where the line PA & PB parallel to x & y respectively.

Let 's' denote the area enclosed by the contour ABPA.



Clearly,

On AP, $y = \eta \Rightarrow dy = 0$

on PB, $x = \xi \Rightarrow dx = 0$

Taking

$\iint_S$ on both sides of eqn (2)

$$\iint_S (wLz - zMw)\,dx\,dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \int_C (u\,dy - v\,dx) = \int_{ABPA} (u\,dy - v\,dx)$$

$$\iint_S (wLz - zMw)\,dx\,dy = \int_A^B (u\,dy - v\,dx) + \int_B^P u\,dy + \int_P^A (-v)\,dx$$

— (3)

Now,

$$\int_P^A (v)\,dx = \int_P^A \left(wbz + w\frac{\partial z}{\partial x} \right) dx$$

$$= \int_P^A \left(wbz + w\frac{\partial z}{\partial x} + z\frac{\partial w}{\partial x} - z\frac{\partial w}{\partial x} \right) dx$$

$$= \int_P^A \left(wbz - z\frac{\partial w}{\partial x} \right) dx + [wz]_P^A$$

$$= \int_P^A \left(wbz - z\frac{\partial w}{\partial x} \right) dx + [wz]_A - [wz]_P \quad \text{(3A)}$$

Now,

$$(3) \Rightarrow \int_P^A (v)\,dx = \int_A^B (u\,dy - v\,dx) + \int_B^P u\,dy - \iint_S (wLz - zMw)\,dx\,dy$$

$$\text{(3A)} \Rightarrow [wz]_P = \int_P^A \left(wbz - z\frac{\partial w}{\partial x} \right) dx + (wz)_A - \int_A^B (u\,dy - v\,dx) - \int_B^P u\,dy + \iint_S (wLz - zMw)\,dx\,dy$$

$$[wz]_P = \int_P^A \left(wbz - z\frac{\partial w}{\partial x} \right) dx + (wz)_A - \int_A^B (u\,dy - v\,dx) - \int_B^P u\,dy + \iint_S (wLz - zMw)\,dx\,dy$$

$$[wz]_P = [wz]_A + \int_P^A z\left(wb - \frac{\partial w}{\partial x} \right) dx - \int_A^B wz(a\,dy - b\,dx) + \int_A^B \left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx \right) - \int_B^P z\left(wa - \frac{\partial w}{\partial y} \right) dy + \iint_S (wLz - zMw)\,dx\,dy \quad (4)$$

Since the function w is arbitrary. we can choose w to satisfies foll condi

i) $M(w) = 0$ on S.

ii) $\frac{\partial w}{\partial y} = aw$ on $x = \xi$

iii) $\frac{\partial w}{\partial x} = bw$ on $y = \eta$

iv) $[w]_p = 1$ (or) $w = 1$ on $P(\xi, \eta)$

$$(4) \Rightarrow [z]_p = [wz]_A - \int_A^B wz(a\,dy - b\,dx) + \int_A^B \left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf(x,y)\,dx\,dy \quad (5)$$

This gives the value of z at any point (ξ, η) when the value of z & $\frac{\partial z}{\partial x}$ are given on the curve AB.

If the value of z and $\frac{\partial z}{\partial y}$ given then consider

$u\,dv + v\,du = uv$

$$\int_A^B z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx = \int_A^B \left(z\frac{\partial w}{\partial y} + w\frac{\partial z}{\partial y} - w\frac{\partial z}{\partial y}\right)dy + \int_A^B \left(w\frac{\partial z}{\partial x} + z\frac{\partial w}{\partial x} - z\frac{\partial w}{\partial x}\right)dx$$

uv

$$= \int_A^B \frac{\partial}{\partial y}(zw)dy - \int_A^B w\frac{\partial z}{\partial y}dy + \int_A^B \frac{\partial}{\partial x}(wz)dx - \int_A^B z\frac{\partial w}{\partial x}dx$$

$u\,dv + v\,du = uv$

$w\partial(z) + \partial(wz)$

$\int_A^B z\partial(w) + \ldots = uv$

$$= \int_A^B \partial(zw) - \int_A^B \left(w\frac{\partial z}{\partial y}dy + z\cdot\frac{\partial w}{\partial x}dx\right)$$

$$\int_A^B z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx = [zw]_B - [zw]_A - \int_A^B \left(w\frac{\partial z}{\partial y}dy + z\frac{\partial w}{\partial x}dx\right) \quad \text{—(6)}$$

sub (6) in (5)

$$[z]_P = \cancel{[wz]_A} - \int_A^B wz\,(a\,dy - b\,dx) + [wz]_B - \cancel{[wz]_A}$$

$$- \int_A^B \left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right) + \iint_S wf(x,y)\,dx\,dy \quad \text{—(7)}$$

(5) + (7) ⇒

$$2[z]_P = [wz]_A + [wz]_B - 2\int_A^B wz(a\,dy - b\,dx) +$$

$$\int_A^B \left[z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right] - \int_A^B \left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right)$$

$$+ 2\iint_S wf(x,y)\,dx\,dy.$$

$$[z]_P = \frac{[wz]_A + [wz]_B}{2} - \int_A^B wz\,(a\,dy - b\,dx)$$

$$+\frac{1}{2}\int_A^B \left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) - \frac{1}{2}\int_A \left(w\frac{\partial z}{\partial y}dy +\right.$$

$$\left. z\frac{\partial w}{\partial x}dx\right) + \iint_S wf(x,y)\,dx\,dy \quad \text{—(8)}$$

∴ Eqn (8) is the value of z at any point (ξ, η) when z, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are given on curve Γ

And eqn (7) is used when z, $\frac{\partial z}{\partial y}$ are given on Γ

(8) Prove that for the equation $\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4} z = 0$, is the Green's function is given by

$w(x, y, \xi, \eta) = J_0 \sqrt{(x-\xi)(y-\eta)}$. where $J_0(z)$ is Bessel's function on 1^{st} kind with order zero.

10 M

Soln:-

Given that $\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{4} z = 0$ — (1)

Here $L = \frac{\partial^2}{\partial x \partial y} + \frac{1}{4}$ then

$a = b = 0$ and $c = 1/4$

∴ adjoint operator of L is self adjoint

$$M(w) = \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{4} w.$$

$$wLz - zMw = \left(w \frac{\partial^2 z}{\partial x \partial y} + w \frac{1}{4} z\right) - \left(z \frac{\partial^2 w}{\partial x \partial y} + z \frac{1}{4} w\right)$$

$$= \frac{\partial}{\partial y}\left(w \frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z \cdot \frac{\partial w}{\partial y}\right)$$

$wLz - zMw = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$, where $U = -z \frac{\partial w}{\partial y}$, $V = w \frac{\partial z}{\partial x}$

$$\iint_S (wLz - zMw)\, dx\, dy = \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\, dy$$

$$= \int_C U\,dy - V\,dx$$

$$= \int_A^B (U\,dy - V\,dx) + \int_B^P U\,dy - \int_P^A V\,dx$$

$$= \int_A^B (u\,dy - v\,dx) - \int_B^P z\frac{\partial w}{\partial y}\,dy - \int_P^A w\frac{\partial z}{\partial x}\,dx - \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$+ \int_P^A z\frac{\partial w}{\partial x}\,dx.$$

$$= \int_A^B (u\,dy - v\,dx) - \int_B^P z\frac{\partial w}{\partial y}\,dy - \int_P^A \frac{d}{dx}(wz)\,dx + \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$= \int_A^B (u\,dy - v\,dx) - \int_B^P z\frac{\partial w}{\partial y}\,dy - (wz)_A + (wz)_P + \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$[wz]_P = -\int_A^B (u\,dy - v\,dx) + \int_B^P z\frac{\partial w}{\partial y}\,dy + [wz]_A - \int_P^A z\frac{\partial w}{\partial x}\,dx$$

$$+ \iint_S (wLz - zMw)\,dx\,dy. \quad \longrightarrow (2)$$

choose w such that,

i) $(w) = 1$ at $P(\xi, \eta)$

ii) $\frac{\partial w}{\partial y} = 0$ on $x = \xi$

iii) $\frac{\partial w}{\partial x} = 0$ on $y = \eta$

iv) $M(w) = 0$, on S

Let $w = W(f)$

where $f^2 = (x-\xi)(y-\eta)$

$$2f\frac{\partial f}{\partial x} = y - \eta$$

$$2f\frac{\partial f}{\partial y} = x - \xi$$

Assume $w = w(f)$ satisfies condition (i) to (iv)

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial f}\cdot\frac{\partial f}{\partial x} = \frac{\partial w}{\partial f}\left(\frac{y-\eta}{2f}\right)$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial f}\left(\frac{y-\eta}{\partial f}\right)\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\left(w_f \cdot f^{-1}(y-\eta)\right)$$

$$= \frac{1}{2}\left[w_{ff} f_y f^{-1}(y-\eta) + w_f (-1) f^{-2} f_y (y-\eta) + w_f f^{-1}\right]$$

$$= \frac{1}{2}\left[w_{ff}\left(\frac{x-\xi}{2f}\right) f^{-1}(y-\eta) - w_f f^{-2}\left(\frac{x-\xi}{2f}\right)(y-\eta) + w_f f^{-1}\right]$$

$$= \frac{1}{2}\left[w_{ff} \frac{f^{-1} f^2 f^{-1}}{2} - w_f f^{+2} \frac{f^{-1} f^{-2}}{2} + w_f f^{-1}\right]$$

$$= \frac{1}{2}\left[w_{ff} \frac{1}{2} - w_f \frac{f^{-1}}{2} + w_f f^{-1}\right]$$

$$= \frac{1}{4}\left[w_{ff} - w_f f^{-1} + 2 w_f f^{-1}\right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{4}\left[w_{ff} + w_f f^{-1}\right]$$

Using (iv), $M(w) = 0$

$$\therefore M(w) = \frac{1}{4}\left[\frac{d^2 w}{df^2} + f^{-1}\frac{dw}{df} + w\right] = 0$$

$$\times 4f^2 \Rightarrow f^2 \frac{d^2 w}{df^2} + f\frac{dw}{df} + f^2 w = 0 \quad \text{——— (5)}$$

where w is w. bessel's function

i.e) $w = J_0(f)$

$w = J_0 \sqrt{(x-\xi)(y-\eta)}$ is a solution

of (2) satisfies (i) to (iv) conditions

i) $M(w) = 0$ on S.

ii) $\frac{\partial w}{\partial y} = a w$ on $x = \xi$

iii) $\frac{\partial w}{\partial x} = bw$ on $y = \eta$

iv) $[w]_p = 1$ (or) $w = 1.0$ on $p(\xi, \eta)$

$\therefore$ Putting the value of w in (2) we get the value of Z.. of given p.d.e.

UNIT-II

Elliptic Differential Equation

24.07.19

5m Derivation of Laplace equation:-

Let two particle of masses m_1 and m_2 be situated at any points P and Q with a distance 'r'.

By Newton's gravitational law

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

where G is gravitational constant.

Let us define a position vector

$\vec{r} = \vec{PQ}$.

Assume that, unit mass at Q & G.

i.e) $m_2 = 1$ & $G = 1$

$$\therefore \boxed{F = \frac{m_1}{r^2}}$$

The force at Q due to the mass at P is given by

$\vec{n} = \frac{\vec{a}}{|\vec{a}|}$

$$\vec{F} = F \cdot \vec{n}$$

$$= F\left(\frac{-\vec{r}}{r}\right)$$

$$= \frac{m_1}{r^2}\left(\frac{-\vec{r}}{r}\right)$$

$$= m_1 \nabla\left(\frac{1}{r}\right)$$

$$\vec{F} = \nabla\left(\frac{m_1}{r}\right) \quad \text{--- (1)}$$

$[\nabla(\frac{1}{r}) = -\frac{\vec{r}}{r^3}$

$\nabla(\frac{1}{r}) = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)(\frac{1}{r})$

$\nabla = \vec{i}\frac{\partial}{\partial x}(\frac{1}{r})$

$r \cdot r = x^2+y^2+z^2$

$r^2 = x^2+y^2+z^2$

$r = x\vec{i}+y\vec{j}+z\vec{k}$

$|r| = \sqrt{x^2+y^2+z^2}$

where $\vec{F}$ is called the indensity of gravitational force.

suppose a partice of the unit mass of the particle moves under a attraction of m_1 at P from ∞ to Q.

Then the workdone is

8.30 9.00 9.30

$$\boxed{W = \int_{\infty}^{r} \vec{F} \cdot d\vec{r}}$$

$$= \int_{\infty}^{r} \nabla \left(\frac{m_1}{r}\right) \vec{dr}$$

$$= \int_{\infty}^{r} \left(\Sigma \frac{\partial}{\partial x} \left(\frac{m_1}{r}\right) \vec{i} \right) \cdot \left(\Sigma dx \vec{i} \right)$$

$$= \int_{\infty}^{r} \Sigma \frac{\partial}{\partial x} \left(\frac{m_1}{r}\right) dx$$

$$= \int_{\infty}^{r} d\left(\frac{m_1}{r}\right)$$

$$= \left(\frac{m_1}{r}\right)_{\infty}^{r} \Rightarrow \frac{m_1}{r} - 0$$

$$\boxed{W = \frac{m_1}{r}}$$

This defines a potential energy & due the particle of P is denoted by

$$\boxed{V = -\frac{m_1}{r}} \quad \text{--- (2)} \qquad V = -W$$

from (1)

$$\Rightarrow \vec{F} = \nabla\left(\frac{m_1}{r}\right)$$

$$\vec{F} = \nabla(-V) \quad \text{--- (3)}$$

Consider the system of masses m_1, m_2, ---- m_n which are h-distance $r_1, \ldots r_n$ respectively.

Then the force of attraction for unit mass due to the system of masses $m_1, m_2, \ldots m_n$ is $\vec{F} = \vec{F_1} + \vec{F_2} + \cdots + \vec{F_n}$

25.7.19

$$\vec{F} = \nabla\left(\frac{m_1}{r_1}\right) + \nabla\left(\frac{m_2}{r_2}\right) + \cdots + \nabla\left(\frac{m_n}{r_n}\right)$$

$$= \sum_{i=1}^{n} \nabla\left(\frac{m_i}{r_i}\right)$$

$$\vec{F} = \nabla\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right)$$

$\left(\frac{1}{r}\right)$ $\frac{\partial r}{\partial r}$ $\frac{1}{r^2}$ r/r $1/r^2$

Then work done is :

$$W = \int_A^B \vec{F} \cdot d\vec{r}$$

$$= \int_{\infty}^{r} \vec{F} \cdot d\vec{r}$$

$$= \int_{\infty}^{r} \nabla\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right) d\vec{r}$$

$$= \int_{\infty}^{r} \left[\sum \frac{\partial}{\partial x}\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right)\vec{i}\right] \cdot \sum dx\,\vec{i}$$

$$= \int_{\infty} \Sigma \frac{\partial}{\partial x}\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right) dx$$

$$= \int_{\infty}^{r} d\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right)$$

$$= \left[\sum_{i=1}^{n} \frac{m_i}{r_i}\right]_{\infty}^{r}$$

$$= \sum_{i=1}^{n} \frac{m_i}{r_i}$$

$$w = -V$$

$$\therefore V = -\sum_{i=1}^{n} \frac{m_i}{r_i}$$

$$\therefore \nabla^2 V = \nabla^2\left(-\sum_{i=1}^{n} \frac{m_i}{r_i}\right)$$

$$= -\sum_{i=1}^{n} \nabla^2\left(\frac{m_i}{r_i}\right) \quad \left[\because \nabla^2\left(\frac{1}{r}\right)=0\right]$$

$$\nabla^2 V = 0 \text{ (laplace equation)} \left[\because \nabla^2\left(\frac{m_i}{r_i}\right) = 0\right]$$

where $\nabla^2 = \nabla \cdot \nabla$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is called laplace equation.

In case the matter of density ρ is continusly distributed in a volume V.

$$V = \iiint_V \frac{\rho(\xi, \eta, \zeta)}{r}\, d\vec{v}$$

are r is calculated by

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

Def:- Gauss Law:-

$$\iint_S \vec{g}\, d\vec{s} = -4\pi G M$$

where $\vec{g}$ = gravitational field ~~of~~ on system.

M = Total mass inside the system.

G = Gravitational Constant.

26.7.19 Derivation of poisson's Equation:-

Let S be a closed surface consisting the particles of masses $m_1, m_2, \ldots m_n$.

Let Q be any point on S and Let $\sum_{i=1}^{n} m_i = m$ be the total mass inside 'S'.

Let $\vec{g_1}, \ldots \vec{g_n}$ be the gravitational field at Q due to $m_1, m_2, \ldots m_n$ respectively in S.

Also,

$$\vec{g} = \sum_{i=1}^{n} \vec{g_i}$$

By Gauss, law,

$$\iint_S \vec{g} \cdot \vec{ds} = -4\pi G M$$

density $\rho = \frac{m \to mass}{V \to volume}$

$$= -4\pi G (\rho V)$$

$m = \rho V$

$$= -4\pi G \iiint_V \rho \, d\vec{v} \quad \text{—} (1)$$

$$= -4\pi G M.$$

where $M = \iiint_V \rho . d\vec{v}$

ρ = The mass density function.

since the gravitational field is a conservative field.

i.e) $\vec{g} = \nabla V$

where V is the scalar potential energy.

But, the Gauss divergence theorem

$$\iint_S \vec{g} \cdot \vec{ds} = \iiint_V \nabla \cdot \vec{g} \cdot d\vec{v} \quad \text{—} (2)$$

From (1) - (2)

$$0 = -4\pi G \iiint_V \rho . d\vec{v} - \iiint_V \nabla \cdot \vec{g} . d\vec{v}$$

$$= \iiint_V (-4\pi G \rho - \nabla \cdot \vec{g}) d\vec{v}$$

$$\therefore -4\pi G \rho - \nabla \cdot \vec{g} = 0$$

$$\nabla \cdot \vec{g} = -4\pi G \rho$$

$$\nabla \cdot (\nabla V) = -4\pi G \rho$$

$$\nabla^2 V = -4\pi G \rho$$

which is the Poisson Equation.

Boundary value problem for Laplace equation:..

Def:-

Boundary value problem for 1^{st} kind : (or)

Birichlet problem Dirichlet Problem

If $f \in C^{(0)}$ and is prescribed on the boundary, C of sum finite region R.

The problem is determined the function $\phi(x, y, z)$ such that,

$\nabla^2 \phi = 0$ in R and satisfying $\phi = f$ on 'c' is called the 1^{st} kind boundary value problem (or) Dirichlet problem.

2^{nd} kind (or) Neumann problem:-

This 2^{nd} kind of boundary value problem is determined by $\phi(x, y, z)$ such that $\nabla^2 \phi = 0$ with in R while $\frac{\partial \phi}{\partial x}$ is satisfied at every point on 'C'.

where $\frac{\partial \phi}{\partial x}$ is the normal derivative of ϕ, then above problem is 2^{nd} kind (or) Neumann problem.

3^{rd} kind (or) Mixed boundary value problem (or) churchills problem

This problem is determine the function $\phi(x,y,z)$ such that $\nabla^2\phi = 0$ with in R, while the boundary condition $\frac{\partial \phi}{\partial x} + h\phi = f$ on 'c' when $h \geq 0$, is satisfied at every point on the boundary 'c'.

This problem is called 3^{rd} kind (or) mixed boundary value problem (or) churchills problem.

? Seperation of variables method

Find the solution of the 2-D laplace eqn $\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ by using separation of variable Method.

soln:- Consider the 2-D laplace equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{---} (1)$$

Assume that the solution of the eqn (1) is of the form

$$U(x,y) = X(x) \cdot Y(y) \quad \text{---} (2)$$

$$(1) \Rightarrow \frac{\partial^2}{\partial x^2}\left(X(x) Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x) Y(y)\right) = 0$$

$$Y(y) \frac{\partial^2 X}{\partial x^2} + X(x) \frac{\partial^2 Y}{\partial y^2} = 0$$

$$Y \frac{\partial^2 X}{\partial x^2} = -X \frac{\partial^2 Y}{\partial y^2}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{-1}{Y}\frac{\partial^2 Y}{\partial y^2} = k$$

where k is a separation parameter.

Now,

$$\frac{\partial^2 X}{\partial x^2} = Xk \quad , \quad \frac{\partial^2 Y}{\partial y^2} = -kY$$

$$\frac{\partial^2 X}{\partial x^2} - kX = 0 \quad , \quad \frac{\partial^2 Y}{\partial y^2} + kY = 0 \quad \text{---} (3)$$

Case(i):-

if $k > 0$

Let $k = p^2$, p is real

(3). $\frac{\partial^2 X}{\partial x^2} - p^2 X = 0$

$(D^2 - p^2) X = 0$

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$\therefore$ C.F is $X(x) = c_1 e^{px} + c_2 e^{-px}$

Next $k = p^2$ in (3)

$$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0$$

$D^2 + p^2 = 0$

$m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$\therefore$ C.F is $= Y(y) = C_3 \cos py + C_4 \sin py$.

$\therefore U(x,y) = (c_1 e^{px} + c_2 e^{-px}) \cdot (C_3 \cos py + C_4 \sin y)$

Case(ii) :- $k<0$

Let $k = -p^2$

$$(3) \Rightarrow \frac{d^2X}{dx^2} - (-p^2)X = 0$$

$$\frac{d^2X}{dx^2} + p^2X = 0$$

$$D^2 + P^2X = 0$$

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$m = \pm ip$$

$\therefore$ C.F is, $X(x) = c_1 \cos px + c_2 \sin px$.

Next, $k = -p^2$ in (3)

$$\frac{d^2Y}{dy^2} + (-p^2)Y = 0$$

$$(D^2 - p^2)Y = 0$$

$$m^2 - p^2 = 0$$

$$m^2 = p^2$$

$$m = \pm P$$

$\therefore$ C.F is $Y(y) = c_3 e^{py} + c_4 e^{-py}$

$\therefore U(x,y) = (c_1 \cos px + c_2 \sin px) \cdot (c_3 e^{py} + c_4 e^{-py})$

Case(iii) $k=0$

$$(3) \Rightarrow \frac{d^2X}{dx^2} = 0 \; ; \; \frac{d^2Y}{dy^2} = 0$$

$$\int\text{ing} \Rightarrow \int \frac{d}{dx}\left(\frac{dX}{dx}\right) dx = 0 \;,\; \int \frac{d}{dy}\left(\frac{dY}{dy}\right) dy = 0$$

$$\therefore \frac{dX}{dx} = c_1 \;,\; \frac{dY}{dy} = c_2$$

sing $\therefore X = xc_1 + c_2$, $Y = yc_3 + c_4$

$\therefore u(x,y) = (xc_1 + c_2) \cdot (yc_3 + c_4)$

Remark:-

Polar Co-Ordinates:-

The Polar Co-Ordinates (r, θ) is defined by $x = r\cos\theta$, $y = r\sin\theta$ and also $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$

Cylindrical Co-Ordinates

cylindrical Co-Ordinates (r, θ, z) is defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and also we have $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$, $z = z$.

Spherical Co-Ordinates:-

(r, θ, ϕ) is defined by $x = r\cos\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and also $r^2 = x^2 + y^2 + z^2$,

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \phi = \tan^{-1}(y/x)$$

Remark:-

i) Laplace Equation in polar Co-Ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

ii) In ~~Cartesian~~ cylindrical Co-Ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(iii) Spherical Co-Ordinates:-

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial u}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

Problem:-

1) show that if the 2-D laplace equation $\nabla^2 u = 0$ is transfermed by plane polar co-ordinates (r, θ) is defined by the relation $x = r\cos\theta$, $y = r\sin\theta$ it takes the form $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

The laplace equation in cartesian co-ordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} (1)$$

The Relation between cartesian and polar - coordinates are

$x = r\cos\theta$, $y = r\sin\theta$.

The relation is $r^2 = x^2 + y^2$ (2), $\theta = \tan^{-1}(y/x)$ (3)

Diff eqn (2) with respect to x & y.

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y$$

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}$$

$$r_x = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta, \quad r_y = \frac{\cancel{r}\sin\theta}{\cancel{r}} = \sin\theta \quad r_z = 0$$

(3) diff w.r.to x & y

$$\theta_x = \frac{1}{1+y^2/x^2} y\left(-1/x^2\right)$$

$$= -\frac{y}{x^2+y^2}$$

$$= -\frac{r\sin\theta}{r^2(\sin^2\theta+\cos^2\theta)}$$

$$\theta_x = -\frac{\sin\theta}{r}$$

since $\theta_y = \frac{1}{1+y^2/x^2}\left(1/x\right)$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{\cancel{r}\cos\theta}{r^{\cancel{2}}}$$

$$\theta_z = 0$$

$$\theta_y = \frac{\cos\theta}{r}$$

since $U = U(r,\theta)$

$$U_x = U_r r_x + U_\theta \theta_x + U_z z_x$$

$$U_x = U_r \cos\theta + U_\theta\left(-\frac{\sin\theta}{r}\right) \quad \text{—} (4)$$

$$U_y = U_r \sin\theta + U_\theta\left(\frac{\cos\theta}{r}\right) \quad \text{—} (5)$$

(4) diff w.r.to x

$$U_{xx} = U_{xr}\, r_x + U_{x\theta}\, \theta_x \quad \text{—— (6)}$$

To find U_{xr}

$$(4) \Rightarrow U_x = U_r \cos\theta - U_\theta \frac{\sin\theta}{r}$$

$$U_{xr} = \cos\theta\, U_{rr} - \sin\theta\left(\frac{1}{r} U_{\theta r} + U_\theta\left(-1/r^2\right)\right) \quad \text{—— (7)}$$

To find $U_{x\theta}$

$$(4) \Rightarrow U_{x\theta} = U_{r\theta} \cos\theta + U_r(-\sin\theta) - \frac{1}{r}\left(U_{\theta\theta}\sin\theta + U_\theta \cos\theta\right)$$

$$U_{x\theta} = U_{r\theta}\cos\theta - U_r \sin\theta - \frac{U_{\theta\theta}\sin\theta}{r} - \frac{U_\theta \cos\theta}{r} \quad \text{—— (8)}$$

sub (7) & (8) in (6)

$$U_{xx} = \left[\cos\theta\, U_{rr} - \sin\theta\left(\frac{1}{r} U_{\theta r} + U_\theta\left(-1/r^2\right)\right)\right]\cos\theta$$

$$+ \left[U_{r\theta}\cos\theta - U_r \sin\theta - \frac{U_{\theta\theta}\sin\theta}{r} - \frac{U_\theta\cos\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right)$$

$$U_{xx} = U_{rr}\cos^2\theta - U_{\theta r}\frac{\sin\theta\cos\theta}{r} + U_\theta \frac{\sin\theta\cos\theta}{r^2}$$

$$- U_{\theta r}\frac{\sin\theta\cos\theta}{r} + \frac{U_r \sin^2\theta}{r} + \frac{U_{\theta\theta}\sin^2\theta}{r^2} +$$

$$\frac{U_\theta \sin\theta\cos\theta}{r^2} \quad \text{—— (A)}$$

|||^{ly}

$$U_{yy} = U_{rr}\sin^2\theta + U_{\theta r}\frac{\sin\theta\cos\theta}{r} - \frac{U_\theta \sin\theta\cos\theta}{r^2}$$

$$+ U_{\theta r}\frac{\sin\theta\cos\theta}{r} + \frac{U_r\cos^2\theta}{r} + \frac{U_{\theta\theta}\cos^2\theta}{r^2}$$

$$- \frac{U_\theta \sin\theta\cos\theta}{r^2} \quad \text{—— (B)}$$

(A) + (B) $\Rightarrow$

$U_{xx} + U_{yy} = U_{rr}(1) + \frac{U_r}{r}(1) + \frac{U_{\theta\theta}}{r^2}(1) = 0$

$\therefore \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0$

2) Show that in cylindrical Co-ordinates (r, θ, z) defined by the relation $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ the laplace equation $\nabla^2 U = 0$ takes the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Soln:- 1st Problem proof → just of one change

Result:-

i) $x^2 R'' + xR' + (m^2x^2 - n^2)R = 0$ is a Bessel's equations of order 'n'. where n is an integer. Then the solution is

$$\boxed{R(x) = A_{mn} J_n(mx) + B_{mn} Y_n(mx)}$$

where J_n and Y_n are Bessel function of 1st and 2nd order respectively. Also know that $Y_n \to \infty$ as $x \to 0$.

ii) $x^2 R'' + xR' + (m^2 x^2)R = 0$ is a bessel equation of order 'zero'.

The solution is $R(x) = A_m J_0(mx) + B_m Y_0(m$

where $Y_0 \to \infty$ as $x \to 0$

iii) $x^2R'' + xR' + (m^2x^2 - n^2)R = 0$ where n is not a integer.

The solution is are

$R(x) = A_{mn} J_n(mx) + B_{mn} J_{(-n)}$

bessel function of 1^{st} kind and also $J_{-n} \to \infty$ as $x \to 0$

iv) $x^2R'' + xR' - n^2R = 0$ is a homogenous linear equation. Then the solution is

$R(x) = A_n r^n + B_n r^{-n}$.

1) Find the soln of laplace equation in cylindrical Co-Ordinates (or) separation of Variable Method.

Soln:-

The laplace equation in cylindrical are (r, θ, z)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{---} (1)$$

Let $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$ is the solution of equation (1)

From (1)

$$\Rightarrow \frac{\partial^2}{\partial r^2}[R(r)\Theta(\theta)Z(z)] + \frac{1}{r}\frac{\partial}{\partial r}[R(r)\Theta(\theta)Z(z)] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[R(r)\Theta(\theta)Z(z)] + \frac{\partial^2}{\partial z^2}[R(r)\Theta(\theta)Z(z)] = 0$$

$$\Rightarrow \Theta Z R'' + \frac{1}{r}\Theta Z R' + \frac{\Theta''}{r^2} Z R + R\Theta Z'' = 0$$

$\div$ by $R\Theta Z$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0 \quad \text{--- (2)}$$

$$\Rightarrow \frac{Z''}{Z} = \frac{-R''}{R} - \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2}\frac{\Theta''}{\Theta} = m^2 \text{ (say)}$$

$$\Rightarrow \frac{Z''}{Z} = m^2 \text{ (say)} \quad \text{--- (3)}$$

Sub eqn (3) in eqn (2) we have

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + m^2 = 0 \quad \text{--- (4)}$$

(3) $\Rightarrow$ $Z'' - m^2 Z = 0$

i.e) $(D^2 - m^2)Z = 0$

Auxillary equation

$p^2 - m^2 = 0$

$p^2 = m^2$

$p = \pm m$

Complementary function is

$$\boxed{Z(z) = c_1 e^{mz} + c_2 e^{-mz}}$$

from (4) $\Rightarrow$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + m^2 = 0$$

$\times^{ly}$ by r^2

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} + m^2 r^2 = 0$$

$\Rightarrow r^2 \frac{R''}{R} + r\frac{R'}{R} + m^2r^2 = -\frac{\theta''}{\theta} = n^2$ (say)

general soln of Bessel eqn of order n

$y(x) = AJ_n(x) + BY_n(x)$

i.e) $r^2R'' + rR' + m^2r^2R = n^2R$ — (5)

and $\theta'' + n^2\theta = 0$ — (6)

$x^2y'' + xy' + (x^2 - n^2)y = 0$ (Bessel eqn)

(5) $\Rightarrow r^2R'' + rR' + (m^2r^2 - n^2)R = 0$

Which is a bessel equation of Order 'n'.

If 'n' is the integer then the solution is

$$\boxed{R(r) = c_3 J_n(mr) + c_4 Y_n(mr)}$$

eqn (6) $\Rightarrow \theta'' + n^2\theta = 0$

i.e) $(D^2 + n^2)\theta = 0$

Auxillary equation is $\theta^2 + n^2 = 0$

$\Rightarrow \theta = \pm in$

Complementary function is

$$\boxed{\Theta(\theta) = c_5 \cos n\theta + c_6 \sin n\theta}$$

$\therefore u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$

$$u = [c_1e^{mz} + c_2e^{-mz}][c_3J_n(mr) + c_4Y_n(mr)][c_5\cos n\theta + c_6\sin n\theta]$$

Bessel

since $Y_n(mr) \to \infty$ as $r \to 0$

$\therefore$ This solution u is unbounded at $r = 0$

in this case we can choose $c_4 = 0$.

$$u = c_3J_n(mr)(c_1e^{mz} + c_2e^{-mz})(c_5\cos n\theta + c_6\sin n\theta)$$

$$u = J_n(mr)(A\cos n\theta + B\sin n\theta)(c_1e^{mz} + c_2e^{-mz})$$

Also $z \to 0$ as $-Z \to \infty$

function this case, We can choose $c_1 = 0$.

$$\therefore u = J_n(mr)\, e^{-mz} (A' \cos n\theta + B' \sin n\theta)$$

Which is the required solution.

RESULT:-

i) $\Theta'' + \cot\theta\, \Theta' + \left(n(n+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$

$$(1-\mu)^2 \frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left(n(n+1) - \frac{m^2}{1-\mu^2}\right)\Theta = 0$$

Where $\mu = \cos\theta$.

This is legendre assumption equation then the solution becomes.

$$\Theta(\theta) = A_{mn} P_n^m(\cos\theta) + B_{mn} Q_n^m(\cos\theta).$$

Where P_n^m and Q_n^m are asoicated legendre Solution of 1st and 2nd kind respectively.

Note that $Q_n^n \to \infty$ as $\theta \to 0$.

ii) $\Theta'' + \cot\theta\, \Theta' + n(n+1)\Theta = 0$

(or)

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + n(n+1)\Theta = 0$$

Where, $\mu = \cos\theta$.

Which is a legendre equation.

Whose solution is

$$\Theta(\theta) = A_n P_n \cos\theta + B_n Q_n \cos\theta$$

(or)

$$\Theta(\mu) = A_n P_n(\mu) + B_n Q_n(\mu)$$

Where P_n and Q_n are legendre function of 1^{st} and 2^{nd} kind.

Now $Q_n \to \infty$ as $\theta \to 0$.

iii) $r^2 R'' + 2rR' + [\lambda^2 r^2 - n(n+1)]R = 0$

where solution is

$$R(r) = \lambda^{-1/2} \left[(A_n)(n+1/2)^{(\lambda r)} + B_n J_{-n+1/2}(\lambda r) \right]$$

Where,

$J_{\pm n+1/2}$ are spherical bessel functions.

Q) Solve the laplace equation spherical Co-Ordinates by using the method of seperation of variable.

COM

The laplace equation in spherical Coordinates (r, θ, ϕ) is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

x^{ly} by r^2 both sides,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \longrightarrow (1)$$

Let $u(r, \theta, \phi) = R(r)\Theta(\theta)\phi(\phi)$ be a solution of (1) — (2)

From (1) & (2) $\Rightarrow$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}(R(r)\Theta(\theta)\phi(\phi))\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial}{\partial \theta} R(r)\Theta(\theta)\phi(\phi)\right] + \frac{1}{\sin^2\theta}\left[\frac{\partial^2}{\partial \phi^2}(R(r)\Theta(\theta)\phi(\phi))\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left[r^2 R' \phi \Theta\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta R\phi\Theta'\right] + \frac{1}{\sin^2\theta} R\Theta\phi'' = 0$$

$$\Rightarrow r^2\Theta\phi R'' + 2r\Theta\phi R' + \frac{R\phi}{\sin\theta}(\sin\theta\, \Theta'') + \frac{1}{\sin\theta} R\phi\,\Theta'\cos\theta + \frac{1}{\sin^2\theta} R\Theta\phi'' = 0$$

$$\Rightarrow r^2\Theta\phi R'' + \Theta\phi R'(2r) + R\phi\Theta'' + R\phi\Theta'\cot\theta + \frac{1}{\sin^2\theta} R\Theta\phi'' = 0$$

$\div$ by $R\Theta\phi$

$$\Rightarrow r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \frac{\Theta'}{\Theta}\cot\theta + \frac{\phi''}{\phi\sin^2\theta} = 0$$

$$\Rightarrow r^2\frac{R''}{R} + 2r\frac{R'}{R} = -\left[\frac{\Theta''}{\Theta} + \frac{\Theta'}{\Theta}\cot\theta + \frac{\phi''}{\phi\sin^2\theta}\right] = n(n+1) \text{ (say)}$$

Now,

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = n(n+1) \quad \text{—— (3)}$$

$$\frac{\Theta''}{\Theta} + \frac{\Theta'}{\Theta}\cot\theta + \frac{1}{\sin^2\theta}\frac{\phi''}{\phi} = -n(n+1) \quad \text{—— (4)}$$

(3) $\Rightarrow r^2R'' + 2rR' - n(n+1)R = 0$

Consider, $z = \log r \Rightarrow e^z = r$

Let $r^2\frac{\partial^2}{\partial r^2} = D(D-1)$, $r\frac{\partial}{\partial r} = D$, $e^z = r$

(3) $\Rightarrow [D(D-1) + 2D - n(n+1)]R = 0$

$$[D^2 + D - n(n+1)]R = 0$$

Auxillary equation

$$m^2+m-n(n+1)=0$$

$$m^2+mn+m-n^2-mn-n=0$$

$$\Rightarrow m(m+n+1)-n(n+m+1)=0$$

$$(m-n)(m+(n+1))=0$$

$$m=n\ ,\ m=-(n+1)$$

$$\therefore R(e^z)=c_1e^{nz}+c_2e^{-(n+1)z}$$

$$R(r)=c_1e^{n\log r}+c_2e^{-(n+1)\log r}$$

$$=c_1r^n+c_2r^{-(n+1)}$$

$$\boxed{R(r)=c_1r^n+c_2\frac{1}{r^{(n+1)}}}$$

(4) $\Rightarrow \dfrac{\Theta''}{\Theta}+\cot\theta\dfrac{\Theta'}{\Theta}+\dfrac{1}{\sin^2\theta}\dfrac{\phi''}{\phi}+n(n+1)=0$

x^{ly} by $\sin^2\theta$

$$\sin^2\theta\left\{\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\frac{1}{\sin^2\theta}\frac{\phi''}{\phi}+n(n+1)\right\}=0$$

and

$$\sin^2\theta\left\{\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+n(n+1)\right\}=-\frac{\phi''}{\phi}=m^2$$

Now, we get,

$$\sin^2\theta\left\{\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+n(n+1)\right\}=m^2 \quad (5)\ \&$$

$$\phi''=-m^2\phi \Rightarrow \phi''+m^2\phi=0 \quad (6)$$

(5) $\Rightarrow \dfrac{\Theta''}{\Theta}+\cot\theta\dfrac{\Theta'}{\Theta}+n(n+1)=m^2/\sin^2\theta$

$$\Rightarrow \Theta''+\cot\theta\,\Theta'+\left\{\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)\right\}\Theta=0$$

Which is legendre's equation

$\therefore \Theta(\theta) = C_3 P_n^m \cos\theta + C_4 Q_n^m \cos\theta$ [result (i)]

Where P_n^m and Q_n^m legendre functions 1^{st} and 2^{nd} kind respectively and $Q_n \to \infty$ as $\theta \to 0$

(6) $\Rightarrow (D^2 + m^2)\phi = 0$

Auxillary equation

$$p^2 + m^2 = 0$$

$$\therefore p = \pm im$$

$\phi(\phi) = C_5 \cos m\phi + C_6 \sin m\phi$

Here if $m=0$, then $\phi(\phi) = C_5$

i.e) The equation (2) the solution of equation (1) doesnot dependens in bounded variable.

This case is called axis symmetric case equation (5) becomes,

$$u(r,\theta,\phi) = \left(C_1 r^n + \frac{C_2}{r^{n+1}}\right) \cdot \left(C_3 P_n^m \cos\theta + C_4 Q_n^m (\cos\theta)\right) \cdot \left(C_5 \cos m\phi + C_6 \sin m\phi\right)$$

For Omitting the unbounded solution choos $C_4 = 0$

$$u(r,\theta,\phi) = \left(C_1 r^n + \frac{C_2}{r^{n+1}}\right)\left(C_3 P_n^m \cos\theta\right)$$
$$(A \cos m\phi + B \sin m\phi)$$

In the axis symmetric case we have $m=0$.

Hence the above solution becomes

$u(r,\theta,\phi) = \left(P_n^m (\cos\theta)\right)\left(A' r^n + B'/_{r^{n+1}}\right)$

Thus the general solution of eqn (1) is given by

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} P_n^m \cos\theta \left(A_n' r^n + \frac{B_n'}{r^{n+1}} \right)$$

∴ This is the general solution of spherical Co-Ordinates.

Interior Dirichlet Problem for circle

Statement :-

A Dirichlet problem for a circle is defined as follows.

To find the values of 'u' at any point in the interior of the circle

$r = a$

Interms of its values and the boundary $r=a$ such that,

u is a single valued and Continuous within and On the circle region and satisfies the equation

$\nabla^2 u = 0$; $0 \le r \le a$; $0 \le \theta \le 2\pi$

subject to $u(a, \theta) = f(\theta)$, $0 \le \theta \le 2\pi$.

Soln:-

The laplace equation polar Co-Ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \longrightarrow (1)$$

Let $u(r, \theta) = R(r)\,\Theta(\theta) \rightarrow (2)$ be a solution of (1) sub (2) in (1) functions variable.

$$\frac{\partial^2}{\partial r^2}\left[R(r)\Theta(\theta)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[R(r)\Theta(\theta)\right] +$$
$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left[R(r)\Theta(\theta)\right] = 0$$

$$\Theta R'' + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$\div$ by $R\Theta$.

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$\times^{ly}$ by r^2.

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = k \text{ (say)}$$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = k \quad \text{—(3)}, \qquad \Theta'' + k\Theta = 0 \quad \text{—(4)}$$

$$(3) \Rightarrow r^2R'' + rR' - kR = 0 \quad \text{—(5)}$$

$$(4) \Rightarrow \Theta'' + k\Theta = 0 \quad \text{—(6)}$$

Case (i):-

If $k>0$, put $k=\lambda^2$

$$(5) \Rightarrow r^2R'' + rR' - \lambda^2R = 0$$

Let $z = \log r$; $r^2\frac{\partial^2}{\partial r^2} = D(D-1)$, $r\,\partial/\partial r = D$.

$$[D(D-1) + D - \lambda^2]R = 0$$

$$[D^2 - \lambda^2]R = 0$$

Auxillary equation,

$$m^2 - \lambda^2 = 0$$

$$m = \pm\lambda$$

$$\therefore R(z) = c_1e^{\lambda z} + c_2e^{-\lambda z}$$

$$R(r) = c_1 e^{\lambda \log r} + c_2 e^{-\lambda \log r}$$

$$R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

$$(6) \Rightarrow \frac{\Theta''}{\Theta} = -\lambda^2 \Rightarrow \Theta'' + \lambda^2 \Theta = 0$$

$Ae^{mx} = Be^{m_2 x}$

$e^{\alpha x}(\cos \beta x - B \sin \beta x)$

Auxillary equation:-

$$m^2 + \lambda^2 = 0$$

$$m = \pm i\lambda$$

$$\therefore \Theta(\theta) = c_3 \cos\lambda\theta + c_4 \sin\lambda\theta$$

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda}) \cdot (c_3 \cos\lambda\theta + c_4 \sin\lambda\theta) \quad (7)$$

Case (ii):-

If $k<0$ put $k = -\lambda^2$

$$(5) \Rightarrow r^2 R'' + rR' + \lambda^2 R = 0$$

$$\Rightarrow [D(D-1) + D + \lambda^2]R = 0$$

$$[D^2 + \lambda^2]R = 0$$

Auxillary equation,

$$m^2 + \lambda^2 = 0$$

$$m = \pm i\lambda$$

$$R(z) = c_5 \cos\lambda z + c_6 \sin\lambda z$$

$$R(r) = c_5 \cos(\lambda \log r) + c_6 \sin(\lambda \log r)$$

$$(6) \Rightarrow \frac{\Theta''}{\Theta} = \lambda^2 \Rightarrow \Theta'' - \lambda^2 \Theta = 0$$

$$(D^2 - \lambda^2)\Theta = 0$$

Auxillary equation

$$m^2 - \lambda^2 = 0$$

$$m = \pm \lambda$$

$$\therefore \Theta(\theta) = c_7 e^{\lambda\theta} + c_8 e^{-\lambda\theta}$$

$$u(r,\theta) = [c_5 \cos(\lambda \log r) + c_6 \sin(\lambda \log r)]$$

$$(c_7 e^{\lambda\theta} + c_8 e^{-\lambda\theta}) \longrightarrow (8)$$

Case(iii):-

If $k = 0$

(5) $\Rightarrow r^2R'' + rR' = 0$

$$[D(D-1)+D]R = 0$$

$$D^2R = 0$$

Auxillary equation

$$m^2 = 0$$

$$m = 0, 0$$

$$R(r) = c_9 \log r + c_{10}$$

(6) $\Rightarrow \theta'' + 0 = 0$

$$\theta'' = 0$$

$$\Rightarrow \int \frac{d}{d\theta}\left(\frac{d\theta}{d\theta}\right) = c_{11}$$

$$\frac{d\theta}{d\theta} = c_{11}$$

$$d\theta = c_{11}\, d\theta$$

$$\int d\theta = \int c_{11}\, d\theta$$

$$\theta = c_{11}\theta + c_{12}$$

$\therefore u(r,\theta) = (c_9 \log r + c_{10})(c_{11}\theta + c_{12})$ — (9)

Now for the interior problem $r=0$ at the Period of the domain of the definition of the Problem.

log 1 = 0

log0 is undefined it is not a real because never

Since, $\log r$ is not defined at $r=0$.

$\therefore$ The Solution of the equation (8) & (9) are not ~~acceptable~~. acceptable.

Then the required solution is obtained by the eqn (7)

i.e] $u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$

Let $r=0$, $u(r,\theta)$ must be finite and hence we can choose $c_2 = 0$.

$u(r,\theta) = (c_1 r^{\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$.

$u(r,\theta) = (A\cos\lambda\theta + B\sin\lambda\theta) r^{\lambda}$.

We know that,

The Periodically Condition

$u(r,\theta) = u(r, \theta+2\pi)$

$u(r,\theta) = r^{\lambda}[A\cos\lambda(\theta+2\pi) + B\sin\lambda(\theta+2\pi)]$

$= r^{\lambda}[A\cos(\lambda\theta+2\pi\lambda) + B\sin(\theta\lambda+2\pi\lambda)]$

$= r^{\lambda}[A(\cos\lambda\theta \cdot \cos 2\pi\lambda - \sin\theta\lambda \cdot \sin 2\pi\lambda)$
$+ B(\sin\lambda\theta \cdot \cos 2\pi\lambda + \cos\lambda\theta \sin 2\pi\lambda)]$

$(A\cos\lambda\theta + B\sin\lambda\theta)\cancel{r^{\lambda}} = \cancel{r^{\lambda}}[A(\cos\lambda\theta \cdot \cos 2\pi\lambda - \sin\theta\lambda \cdot \sin 2\pi\lambda) + B(\sin\lambda\theta \cdot \cos 2\pi\lambda + \cos\lambda\theta \sin 2\pi\lambda)]$

$\Rightarrow A[\cos\lambda\theta - \cos\lambda\theta \cdot \cos 2\pi\lambda + \sin\lambda\theta \cdot \sin 2\pi\lambda) + B(\sin\lambda\theta - \sin\lambda\theta \cdot \cos 2\pi\lambda - \cos\lambda\theta \cdot \sin 2\pi\lambda) = 0$

$\sin 2\theta = 2\sin\theta\cos\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$\Rightarrow A[\cos\lambda\theta(1-\cos 2\pi\lambda) + \sin\lambda\theta \cdot \sin 2\pi\lambda) + B[\sin\lambda\theta(1-\cos 2\pi\lambda) - \cos\lambda\theta \sin 2\pi\lambda] = 0$

$\Rightarrow A[2(\cos\lambda\theta \cdot \sin^2\lambda\pi) + 2\sin\lambda\theta \cdot \sin\pi\lambda \cdot \cos\pi\lambda] + B[2\sin\lambda\theta \cdot \sin^2\lambda\pi - 2\sin\pi\lambda \cdot \cos\pi\lambda \cdot \cos\lambda\theta] = 0$

$\Rightarrow 2\sin\lambda\pi [\{ A(\cos\lambda\theta \cdot \sin\pi\lambda + \sin\lambda\theta \cdot \cos\pi\lambda) +$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ $\quad B(\sin\lambda\theta \cdot \sin\pi\lambda - \cos\pi\lambda \cos\lambda\theta \}] = 0$

$\Rightarrow 2\sin\lambda\pi \{ A\sin(\lambda\theta + \lambda\pi) + B(-\cos(\lambda\theta + \lambda\pi)) \} = 0$

$\Rightarrow 2\sin\lambda\pi \{ A\sin(\lambda\theta + \lambda\pi) + B(-\cos(\lambda\theta + \lambda\pi) \} = 0$

$$2\sin\lambda\pi = 0$$

$$\lambda\pi = n\pi \;;\; n = 0,1,2,3\ldots$$

$$\boxed{\lambda = n.}$$

$\therefore \boxed{u(r,\theta) = (A\cos n\theta + B\sin n\theta) r^n} \;;\; n = 0,1,2\ldots$

In general,

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \quad \text{--- (10)}$$

Where $A_0 = a_0/2$; $A_n = a_n$; $B_n = b_n$ $\forall$ $n = 1,2,\ldots$

Which is the full range fourier series,

Given that

$$u(a,\theta) = f(\theta)$$

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (r^n a_n \cos n\theta + r^n b_n \sin n\theta)$$

$$u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a^n a_n \cos n\theta + a^n b_n \sin n\theta) \rightarrow (11)$$

Where the fourier co-efficient series are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\,d\phi \;;\; a^n a_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\,d\phi$$

$$a_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \cos n\phi \, d\phi$$

$$b_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \sin n\phi \, d\phi$$

$x + x^2 + x^3$

$x(1 + x + x^2 + \cdots)$

From eqn (10)

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \, d\phi + \sum_{n=1}^{\infty} r^n \left[\left\{ \frac{\cos n\theta}{\pi a^n} \int_0^{2\pi} f(\phi) \cos n\phi \, d\phi + \frac{\sin n\theta}{\pi a^n} \int_0^{2\pi} f(\phi) \sin n\phi \, d\phi \right\} \right.$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} \left[\frac{f(\phi)}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(\phi) \cos n\phi \cos n\theta + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(\phi) \sin n\phi \sin n\theta \right\} \right) d\phi \right]$$

$$u(r,\theta) = \frac{1}{\pi} \int_0^{2\pi} \frac{f(\phi)}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(\phi) \left[\cos n\phi . \underbrace{\cos n\theta + \sin n\phi \sin n\theta}_{\cos(A-B)}) \right] \Big\} d\phi .$$

$$u(r,\theta) = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\phi - \theta) \right\} d\phi \quad (12)$$

$$\text{Let } C = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\phi - \theta)$$

$$\text{Assume, } S = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \sin n(\phi - \theta)$$

$$C + iS = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left\{ \cos n(\phi - \theta) + i \sin n(\phi - \theta) \right] \Big\}$$

$$= \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\phi - \theta)} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{r}{a}\right) e^{i(\phi - \theta)} \right]^n$$

$$1 + x^2 + x^3 + \cdots = (1-x)^{-1}$$

$$= \frac{r}{a} e^{i(\phi-\theta)} + \left[\left(\frac{r}{a}\right) e^{i(\phi-\theta)}\right] + \cdots$$

$$= \left(\frac{r}{a} e^{i(\phi-\theta)}\right)\left\{1 - \left(\frac{r}{a} e^{i(\phi-\theta)}\right)\right\}^{-1}$$

$$S_n = \left(r/a\right) e^{i(\phi-\theta)} \Big/ \left[1 - \left(\frac{r}{a}\right) e^{i(\phi-\theta)}\right]$$

$$C + iS = \frac{(r/a)\, e^{i(\phi-\theta)}}{\left(1-(r/a)\, e^{i(\phi-\theta)}\right)} \times \frac{1-(r/a)\, e^{-i(\phi-\theta)}}{1-\left(\frac{r}{a}\right) e^{-i(\phi-\theta)}}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{\left(\frac{r}{a}\right) e^{i(\phi-\theta)} - (r/a)^2}{\left[1 - \left(\frac{r}{a}\right)\left(\cos(\phi-\theta) - i\sin(\phi-\theta)\right)\right] \cdot \left[1 - \frac{r}{a}\left(\cos(\phi-\theta) + i\sin(\phi-\theta)\right)\right]}$$

$(a-b)^2$

$$= \frac{\left(\frac{r}{a}\right)\left[\cos(\phi-\theta) + i\sin(\phi-\theta)\right] - \left(\frac{r}{a}\right)^2}{\left[1 - \left(\frac{r}{a}\right)\cos(\phi-\theta)\right]^2 + \left[\left(\frac{r}{a}\right)\sin(\phi-\theta)\right]^2}$$

$$= \frac{\left(\frac{r}{a}\right)\left[\cos(\phi-\theta) + i\sin(\phi-\theta)\right] - (r/a)^2}{1 + \frac{r^2}{a^2}\cos^2(\phi-\theta) - \frac{2r}{a}\cos(\phi-\theta) + r^2/a^2 \sin^2(\phi-\theta)}$$

$$C + iS = \frac{\left(\frac{r}{a}\right)\cos(\phi-\theta) - \frac{r^2}{a^2} + i\left(\frac{r}{a}\right)\sin(\phi-\theta)}{1 + \frac{r^2}{a^2} - 2\left(r/a\right)\cos(\phi-\theta)}$$

Equating real part,

$$c = \frac{\left(\frac{r}{a}\right)\cos(\phi-\theta) - \frac{r^2}{a^2}}{1+\frac{r^2}{a^2} - 2\frac{r}{a}\cos(\phi-\theta)}$$

$$c + 1/2 = \frac{(r/a)\cos(\phi-\theta) - r^2/a^2}{1+\frac{r^2}{a^2} - \frac{2r}{a}\cos(\phi-\theta)} + 1/2$$

$$c + 1/2 = \frac{2\left[(r/a)\cancel{\cos(\phi-\theta)} - r^2/a^2\right] + \left[1+\frac{r^2}{a^2} - \frac{2r}{a}\cancel{\cos(\phi-\theta)}\right]}{2\left[1+\frac{r^2}{a^2} - 2\frac{r}{a}\cos(\phi-\theta)\right]}$$

$$= \frac{1-(r/a)^2}{2\left[1+(r/a)^2 - 2\,r/a\cos(\phi-\theta)\right]}$$

$$= \frac{1-(r/a)^2}{2/a^2\,(a^2+r^2-2ar\cos(\phi-\theta))}$$

$$= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\phi-\theta))} \quad \text{— (13)}$$

sub (13) in (12)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[\frac{a^2-r^2}{a^2-2ar\cos(\phi-\theta)+r^2}\right]d\phi$$

This is called a interior dirichlet problem for a circle.

Exterior Dirichlet Problem for Circle:-

5M

statement:-

The Exterior Dirichlet problem is described by the $\nabla^2\phi = 0$ for $0 \le \theta \le 2\pi$, $r \ge a$ and $\phi(a,\theta) = f(\theta)$ where $r = a$. Where $f(\theta)$ is continuous function of θ an $r = a$ and ϕ must be bounded as $r \to \infty$.

Proof:-

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

Now, has $r \to \infty$ we require ϕ to be bounded. $\therefore$ Take $C_n = 0$, $D_n = 1$

$$\phi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Now, $C + iS = \frac{a}{r} e^{i(\phi-\theta)} + \left[(a/r)\, e^{i(\phi-\theta)}\right]^2 + \cdots$

$$= \left(\frac{a}{r}\right) e^{i(\phi-\theta)}\left[1 + \left(\frac{a}{r}\right) e^{i(\phi-\theta)} + \cdots\right]$$

$$= (a/r)\, e^{i(\phi-\theta)}\left[1 - \left(\frac{a}{r}\right) e^{i(\phi-\theta)}\right]^{-1}$$

$$= \frac{(a/r)\, e^{i(\phi-\theta)}}{1 - (a/r)\, e^{i(\phi-\theta)}}$$

$$= \frac{\left(\frac{a}{r}\right) e^{i(\phi-\theta)}}{1 - \left(\frac{a}{r}\right) e^{i(\phi-\theta)}} \times \frac{1 - \left(\frac{a}{r}\right) e^{-i(\phi-\theta)}}{1 - \left(\frac{a}{r}\right) e^{-i(\phi-\theta)}}$$

by (1), (2) $\Rightarrow$ $\phi(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi-\theta)\right\} d\phi$

$$= \frac{\left(\frac{a}{r}\right) e^{i(\phi-\theta)} - (a/r)^2}{\left(1-\left(\frac{a}{r}\right) e^{i(\phi-\theta)}\right)\left[1-\left(\frac{a}{r}\right) e^{-i(\phi-\theta)}\right]}$$

$$\phi(r,\theta) = \frac{\left(\frac{a}{r}\right) e^{i(\phi-\theta)} - (a/r)^2}{\left[1-\left(\frac{a}{r}\right)\left[\cos(\phi-\theta)+i\sin(\phi-\theta)\right]\right]\left[1-\left(\frac{a}{r}\right)\left[\cos(\phi-\theta)-i\sin(\phi-\theta)\right]\right]}$$

$$= \frac{\left(\frac{a}{r}\right)\left[\cos(\phi-\theta)+i\sin(\phi-\theta)\right] - (a/r)^2}{\left[1-\left(\frac{a}{r}\right)\cos(\phi-\theta)\right]^2 + \left[\left(\frac{a}{r}\right)\sin(\phi-\theta)\right]^2}$$

$(a-b)^2$ formula

$$= \frac{\left(\frac{a}{r}\right)\cos(\phi-\theta) - \left(\frac{a}{r}\right)^2 + i\left(\frac{a}{r}\right)\sin(\phi-\theta)}{1+\left(\frac{a}{r}\right)^2\cos^2(\phi-\theta) - 2\left(\frac{a}{r}\right)\cos(\phi-\theta) + \frac{a^2}{r^2}\sin^2(\phi-\theta)}$$

$$= \frac{\left(\frac{a}{r}\right)\cos(\phi-\theta) - (a/r)^2 + i\left(\frac{a}{r}\right)\sin(\phi-\theta)}{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos(\phi-\theta)}.$$

Equating the real parts

$$C = \frac{\left(\frac{a}{r}\right)\cos(\phi-\theta) - \left(\frac{a}{r}\right)^2}{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos(\phi-\theta)}$$

$$c + 1/2 = \frac{\left(\frac{a}{r}\right)\cos(\phi-\theta) - \left(\frac{a}{r}\right)^2}{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos(\phi-\theta)} + 1/2$$

$$= \frac{\cancel{2\left(\frac{a}{r}\right)\cos(\phi-\theta)} - 2\frac{a^2}{r^2} + 1 + \frac{a^2}{r^2} \cancel{- 2\left(\frac{a}{r}\right)\cos(\phi-\theta)}}{2\left[1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos(\phi-\theta)\right]}$$

$$c + 1/2 = \frac{1 - a^2/r^2}{2/r^2\left[r^2 + a^2 - 2ar\cos(\phi-\theta)\right]}$$

$$= \frac{r^2 - a^2}{2[r^2 + a^2 - 2ar\cos(\phi - \theta)]}$$

Sub (6) in (5)

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[\frac{r^2 - a^2}{r^2 + a^2 - 2ar\cos(\phi-\theta)}\right] d\phi$$

This is the solution of Exterior dirchlet for a circle.

Interior Neumann for a Circle :-

Statement :-

(10) 2M (Find the value of u at any point in the interior of the circle $r=a$ such that $\nabla^2 u = 0$, $0 \le r \le a$ and $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = g(\theta)$ on $r=a$.)

Proof :- Given $\nabla^2 u = 0$ —(1)

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = g(\theta) \quad \text{—(2)}$$

Then by the method of seperation of Variables the general solution of equation (1) is given by

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(C_n r^n + D_n r^{-n}\right)(A_n\cos n\theta + B_n \sin n\theta).$$

This gives u is unbounded. where $r=0$

where $D_n = 0$

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$= \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

$$= A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \longrightarrow (3)$$

Where $A_0 = \frac{a_0}{2}$, $A_n = a_n$, $B_n = b_n$ $\forall n = 1, 2, \ldots$

differentiable partially w.r.to r.

$$\frac{\partial u}{\partial r} = 0 + \sum_{n=1}^{\infty} n r^{n-1} (a_n \cos n\theta + b_n \sin n\theta)$$

Given that $\frac{\partial u(a,\theta)}{\partial r} = g(\theta)$, where $r = a$

$$\Rightarrow g(\theta) = \left(\frac{\partial u}{\partial r}\right)_{r=a}$$

$$= \sum_{n=1}^{\infty} n a^{n-1} (a_n \cos n\theta + b_n \sin n\theta).$$

$$g(\theta) = \sum_{n=1}^{\infty} (n a^{n-1} a_n \cos n\theta + na^{n-1} b_n \sin n\theta)$$

Which is the full range fourier series.

whose fourier co-efficient are, $\underline{a_0 = 0.}$

$$n a^{n-1} a_n = \frac{1}{\pi} \int_0^{2\pi} g(\phi) \cos n\phi \, d\phi$$

$$\Rightarrow a_n = \frac{1}{n a^{n-1} \pi} \int_0^{2\pi} g(\phi) \cos n\phi \, d\phi \quad \text{— (i)}$$

$$n a^{n-1} b_n = \frac{1}{\pi} \int_0^{2\pi} g(\phi) \sin n\phi \, d\phi$$

$$b_n = \frac{1}{n a^{n-1} \pi} \int_0^{2\pi} g(\phi) \sin n\phi \, d\phi \quad \text{— (ii)}$$

sub (i) and (ii) in (3)

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[\frac{1}{n a^{n-1} \pi} \int_0^{2\pi} g(\phi) \cos n\phi \cdot \cos n\theta \, d\phi + \frac{1}{n a^{n-1} \pi} \int_0^{2\pi} g(\phi) \sin n\phi \cdot \sin n\theta \cdot d\phi\right]$$

$$\underset{u(r,\theta)}{\phi(r,\theta)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \frac{1}{n a^{n-1} \pi} \left[\int_0^{2\pi} g(\phi) (\cos n\phi \cos n\theta + \sin n\phi \sin n\theta \right] d\phi .$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \frac{1}{n a^{n-1} \pi} \left[\int_0^{2\pi} g(\phi) \cos n(\phi - \theta)\, d\phi \right]$$

$\frac{1}{na^{n-1}} = \frac{a}{na^n}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \frac{a}{n a^n \pi} \left[\int_0^{2\pi} g(\phi) \cdot \cos n(\phi - \theta)\, d\phi \right]$$

$$\underset{u(r,\theta)}{\cancel{\phi(r,\theta)}} = \frac{a_0}{2} + \frac{a}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{1}{n} \int_0^{2\pi} g(\phi) \cos n(\phi - \theta)\, d\phi .$$

$$= \frac{a_0}{2} + \frac{a}{\pi} \int_0^{2\pi} g(\phi) \left[\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\frac{1}{n}\right) \cos n(\phi - \theta) \right] d\phi \quad \text{--- (A)}$$

Let $C = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\frac{1}{n}\right) \cos n(\phi - \theta)$

$$S = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\frac{1}{n}\right) \sin n(\phi - \theta) .$$

$$\left[\frac{\partial u}{\partial n} - \frac{\partial u}{\partial r} = g(\theta) \text{ on } r = a \right]$$

$$C + iS = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\frac{1}{n}\right) \left[\cos n(\phi - \theta) + i \sin n(\phi - \theta) \right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\frac{1}{n}\right) e^{in(\phi - \theta)} \qquad e^{in\theta}$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{r}{a}\right) e^{i(\phi - \theta)} \right]^n \frac{1}{n}$$

$$= \left(\frac{r}{a}\right) e^{i(\phi - \theta)} + \left[\left(\frac{r}{a}\right) e^{i(\phi - \theta)} \right]^2 \frac{1}{2} + \left[\left(\frac{r}{a}\right) e^{i(\phi - \theta)} \right]^3 \frac{1}{3} + \cdots$$

$$c+is = -\log\left[1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right]$$

Where $z = 1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}$

$1+x+x^2$

$$e^{c+is} = e^{-\log z}$$

$\log m^n = n\log m$

$$= z^{-1}$$

$$= \frac{1}{z}$$

$$z = \frac{1}{e^{c+is}} = e^{-(c+is)}$$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

$$= e^{-c}\cdot e^{-is}$$

$$|z| = |e^{-c}|\,|e^{-is}|$$

Take real Part

$$|z| = |e^{-c}| = e^{-c}$$

$$\Rightarrow \log|z| = \log e^{-c}$$

$$\Rightarrow \log|z| = -c$$

$$c = -\log|z|$$

$\Rightarrow c_n = -\log n|z|$

$$c = -\log\left|1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right|$$

$$= -\log\left|1-\left(\frac{r}{a}\right)\left(\cos(\phi-\theta)+i\sin(\phi-\theta)\right)\right|$$

$$= -\log\left|\left(1-\left(\frac{r}{a}\right)\cos(\phi-\theta)\right)-\left(\frac{ir}{a}\right)\sin(\phi-\theta)\right|$$

$$= -\log\sqrt{\left(1-\left(\frac{r}{a}\right)\cos(\phi-\theta)\right)^2+\left(\frac{r}{a}\right)^2\sin^2(\phi-\theta)}$$

$(a-b)^2$

$a^2 + b^2 - 2ab$

$$= -\log\sqrt{1-\frac{2r}{a}\cos(\phi-\theta)+\left(r/a\right)^2}$$

$$= -\log \sqrt{\frac{1}{a^2}\left(a^2 - 2ar\cos(\phi-\theta) + r^2\right.}$$

$$= -\log\left(\frac{1}{a}\right)\sqrt{a^2+r^2-2ar\cos(\phi-\theta)} \quad \text{— (B)}$$

Sub in eqn (A)

$$\underset{u(r,\theta)}{\phi(r,\theta)} = \frac{a_0}{2} + \frac{a}{\pi}\int_0^{2\pi} g(\phi)\left[-\ln\left(\frac{1}{a}\right)\sqrt{a^2+r^2-2ar\cos(\phi-\theta)}\right]d\phi$$

$$\underset{u(r,\theta)}{\phi(r,\theta)} = \frac{a_0}{2} - \frac{a}{\pi}\int_0^{2\pi} g(\phi)\left[\ln(1/a)\sqrt{a^2+r^2-2ar\cos(\phi-\theta)}\right]d\phi$$

This is the interior Neumann for a circle.

RESULT:-

$$\int_{-1}^{1}\left[P_n^m(\mu)\right]^2 d\mu = \frac{2}{2n+1}\frac{(m+n)!}{(m-n)!}$$

Interior Dirichlet problem for a sphere:.

13/03/19 Tuesday

Statement:-

Find the value of u at any point in the interior of sphere $r=a$ such that

$\nabla^2 u = 0 \rightarrow (1) \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi$ and

$u(a,\theta,\phi) = f(\theta,\phi)$ on $r=a$.

Proof:-

By using Seperation variable method.

The general solution of eqn (1) in spherical co-ordinates is given by

$$u(r,\theta,\phi) = \left[c_1 r^n + \frac{c_2}{r^{n+1}}\right]\left[c_3 P_n^m \cos\theta + c_4 Q_n^m \cos\theta\right]$$
$$\left[c_5 \cos m\phi + c_6 \sin m\phi\right] \quad \text{---(2)}$$

where $P_n^m \cos\theta$ and $Q_n^m \cos\theta$ are legendre function of 1^{st} and 2^{nd} kind respectively.

since $Q_n^m \cos\theta \to \infty$ as $\theta \to 0$.

We can choose $c_4 = 0$.

since eqn (2) is unbounded when $r = 0$.

$$\Rightarrow u(r,\theta,\phi) = r^n P_n^m \cos\theta \left[A_{nm} \cos m\phi + B_{nm} \sin m\phi\right]$$

using given boundary condition

$u(a,\theta,\phi) = f(\theta,\phi)$ where $r = a$.

$$\Rightarrow f(\theta,\phi) = u(a,\theta,\phi)$$

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a^n P_n^m \cos\theta \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi\right] \quad \text{---(3)}$$

In particular,

$$f(\theta,\phi) = a^n P_n^m \cos\theta \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi\right]$$

Let $\mu = \cos\theta$.

$$f(\theta,\phi) = a^n P_n^m (\mu)\left[A_{mn} \cos m\phi + B_{mn} \sin m\phi\right]$$

Integrate both sides with respect to μ, $-1 \le \mu \le 1$ and multiply by $\cos m\phi \, P_n^m(\mu)$

We get,

$$\int_{-1}^{1} f(\theta,\phi) \cos m\phi \, P_n^m(\mu) \, d\mu = \int_{-1}^{1} a^n \left[P_n^m(\mu)\right]^2 \left[A_{mn} \cos^2 m\phi + B_{mn} \cos m\phi \cdot \sin m\phi\right] d\mu$$

$$2a^n\left[A_{mn}\cos^2 m\phi + B_{mn}\sin m\phi\cdot\cos m\phi\right]\int_{-1}^{1}[P_n^m(\mu)]^2 d\mu$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \left[A_{mn}\left(\frac{1+\cos 2m\phi}{2}\right) + B_{mn}\left(\frac{\sin 2m\phi}{2}\right)\right]$$

$$\left[\frac{2}{2n+1}\cdot\frac{(m+n)!}{(m-n)!}\right]\cdot a^n$$

Integrate with respect to ϕ, $0 \le \phi \le 2\pi$.

We have,

$$\int_0^{2\pi}\int_{-1}^{1} f(\theta,\phi)\cos m\phi\, P_n^m(\mu)\,d\mu\, d\phi = \frac{2a^n(m+n)!}{(2n+1)(m-n)!}\int_0^{2\pi}\left[A_{mn}\left(\frac{1+\cos 2m\phi}{2}\right) + B_{mn}\left(\frac{\sin 2m\phi}{2}\right)\right]d\phi.$$

$$= \frac{2a^n(m+n)!}{(2n+1)(m-n)!}\left\{\frac{A_{mn}}{2}\left[\phi + \frac{\sin 2m\phi}{2m}\right]_0^{2\pi} + \frac{B_{mn}}{2}\left[-\frac{\cos 2m\phi}{2m}\right]_0^{2\pi}\right.$$

$\sin 2\pi = 0$

$$= \frac{2a^n(m+n)!}{(2n+1)(m-n)!}\left[\frac{A_{mn}}{\cancel{2}}(\cancel{2}\pi) + \frac{B_{mn}}{2}\left(\cancel{\frac{-1}{2m}} + \cancel{\frac{1}{2m}}\right)\right]$$

$$= \frac{2a^n(m+n)!}{(2n+1)(m-n)!}A_{mn}\pi$$

$$A_{mn} = \frac{1}{2\pi}\frac{(2n+1)(m-n)!}{a^n(m+n)!}\int_0^{2\pi}\int_{-1}^{1} f(\theta,\phi)\cos m\phi\, P_n^m(\mu)\,d\mu\, d\phi, \text{ where } \mu = \cos\theta$$

$$B_{mn} = \frac{1}{2\pi}\frac{(2n+1)(m-n)!}{a^n(m+n)!}\int_0^{2\pi}\int_{-1}^{1} f(\theta,\phi)\sin m\phi\, P_n^m(\mu)\,d\mu\, d\phi.$$

eqn ③ $\Rightarrow$

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n P_n^m(\mu)\cdot\left[\frac{(2n+1)(m-n)!}{(2\pi a^n(m+n)!}\int_0^{2\pi}\int_{-1}^{1} f(\psi,\chi)\cos m\chi\right.$$

$\phi.\ \cos m\phi\ P_n^m(\mu')\,d\mu'\,d\psi + \dfrac{(2n+1)(m-n)!}{2\pi a^n (m+n)!}\displaystyle\int_0^{2\pi}\int_{-1}^{1} f(\psi,\mu')$

$\sin m\psi\ P_n^m(\mu')\ \sin m\phi\ d\mu'\, d\psi.$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} f(\psi,\mu')\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r}{a}\right)^n \frac{(2n+1)(m-n)!}{(m+n)!} P_n^m(\mu)$$

$$P_n^m(\mu')\left[\cos m\psi \cos m\phi + \sin m\psi \sin m\phi\right] d\mu'\, d\psi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} f(\psi,\gamma)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r}{a}\right)^n \frac{(2n+1)(m-n)!}{(m+n)!}$$

$$P_n^m \cos\theta \cdot P_n^m(\cos\gamma)\cos m(\psi-\phi)\, d\psi\, d\gamma$$

$$u(r,\theta,\phi) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r}{a}\right)^n \frac{(2n+1)(m-n)!}{(m+n)!}\int_0^{2\pi}\int_{-1}^{1}$$

$$f(\psi,\gamma)\, P_n^m \cos\theta \cdot P_n^m(\cos\gamma).$$

$$\cos m(\gamma-\phi)\, d\psi \cdot d\gamma.$$

which is the exterior dirchlet problem for a sphere.

$\theta = \gamma,\quad \phi = \psi$

$\varphi \quad \psi$

$P_{mn}^m \rightarrow$

$\mu' = \cos\gamma.$