Real Analysis - I

Paper Code : 19UMAP2

UNIT - I

Functions - Real valude functions - equivalence - Countability - Real numbers - least upper boundes (Section 1.3 to 1.7) - Sequance of Real numbers - deffinition of sequences of subsequences - Limit of a sequence - Convergence sequence - divergence sequence (section 2.1 to 2.4)

UNIT - II

bounded sequences - monotonic seauences - operation on convergence sequences - operation on divergence sequences - limit superior and limit unferior - caushy sequences (section 2.5 to 2.10)

UNIT - III

Series of Real analysis - Convergent and divergent of series - alternative series - conditional convergent and absolute convergent rearrangement of series - test for absolute convergent - Series whose term form a non - increasing sequence (sec 3.1 to 3.7)

UNIT - IV

Limits and metric spaces - Limit of a fu
on the real line - metric spaces - limits in metric
spaces (section 4.1 to 4.3)

UNIT - V

continuous function of metric spaces - Fun
continuous at a point on the real line - reformulati
functions continuous on a metric space - open sets
closed sets - dis continous function on R^1 (section
(sec. 5.1 - 5.6)

Text book :

1. methods of Real analysis by - Richard R. Goldberg

Reference book :

1. A first course in Real analysis - Sterling K. Barberian
2. mathematical analysis - TOM.M. APostel
3. Real analysis - M.S Rangachari

UNIT - I

Definition : 1 [Cartesion product :]

If A and B are sets, then the Cartesion Product of A & B (denoted by $A \times B$) is the set of all ordered pairs (a, b), whose $a \in A$ and $b \in B$

Definition : 2

Let A and B be any two sets A function 'f' from 'A' into 'B' is a subset of $A \times B$ (and hence is a set of order pairs (a, b)) with the property that, each $a \in A$ belongs to precisely one pair (a, b).

We usually write, $y = f(x)$ Than 'y' is called the image of 'x' under 'f'

If 'f' $A \to B$, the set 'A' is called the domain of 'f' and the set 'B' is called range (codomain) of 'f'

ie) $\{ b \in B \mid b = f(a) \ \forall \ a \in A \}$

Function :

In to each $x \in X$ (domain) there corresponds one and only value of $y \in Y$ (codomain) then we say that y is a function of x

Theorem : 1

If $f: A \to B$ and if $X \subset B$; $Y \subset B$ then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

(or)

The inverse image of the union of sets is the inverse images.

Proof

let, $a \in f^{-1}(X \cup Y)$

$\Rightarrow f(a) \in X \cup Y$

$\Rightarrow f(a) \in X$ (or) $f(a) \in Y$

$\Rightarrow a \in f^{-1}(X)$ (or) $a \in f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X) \cup f^{-1}(Y) \to (1)$.

Conversely,

let $a \in f^{-1}(X) \cup f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X)$ (or) $a \in f^{-1}(Y)$

$\Rightarrow f(a) \in X$ (or) $f(a) \in Y$

$\Rightarrow$ ...

From ① and ② we get

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$$

Hence proved.

Theorem : 2

If $f : A \to B$ and if $X \subset B$, $Y \subset B$ Then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

(or)

Prove that the inverse image of the intersection of the inverse images :

Proof :

let $a \in f^{-1}(X \cap Y)$

$\Rightarrow f(a) \in X \cap Y$

$\Rightarrow f(a) \in X$ and $f(a) \in Y$

$\Rightarrow a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X) \cap f^{-1}(Y) \to ①$.

Conversely

let $a \in f^{-1}(X) \cap f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$

$\Rightarrow f(a) \in X$ and $f(a) \in Y$

$\Rightarrow f(a) \in X \cap Y$

$\Rightarrow a \in f^{-1}(X \cap Y)$.

From ① and ② we get

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(X)$$

Hence proved.

Theorem : 3

If $f: A \to B$ and $X \subset A$; $Y \subset A$ then,

$f(X \cup Y) = f(X) \cup f(Y)$

Proof :

Let $b \in f(X \cup Y)$ then $b = f(a)$ $\forall a \in X \cup Y$

Either $a \in X$ or $a \in Y \Rightarrow f^{-1}(b) \in X$ (or) $f^{-1}(b) \in Y$
$\Rightarrow$

Thus, either $b \in f(X)$ (or) $b \in f(Y)$

hence, $b \in f(X) \cup f(Y)$

$f(X \cup Y) \subset f(X) \cup f(Y) \to$ ①.

Conversely;

If $c \in f(X) \cup f(Y)$

Then either $c \in f(X)$ (or) $c \in f(Y)$

Then 'c' is the image of some point in 'x'

(or)

'c' is the image of some point in 'y'

hence,

'c' is the image of the point $X \cup Y$

ie) $c \in f(X \cup Y) \to$ ②

From ① and ②.

$f(X) \cup f(Y) \subset f(X \cup Y) \to$ ③

$\therefore f(X) \cup f(Y) = f(X \cup Y)$

hence proved.

note :

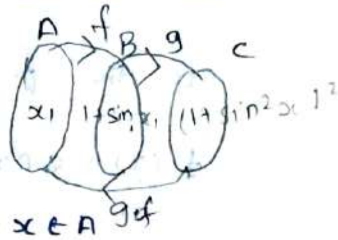
If $f : A \to B$ and $X \subset A$; $Y \subset A$ then $f(X \cap Y) = f$

need not be true .

The composition of function :

If $f : A \to B$ and $g : B \to C$ then we define

the function $g_o f$ by



$$g_o f(x) = g\,[f(x)] \quad \forall\, x \in A$$

ie) The image of x with respect to gof is defined to be the image of $f(x)$ under 'g'. The function gof is called composition of 'f' with 'g'

Ex :

$gof : A \to C$

if, $f(x) = 1 + \sin x$

$g(x) = x^2$

$\Rightarrow gof(x) = g\,[f(x)]$

$= g\,[1 + \sin x)$

$= (1 + \sin x)^2$

$gof = 1 + \sin^2 x + 2\sin x$

Real - valued functions

[illegible]

is called a real - valued function

[illegible]

[illegible] valued functions.

Definition :

1] If $f : A \to R$ and 'c' is a real number ($c \in R$)

the function cf is defined by

$$(cf)(x) = c\,[f(x)] \quad \forall\, x \in A$$

2] If $f : A \to R$ and $g : A \to R$, then maximum (f, g)

is the function defined by

$$\text{maximum } (f, g)(x) = \text{maximum } [f(x), g(x)] \quad \forall x \in A$$

and,

$$\text{minimum of } (f, g)(x) = \text{minimum } [f(x), g(x)] \quad \forall x \in A$$

note :

i) If $f : A \to R$ then $|f|$ is the function is defined by

$$|f|(x) = |f(x)| \quad \forall\, x \in A$$

ii) Maximum of $(f, g) = \dfrac{|f-g| + f + g}{2}$

iii) minimum of $(f, g) = \dfrac{-|f-g| + f + g}{2}$

definition [characteristic function]

If $A \subset S$, then ψ_A (is called the characteristic function of A) is defiend as

$$\psi_A(x) = 1 \quad \text{if } (x \in A)$$

$$\psi_A(x) = 0 \quad \text{if } (x \in A')$$

Note :

The characteristic function of $A, B \subset S$

i) $\psi_{A\cup B}$ = maximum (ψ_A, ψ_B)

ii) $\psi_{A\cap B}$ = minimum (ψ_A, ψ_B)

iii) $\psi_{A-B} = \psi_A - \psi_B$

iv) $\psi_{A'} = 1 - \psi_A$

v) $\psi_S = 1$; $\psi_\phi = 0$

$\therefore$

$\psi_{A\cup B}(x)$ = maximum $(\psi_A, \psi_B)(x)$

$= 1 \quad \forall x \in A\cup B$

$\psi_{A\cap B}(x)$ = minimum $(\psi_A, \psi_B)(x)$

$= 0 \quad \forall x \in A\cap B$

Equivalance and Countability :

Definition : [1-1]

If $f : A \to B$ then 'f' is called 1-1 (one-to-o

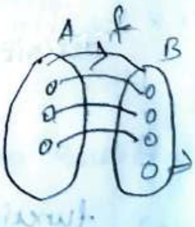
if

$f(a_1) = f(a_2) \Rightarrow$

$a_1 = a_2$

(or)

$f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$



Ex :

1) The function 'f' is defind by

$f(x) = x^2 \quad \forall -\infty < x < \infty$

is not 1-1

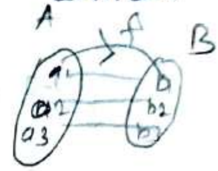
ii) The function 'f' is defined by

$$f(x) = x^2 \quad \forall\ 0 < x < \infty$$

is not 1-1

Definition [Inverse function]

If $f : A \to B$ and f is 1-1 then the function f^{-1} (called inverse function) is defined by



if $f(a) = b$ then $f^{-1}(b) = a$

Then the domain of f^{-1} is the range of 'f' and the range of f^{-1} is the domain of 'f'

Definition : [Equivalent :]

If $f : A \to B$ and f is 1-1 then 'f' is called a 1-1 correspondence (between A and B)

If there exist a 1-1 correspondance between the sets A and B then A and B are called equivalent

Ex:

1) Every set 'A' is equivalent to itself

2] If A and B are equivalent then B and A are equiv

3] If A and B are equivalent and B and c are equival

are equivalent

Countable :

The set 'A' is called countable (or) denume[rable]

If A is equalent to the set I of positive

integers

An uncountable set is an infinite se[t]

which is not countable

Thus 'A' is countable there exist ($\exists$) a

1-1 function from 'I' onto 'A'

Ex:

The set of all integers is Countable

$$Z = \{0, \pm 1, \pm 2, \pm 3 \dots\}$$

$$Z = \{0, -1, 1, -2, 2, -3, 3, -4, 4 \dots\}$$

now,

$$I = \{1, 2, 3 \dots\}$$

$$f : Z \rightarrow I$$

0, 1, 2

$$f(x) = \begin{cases} \left(\frac{n-1}{2}\right) & \text{if 'n' is } \cancel{\text{called}} \text{ odd} \\ -(n/2) & \text{if 'n' is even} \end{cases}$$

I is equivalent to 'Z'

ie) Z is equivalent to I

Z is countable.

Theorem (1)

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If A_1, A_2, A_3 .... are countable sets then union

$\bigcup_{n=1}^{\infty} A_n$ is also countable.

(or)

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Prove that the countable union of countable set is countable.

Proof :

Let A_1, A_2, A_3 ..... are countable sets.

To prove $\bigcup_{n=1}^{\infty} A_n$ is countable :-

we can write

$$A_1 = a_1^1, a_2^1, a_3^1 \ldots\ldots$$

$$A_2 = a_1^2, a_2^2, a_3^2 \ldots\ldots$$

$$A_3 = a_1^3, a_2^3, a_3^3 \ldots\ldots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Union of sets $= \bigcup_{n=1}^{\infty} A_n$

Define the height of the elements $a_k^j = j+k$

element of height '2' is a_1^1

element of height '3' is a_1^2, a_2^1

element of height '4' is $a_3^1, a_2^2, a_1^3, \ldots\ldots$

arrange the above elements according to their heights.

ie) $a_1^1, a_1^2, a_2^1, a_3^1, a_2^2, a_1^3 \ldots.$

Thus

$$\begin{array}{ccccc}
a_1^1 & a_2^1 & a_3^1 & a_4^1 & \cdots\cdots \\
a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots\cdots \\
a_1^3 & a_2^3 & a_3^3 & a_4^3 & \cdots\cdots \\
a_1^4 & a_2^4 & a_3^4 & a_4^4 & \cdots\cdots
\end{array}$$

clearly, the set is countable.

$\therefore \bigcup_{n=1}^{\infty} A_n$ is countable

Hence proved.

Theorem : 2

5m

Prove that, the set of all rational number is countable

Proof:

Let $E_1 = \{ \frac{0}{1}, \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \ldots\ldots \}$

$E_2 = \{ 0/2, \pm 1/2, \pm 2/2, \pm 3/2, \ldots\ldots \}$

$E_3 = \{ 0/3, \pm 1/3, \pm 2/3, \pm 3/$

$\vdots \qquad \vdots$

$\bigcup_{n=1} E_n$

we known that

"countable union of countable set is countable"

$\Rightarrow \bigcup_{n=1}^{\infty} E_n$ is countable

$\therefore$ The set of alterational no. is countable

Theorem : 3

If 'B' is an infinite subset of the countable sets 'A'

Then 'B' is countable

Proof :

Let $A = \{ a_1, a_2, a_3, a_4, \ldots \}$

let 'B' be the subset of 'A'

Let n_1 be the smallest subscript for which

$a_{n_1} \in B$. and

n_2 be the smallest subscript for which

$a_{n_2} \in B$

and so on.

$\therefore B = \{ a_{n_1}, a_{n_2}, a_{n_3}, \ldots \}$

The elements of 'B' are labled with $1, 2, 3, \ldots$

and so, 'B' is countable.

hence, 'B' is countable.

Theorem : 4

Prove that the set of all rational numbers in $[0,1]$ is countable.

Proof :

Let , $A = [0,1]$

⇒ 'B' be the subset of rational number

we know that

"Infinite subset of Countable. set is Countab

we get :

'B' is countable ⇒ 'A' is Countable.

∴ The set of all rationals in $[0,1]$ is Counta

Hence ~~the~~ proof

Rial Numbers :

Theorem : 5

Prove that the set $[0,1] = \{ x \mid 0 \leq x \leq 1 \}$ is uncountab

Proof :

we prove that $[0,1]$ is uncountable Suppose $[0,1]$ is countable,

Let $[0,1] = \{ x_1, x_2, x_3 \ldots \}$

where, every no. in $[0,1]$ occurs among the 'x_i'

empanding each 'x_i' in decimals

$$\Rightarrow x_1 = 0.a_1^1\, a_2^1\, a_3^1 \ldots\ldots$$

$$x_2 = 0.a_1^2\, a_2^2\, a_3^2 \ldots\ldots$$

$$x_3 = 0.a_1^3\, a_2^3\, a_3^3 \ldots\ldots$$

$$\vdots$$

$$x_n = 0.a_n^n,\, a_2^n,\, a_3^n \ldots$$

$$\vdots$$

Let 'b_1' be the any integer (from 0 to 8) such that $b_1 \neq a_1^1$

Let 'b_2' be the any integer (from 0 to 8) $\ni: b_2 \neq a_2^2$

In general

Let 'b_n' be the any integer (from 0 to 8) $\ni: b_n \neq a_n^n$

Let, $y = 0.b_1 b_2 b_3 \ldots\ldots$

then for any 'n'

The decimal expansion for 'y' differs from the decimal expansion for 'x_n'

since, $b_n \neq a_n^n$

$$\Rightarrow y \neq x_1;\ y \neq x_2;\ y \neq x_3, \ldots\ldots$$

$$\Rightarrow y \notin \{x_1, x_2, x_3 \ldots\ldots\}$$

but, $y \in [0,1]$, which is $\Rightarrow \Leftarrow$ to our assumption

$\therefore [0,1]$ is uncountable

Theorem : 6

(X) Prove that the set R of all real numbers is uncountable

Proof :

Let us assume that,

'R' is countable.

Let, [0,1] ⊂ R

now,

[0,1] is an infinite subset of the countable set countable,

∴ [0,1] is countable.

We know that,

[0,1] is uncountable

[construction]

which is ⇒ ∉ to our assumption

The set of all real numbers is uncountable.

Hence the theorem,

Theorem : 7

Prove that the set of all irrational number is uncountable

Proof :

The set 'Q' of all rationals is countable

The set 'R' of all real numbers is uncountable

The set of all irrationals is equal to (R - Q)

[uncountable - countable = countable]

The set of all irrationals is uncountable.

Hence the prove

Theorem : 8.

Prove that if 'B' is countable, subset of the uncountable set A Then (A - B) is uncountable.

Proof :

The A is uncountable

The subset B of A is countable

The set value (A-B) is uncountable.

[uncountable - countable = uncountable.

Definition : [Binary Expansions :-]

The Binary expansion for a real value 'x' uses only the digits '0' and 1

Ex:-

$0 \cdot a_1 a_2 a_3 \ldots$ means $\frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots\ldots$

$0 \cdot 1000\ldots = \frac{1}{2^1} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \ldots\ldots$

$0.1000 = 1/2$

$0 \cdot 110100\ldots = \frac{1}{2^1} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{0}{2^6} \ldots$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$

$= \frac{8+4+1}{16} = 13/16.$

$$0.00010 = \frac{0}{2^1} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \ldots\ldots$$

$$= \frac{1}{16}.$$

Definition : [Ternary Expansions :-]

The Ternary expansion a real number 'x' uses the digits $0, 1, 2$

Thus, $x = 0 \cdot b_1 b_2 b_3 \ldots\ldots$

means $x = \frac{b_1}{3} + \frac{b_2}{3^2} + \frac{b_3}{3^3}$

Ex :-

$$0.100\ldots = \frac{1}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \ldots$$

$$= 1/3$$

$$0.0222 = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \ldots$$

$$= \frac{2}{9} + \frac{2}{27} + \frac{2}{81} \quad = \frac{2}{3^2}\left(1 + 1/3 + 1/3^2 + \ldots\right)$$

$$= \frac{2}{3 \cdot 3}\left(\frac{1}{1 - 1/3}\right) = \frac{2}{3 \cdot 3}\left(\frac{1}{2/3}\right)$$

$$= 1/3$$

Definition [Cantor set] :-

The Cantor set 'K' is the set of all numbers $x \in [0, 1]$ which have the ternary expansion without the digit 1.

Least upper bounds :

Definition :. [Bounded]

The subset $A \subset R$ is set to be bounded above if there is a no. $N \in R$ such that $\boxed{x \le N}$ for every $x \in A$

The subset $A \subseteq R$ is said to be bounded below if here is a no. $M \in R$ such that $\boxed{M \le x}$. for every $x \in A$.

If 'A' is bouth bounded below and bounded above, we say that 'A' is bounded.

Definition [upper bound and Lower bounde :.]

If $A \subset R$ is bounded above, then 'N' is called an upper bound for 'A'. If $\boxed{x \le N}$ $\forall\ x \in A$

If $A \subset R$ is bounded below, then 'M' is called an lower bounded for 'A' if $\boxed{M \le x}$ $\forall\ x \in A$

Definition [Least upper bounds] :. [LUB] (or) Supremum

let subset A of R be bounded above, the number 'L' is called ...

If,

i] 'L' is an upper bounded for 'A'

ii] no. number smaller then 'L' is an upper bound for 'A'

Definition [Greatest lower bound] :- [GLB] (or infimum)

let $A \subset R$ be bounded below, the number 'L' Called the greatest lower bound for 'A'. If,

i] 'L' is an lower bounded for 'A'

ii] no. number greater then 'L', is an lower bound for 'A'

Ex: check wheather the sets bounded (or) not.

1] = $\{ \frac{1}{n} ; n \in N \}$

2] $\{ (-1)^n (1/n) ; n \in N \}$

3] $\{ (-1)^n (n) ; n \in N \}$

4] $\{ (-1)^n (1/n) + 1 ; n \in N \}$

5) $\{ \frac{4n+3}{n} ; n \in N \}$

6) $\{ \frac{n}{n+1} ; n \in N \}$

7) $\{ -\frac{n-1}{n} ; n \in N \}$

1 $\{ \frac{1}{n} ; n \in N \}$

Sol:

Let $S = \{ 1/n ; n \in N \}$

$= \{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \dots \}$

$S = \{ 1, 1/2, 1/3 \dots \}$.

UB $= 1$, L.B $= 0$.

It's s is bounded.

2 $\{ (-1)^n (1/n) ; n \in N \}$

Let $S = \{ (-1)^n (\frac{1}{n}) ; n \in N \}$

$S = \{ (-1)^1 (1/1), (-1)^2 (1/2), (-1)^3 (1/3) \dots \}$

$= \{ -1, 1/2, -1/3, 1/4, -1/5, 1/6 \dots \}$

$\sup(S) = 1/2$

$\inf(S) = -1$

3 $\{ (-1)^n (n) ; n \in N \}$

$S = \{ (-1)^n (n) ; n \in N \}$

$= \{ -1, 2, -3, 4, -5, 6, -7 \dots \}$

$= \{ \dots -7, -5, -3, -1, 2, 4, 5, 6 \dots \}$

The set is 's' neighes bounded not bounded $\therefore$ s is unboun

4 $\{ 1+(-1)^n (1/n) ; n \in N \}$

$S = \{ (1-1), (1+1/2), (1-1/3), (1+1/4), (1-1/5) \ldots \}$

$= \{ 0, 3/2, 2/3, 5/4, \frac{4}{5} \ldots \}$

Inf $(S) = 0$.

Sup $(S) = 3/2$

'S' is bounded.

5 $\{ \frac{4n+3}{n} ; n \in N \}$

$S = \frac{4n+3}{n} = \frac{4n}{n} = 3/n$

$S = \{ 4 + \frac{3}{n} \} ; n \in N.$

$= \{ 7, 11/2, 5, 19/4, 23/5, 9/2 \ldots \}$

Sup $= 7$

Inf $= 4$

S is bounded.

6 $\{ \frac{n}{n+1} ; n \in N \}$

$S = \{ 1/2, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots \}$

Sup$(S) = 1$ Inf $(S) = 0.5$

S is bounded

7 $S = \{ -\frac{(n+1)}{n} ; n \in N \}$

$= \{ -2/1 , -\frac{3}{2} , -\frac{4}{3} , -\frac{5}{4} , -\frac{6}{5} \ldots \}$

$Inf(S) = -2$ $\quad$ $Sup(S) = -1$

'S' is bounded.

8 set of all 'R'

Real number set

'R' is unbounded.

Least upper bound axiom :-

If 'A' is any non-empty sub set of 'R' that is bounded above then 'A' has a least upper bound in 'R'

Theorem : 9

If 'A' is any non-empty subset of 'R' that is bounded below, then 'A' has a greatest lower bounded in 'R'

Proof:

Let $B \subset R$ be the set of all $x \in R$ such that $(-x) \in A$

[ie) the elements of 'B' are the negatives of the element of 'A']

If 'M' is a lower bound for 'A'

then, $(-M)$ is a upper bound for 'B'

For if, $x \in B$ then $(-x) \in A$ and so $M \le -x, \forall x$

Hence 'B' is bounded above. so that,

by the definition (if $A \subset R$ is belowed aboved $\Rightarrow$ A has l.u upper Bound:

$\therefore$ A has a Greatest lower bound.

— x ——— x ——— x ——— x ——— x ———

UNIT - II

$\ni$ - such that

$\exists$: There exist

$\Leftrightarrow$: if and only if

Bounded sequences:

Definition :

we say that sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above If the rang of $\{s_n\}$ is bounded above.

11^{ly}

similarly,

we say than the sequence $\{s_n\}_1^{\infty}$ is bounded below,

If the range of $\{s_n\}_1^{\infty}$ is bounded below

Thus $\{s_n\}_1^{\infty}$ is bounded if and only if There exists $M \in R$ such that

$$|s_n| \leq M \quad (n \in I)$$

Note :

i] we know that

The sequence of real numbers $\{s_n\}_1^{\infty}$ is a function from I into 'R' we see that the range of $\{s_n\}_1^{\infty}$ (namely ($\{s_1, s_2, s_3 \ldots\}$) is a subset of 'R' ii] If sequence divergents to ∞ (or, $-\infty$) then the sequence is not bounded.

Ex :

i] The sequence $\{1, -2, 3, -4, 5, -6 \dots\}$ is oscillat and which is neither bounded below nor bounded above, Therefore the sequence is unbounded

ii] The sequence $s_n = \{1, -1, 1, -1, 1, -1 \dots\}$ is oscilla and its bounded (bounded Above $= +1$; bounded below $=$

iii] The sequence $s_n = \{1, 2, 1, 3, 1, 4, 1, 5 \dots\}$ is unbounded (bounded below $= 1$ and bounded above $=$

Theorem : 1

5M 4a (+) vvv I

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ is convergent then $\{s_n\}_1^{\infty}$ is bounded.

(or)

Prove that, Every convergent sequence is bounded.

proof:

Given that it is

$\{s_n\}$ is convergent

$\therefore \{s_n\}$ is convergent $\Rightarrow s_n \to L$ as $n \to \infty$

if Suppose, $\epsilon = 1$ There exist $N \in I$

Such that, $n \geq N \Rightarrow \boxed{|s_n - L| < \epsilon}$

$\Rightarrow |s_n - L| < 1$

$\Rightarrow \boxed{|s_n| < L + 1}$

$\because n = N_1 + N_2 + N_3 \dots\dots \Rightarrow |s_n| < L + 1 \to (1).$

Let,
$$M = \max\{|s_1|, |s_2|, |s_3|, \dots |s_{n-1}|\}$$

$n = 1, 2, 3, \dots (N-1) \Rightarrow |s_n| < M \to (2).$

From (1) and (2). we get

$n = 1, 2, 3, \dots N, N+1, N+2 \dots\dots$

$\Rightarrow |s_n| < L + 1 + M$

$\therefore \boxed{|s_n| < (L+M) + 1}$

$\therefore \{s_n\}$ is bounded.

Hence Every convergent sequence is bounded.

Hence proved.

Def^n [Monotone Sequence]

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers

if, $s_1 \leq s_2 \leq s_3 \leq 4 \leq \dots\dots \leq s_n \leq s_{n+1} \dots\dots$

Then $\{s_n\}$ is called non-decreasing sequence.

similarly,

If $s_1 \geq s_2 \geq s_3 \geq \dots\dots \geq s_n \geq s_{n+1} \dots$

Then $\{s_n\}_{n=1}^{\infty}$ is called non-increasing sequence.

A monotone sequence is a sequace which is eaither non-increasing (or) non-decreasing (or) both.

Theorem : 2.

Prove that a non-decreasing sequence which is bounded above is convergent.

Proof:

Let s_n be a non-decresing sequence and bounded above.

To prove that $\{s_n\}$ is convergent.

Given

$\Rightarrow s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n \leq \ldots$

Consider the set, $A = \{s_1, s_2, s_3, \ldots\}$

A is a non-empty subset of $\mathbb{R}$

which is bounded above.

by least upper bound axiom,

"If ~~A is~~ $A \subset \mathbb{R}$ which is bounded above

Then 'A' has got least upper bound in $\mathbb{R}$

$\Rightarrow$ 'A' Must be Least upper bound.

let M = Least upper bound $\{s_1, s_2, s_3, \ldots\}$

M = Least upper bound for 'A'

we will prove $s_n \to L$ as $n \to \infty$

Given $\epsilon > 0$, the number $(M - \epsilon)$ is not an upper bound for 'A'

Hence for some $N \in I$.

$s_n > (M - \epsilon)$; $n \geq N$ $\to$ (1).

but,

'M' is an upper bound for 'A'

$\Rightarrow s_n \leq M$; $n \in I$

$\Rightarrow$ $s_n < M + \epsilon$ $\to$ (2).

from (1) & (2)

$\Rightarrow (M - \epsilon) < s_n < (M + \epsilon)$

$\Rightarrow -\epsilon < s_n - M < \epsilon$

$\Rightarrow |s_n - M| < \epsilon$

$\therefore s_n \to M$ as $n \to \infty$

hence the sequence $\{s_n\}$ is convergent.

Hence Proved.

Theorem: 3

VVI

I: Prove that a non-increasing sequence which is bounded below is convergent.

Proof:

Let s_n be a non-increasing sequence and bounded below.

To prove that $\{s_n\}$ is convergent

Given,

$\Rightarrow s_1 \geq s_2 \geq s_3 \geq \dots \dots \geq s_n \geq \dots$

Consider the set $A = \{s_1, s_2, s_3 \dots\}$

A is non-empty subset of 'R'

which is bounded below

by greatest lower bound axiom

"If $A \subset R$ which is bounded below.

Then 'A' has get greatest lower bound in 'R'

$\Rightarrow$ 'A' must be greatest lower bound.

let, M = greatest lower bound $\{s_1, s_2, s_3 \dots\}$

$\Rightarrow M$ = greatest lower bound for A.

we will prove $s_n \to 1$ as $n \to \infty$

given

$\epsilon > 0$ the number $(M+\epsilon)$ is not an lower bound for A.

hence, for some $N \in I$

$s_n < (M+\epsilon)$; $n \geq N$ $\to$ (1).

but

'M' is an greatest lower bound for A.

$\Rightarrow s_n \geq M$; $n \in I$

$\Rightarrow s_n > M - \epsilon \to$ (2).

from (1) and (2)

$\Rightarrow (M - \epsilon) < s_n < (M+\epsilon)$

$\Rightarrow -\epsilon < s_n - M < \epsilon$.

$\Rightarrow |s_n - M| < \epsilon$.

$\therefore s_n \to M$ as $n \to \infty$

hence the sequence $\{s_n\}$ is convergent

Hence Proof.

Theorem : 4.

A non-decreasing sequence, which is not bounded above then the sequence divergent to infinity

Proof:

Let $\{s_n\}_1^\infty$ be a non-decreasing sequence and not bounded above.

Prove that $\{s_n\}$ divergent to '∞'

For, any $M > 0$, $\exists : N \in I \ni :$

$n \geq N \Rightarrow \boxed{s_n > M}$

Now,

$\{s_n\}$ is not bounded above.

$\Rightarrow \{s_n\}$ does not have upper bound

$\Rightarrow$ 'M' is not an upper bound of $\{s_1, s_2, s_3 \ldots\}$

$\therefore \exists N \in I \ni : s_n > M$.

hence,

$\{s_n\}_1^\infty$ is divergent to '∞'

Theorem: 5

A non-increasing sequence which is not bounded below then the sequence divergent to minus infinity.

Let $\{s_n\}_1^\infty$ be a non-increasing sequence and not bounded below.

For any $M > 0$, $\exists N \in I^+$:

$n \geq N \Rightarrow \boxed{s_n < -M}$

Now,

$\{s_n\}$ is not bounded below.

$\Rightarrow \{s_n\}$ does not have lower bound

$\Rightarrow$ M is not an lower bound of $\{s_1, s_2, s_3 \ldots\}$

$\exists N \in I^+ ; s_n < -M$

hence $\{s_n\}_1^\infty$ is divergence to $-\infty$.

Theorem: 6.

-X- 10 M

The sequence $\{(1+\frac{1}{n})^n\}_{n=1}^\infty$ is convergent

10M.

Proof.

Let $s_n = \{(1+\frac{1}{n})^n\}_{n=1}^\infty$

To prove that

1^{st} we will prove $\{s_n\}$ is non decresing :-

The Binomial expansion of $\{s_n\}$ is.

$$S_n = 1 + nC_1\left(\frac{1}{n}\right) + nC_2\left(\frac{1}{n}\right)^2 + nC_3\left(\frac{1}{n}\right)^3 + \dots + nC_n\left(\frac{1}{n}\right)^n$$

$$= 1 + \frac{n}{1!}\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \dots$$

$$+ \frac{n(n-1)(n-2)(n-3)\dots 3\cdot 2\cdot 1}{n!}\left(\frac{1}{n^n}\right)$$

$$= n + \frac{\cancel{n}}{1!}\left(\frac{1}{\cancel{n}}\right) + \frac{n^2(1-\frac{1}{n})}{2!}\left(\frac{1}{n^2}\right) + \frac{n^3(1-\frac{1}{n})(1-\frac{2}{n})}{3}\frac{1}{n^3} + \dots$$

$$+ \frac{(n^n)(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\dots}{n!}$$

$$= n + 1 + \frac{(1-\frac{1}{n})}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})}{4!} + \dots$$

$$\frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\dots 3\cdot 2\cdot 1}{n!}$$

we can take $\{S_n\}$ has $(k+1)$-th term

sub $n = n+1$ $\quad t_{k+1} = \frac{(1-\frac{1}{n+1})(1-\frac{2}{n+1})(1-\frac{3}{n+1})\dots(1-\frac{k}{n+1})}{k!}$

$$t_{k+1} = \frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\dots(1-\frac{(k+1)}{n})}{k!}$$

Compare (t_{k+1}) and (t_{k+1})

$$n < n+1$$

$$\Rightarrow \frac{1}{n} > \frac{1}{n+1}$$

$$\Rightarrow \frac{-1}{n} < \frac{-1}{n+1}$$

$$\Rightarrow 1-\frac{1}{n} < 1-\frac{1}{n+1}$$

lll^{ly}

$$1-\frac{2}{n} < 1-\frac{2}{n+1}$$

$$1-\frac{3}{n} < 1-\frac{3}{n+1}$$

$$\vdots$$

$$\frac{1-(k+1)}{n} < \frac{1-(k+1)}{n+1}$$

$$t(k+1) < t^{-1}(k+1)$$

$$s_n < s_{n+1}$$

$(\frac{1}{n^n})$

$\{s_n\}$ is a non-decreasing sequence.

Next we will prove that

The $\{s_n\}$ is bounded above

$$\{s_n\} = \left\{(1+\tfrac{1}{n})^n\right\}_{n=1}^{\infty}$$

$$s_n = 1+\frac{n}{n!}\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \dots + n \text{ times}$$

$$= 1+\frac{n}{1!}\left(\frac{1}{n}\right) + \frac{n^2(1-1/n)}{2!}\left(\frac{1}{n^2}\right) + \frac{n^3(1-1/n)(1-2/n)}{3!}\left(\frac{1}{n^3}\right) + \dots + n \text{ times}$$

$$s_n = 1+1+\frac{(1-1/n)}{2!} + \frac{(1-1/n)(1-2/n)}{3!} + \dots + n \text{ times}$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots + n \text{ times.}$$

[taking limit as $n \to \infty$]

$\leq 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \dots + n$ times

$S_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ n times.

$$S_n < 1 + \frac{1}{1 - \frac{1}{2}}$$

$$< 1 + \frac{1}{\frac{2-1}{2}}$$

$$< 1 + \frac{1}{1/2}$$

$$< 1 + 2$$

$$\boxed{S_n < 3}$$

$\{S_n\}$ has bounded above.

The sequence $\{S_n\}$ is non-decreasing and bounded above

by a know That

$$\{S_n\}_1^\infty = \{(1 + 1/n)^n\}_1^\infty \text{ is Convergent}$$

UNIT - III

series of real numbers :

Definition :

The infinite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $\langle \{a_n\}_1^{\infty}, \{s_n\}_{n=1}^{\infty} \rangle$

where

$\{a_n\}_{n=1}^{\infty}$ is a sequence of real number

and

$s_n = a_1 + a_2 + a_3 + \dots + a_n \quad (n \in I)$ The number a_n is called the n^{th} form ~~partical n^{th} form~~ of the series & the number s_n is called the n^{th} partical sum of the series

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partical sum $s_n = a_1 + a_2 + a_3 + \dots + a_n$ $(n \in I)$

If the sequence $\{s_n\}_{n=1}$ convergent to $A \in$ and we say convergente to A)

Divergents series: If $\{s_n\}_{n=1}^{\infty}$ is divergents we say that the series $\sum_{n=1}^{\infty} a_n$ in divergents.

$$\sum a_n = 1-1+\cdots+(-1)^{n+1}+\cdots \text{ is}$$
$$= \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Theorem: 1

Statement:

If $\sum_{n=1}^{\infty} a_n$ convergent to 'A' and $\sum_{n=1}^{\infty} b_n$ convergent to 'B' Then,

i] $\sum_{n=1}^{\infty} (a_n+b_n)$ convergent to $(A+B)$

ii] $\sum_{n=1}^{\infty} (a_n-b_n)$ convergent to $(A-B)$

iii] If $c \in R$, then $\sum_{n=1}^{\infty} (c\,a_n)$ convergence to (cA)

Proof:

Let $s_n = \sum_{k=1}^{n} a_k$

and

$$t_n = \sum_{k=1}^{n} b_k$$

now,

$\sum_{n=1}^{\infty} a_n$ convergent to A and $\sum_{n=1}^{\infty} b_n$ ~~diverges~~ convergent to B

ie) $S_n \to A$ as $n \to \infty$ and $t_n \to B$ as $n \to \infty$

i) $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

$$\sum_{k=1}^{\infty} (a_k + b_k) = S_n + t_n$$

we know that

"If S_n and t_n are sequence of real number and

$S_n \to L$ and $t_n \to M$.

$\therefore \{S_n + t_n\} \to (L+M)$ as $n \to \infty$

$\Rightarrow \sum_{n=1}^{\infty} a_k + b_k = A+B$

ii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} [a_n + (-b_n)]$.

$$= A + (-B)$$

$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$ by case (i)

If $\sum_{n=1}^{\infty} a_k$ convergent and given $\epsilon > 0$ $\exists n \in I$

$\ni$: $\left| \sum_{n=1}^{\infty} a_k \right| < \epsilon$

Proof:

Let $\sum_{k=1}^{\infty} a_k$ is convergent

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is convergent.

where, $s_n = a_1 + a_2 + a_3 + \dots + a_n$

we know that

" Every convergent sequence is Cauchy sequence"

$\Leftrightarrow \{s_n\}_{n=1}^{\infty}$ is cauchy Sequence.

$\Leftrightarrow$ Given $\epsilon > 0$ $\exists N, M \in I \ni$:

$|s_n - s_m| < \epsilon \quad \forall n \geq N$ and $m \geq M$.

$\Leftrightarrow |(a_1 + a_2 + a_3 + \dots + a_m + a_{m+1} + \dots a_n) - (a_1 + a_2 + a_3 + \dots a_m)| < \epsilon$ $\forall m, n \geq N$

$\Leftrightarrow |a_{m+1} + a_{m+2} + a_{m+3} + \dots + a_n| < \epsilon \quad \forall m, n \geq N$

$\Leftrightarrow \left| \sum_{k=m+1}^{\infty} a_k \right| < \epsilon$

Theorem : 5

5m
*: a] If $0 < x < 1$ then $\sum x^n$ convergent to $\left(\frac{1}{1-x}\right)$

b] if $x \geq 1$ $(1 \leq x \leq \infty)$ then $\sum_{1}^{\infty} x^n$ divergent

Proof:

[a] Given that, $\sum_{n=1}^{\infty} x^n$ is convergent series.

with $0 < x < 1$

consider, the n^{th} partial sum.

$s_n = a_1 + a_2 + a_3 + \dots + a_n$

$s_n = \frac{1-x^{n+1}}{1-x}$ if $x < 1$

$\Rightarrow s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$

$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1-x}\right) - \lim_{n\to\infty} \left(\frac{x^{n+1}}{1-x}\right)$

$$\boxed{\lim_{n\to\infty} s_n = \left(\frac{1}{1-x}\right) - \lim_{n\to\infty} \left(\frac{x^{n+1}}{1-x}\right)} \quad \rightarrow (1).$$

by known theorem

"If $0 < x < 1$, The sequence $\{x^n\}$ convergent to '0'

$$\Rightarrow \lim_{n\to\infty} \{x^{n+1}\} = 0.$$

$$(1) \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-x} - 0.$$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-x}$$

$\{S_n\}_{n=1}^{\infty} = \sum_{1}^{\infty} x^n$ is convergent to $\left(\frac{1}{1-x}\right)$

b] If $x \geq 1$

we know that

" $1 < x < \infty$ $(x \geq 1)$ the sequence $\{x^n\}$ divergent to '∞'

$$(1) \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-x} - \infty$$

$\left\{ \lim_{n\to\infty} x^{n+1} = \infty \right.$

$$\boxed{\lim_{n\to\infty} S_n = \infty}$$

$\sum_{1}^{\infty} x^n$ is divergent when $x \geq 1$

Hence the theorem.

Theorem: 6

Jm

.x. Prove that $\sum\left(\frac{1}{n}\right)$ is divergent

Proof:

of given series.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

here $a = 1/n$

Consider the sub sequence of the sequence $\{S_n\}_{n=1}^{\infty}$

is $S_1, S_2, S_4 \dots S_{2^n}$

ie) $S_{2^0}, S_{2^1}, S_{2^2}, S_{2^3} \dots S_{2^n}$

Now, $S_1 = a_1$

$S_1 = \frac{1}{1}$.

$\boxed{S_1 = 1}$

$\Rightarrow S_2 = a_1 + a_2 = \frac{1}{1} + \frac{1}{2} =$

$S_2 = 1 + \frac{1}{2}$.

$\boxed{S_2 = \frac{3}{2}}$

$\Rightarrow S_4 = a_1 + a_2 + a_3 + a_4$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$= \frac{3}{2} + \frac{1}{3} + \frac{1}{4}$.

$> \frac{3}{2} + \frac{1}{3} + \frac{1}{4}$

$> \frac{3}{2} + \frac{1}{2}$

$$\boxed{S_4 > \frac{4}{2}}$$

$$\boxed{S_{2^2} > 2}$$

$$S_8 = S_{2^3} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 2 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$> 2 + \frac{1}{2}$$

$$\boxed{S_{2^3} > \frac{5}{2}}$$

The given therem is

$$\boxed{S_{2^n} > \frac{n+2}{2}}$$

hence $\{S_{2^n}\}$ is a divergent sub sequence.

we know that

"all subsequence of divergent sequence is divergent"

$\{S_n\}_{n=1}^{\infty}$ is divergent sequence.

hence,

$\sum_{n=1}^{\infty} (1/n)$ is divergent sequence.

Note :-

i] If $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative numbers.

we sometimes write $\sum_{n=1}^{\infty} a_n < \infty$

ii] If $\sum_{n=1}^{\infty} a_n$ is divergent series

we write $\sum_{n=1}^{\infty} a_n = \infty$.

Theorem : 7

Imp :-

If $\sum a_n$ is a divergent series of +ve numbers Then there is a sequence $\{t_n\}_{n=1}^{\infty}$ of +ve numbers which convergent to '0' but for which.

$\sum_{n=1}^{\infty} (t_n)(a_n)$ still divergents.

Proof :-

Let

$s_n = a_1 + a_2 + a_3 + \dots + a_n$ be the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$

first we have to show that the series $\sum_{k=1}^{\infty} \left(\frac{s_{k+1} - s_k}{s_{k+1}}\right)$ is divergent series.

It $z \neq x$ choose $\epsilon < |z - x|$

For any $m \in I$, choose $n \in I$

such that;

$$\boxed{s_{n+1} > 2 s_m}$$

(This is possible since by hypothesis $\{s_k\}_{k=1}^{\infty}$ divergent to '∞')

now,

Given That, $\sum_{n=1}^{\infty} a_n$ is divergent

now, $\{s_k\}_{k=1}^{\infty}$ is non-decreasing

hence,

$$\sum_{k=m}^{n} \left(\frac{s_{k+1} - s_k}{s_{k+1}}\right) \geq \sum_{k=m}^{n} \frac{s_{k+1} - s_k}{s_{n+1}}$$

$$= \frac{1}{s_{n+1}} \sum_{k=m}^{n} (s_{k+1} - s_k)$$

$$= \frac{1}{s_{n+1}} \Big[(s_{m+1} - s_m) + (s_{m+2} - s_{m+1}) + \dots + (s_{n+1} - s_n) \Big]$$

$$= \frac{1}{s_{n+1}} \left[s_{n+1} - s_n \right]$$

$$> \frac{1}{s_{n+1}} \left[s_{n+1} - \frac{s_{n+1}}{2} \right]$$

For any $m, n \in I$

$s_{n+1} > 2 s_m$

$$\frac{S_{n+1}}{2} > S_m$$

$$\geq \frac{1}{S_{n+1}}\left(\frac{S_{n+1}}{2}\right).$$

$$\boxed{\sum_{k=m}^{n}\left(\frac{S_{k+1}-S_k}{S_{k+1}}\right) \geq \tfrac{1}{2}}$$

we know that

$\sum a_k$ is convergent

Then given $\epsilon > 0$ $\exists$ (+ve) Integer n $\ni$:

$$\left|\sum_{k=1}^{n} a_k\right| < \epsilon \quad \forall\, m, n \geq N.$$

From the above result, the series $\sum_{k=m}^{n}\left(\frac{S_{k+1}-S_k}{S_{k+1}}\right)$ is divergent to '∞'

$$\Rightarrow \sum_{k=1}^{\infty}\left(\frac{S_{k+1}-S_k}{S_{k+1}}\right) = \infty$$

we know that any subsequence of divergent seqnon is divergent $[a_{k+1} = S_{k+1} - S_k]$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{a_{k+1}}{S_{k+1}} = \infty$$

$$\Rightarrow \sum_{k=2}^{\infty} \frac{a_k}{S_k} = \infty$$

$$\Rightarrow \sum_{k=2}^{\infty} a_k(\epsilon_k) = \infty \quad \text{where, } \epsilon_k = \frac{1}{S_k}$$

$$\Rightarrow \sum_{n=2}^{\infty} \epsilon_n a_n = \infty \quad \text{by } k \to \infty \Rightarrow \epsilon_k \to 0$$

$\epsilon_k \to 0$ as $k \to \infty$.

It $z \neq x$ choose $\epsilon < |z - x|$

UNIT - IV

LIMITS and METRIC SPACES

Limit of a function on the real line :-

Definition

we say that $f(x)$ approachs L

(where, $L \in R$) as 'x' approaches 'a'

If given $\epsilon > 0$ there exist $\delta > 0$ such that

$$\boxed{|f(x) - L| < \epsilon \quad (0 < |x-a| < \delta)}$$

In this case, we write

$$\boxed{\lim_{x \to a} f(x) = L} \quad (\text{or}) \quad \boxed{f(x) \to L \text{ as } x \to a}$$

Ex :.

if, $f(x) = x^2 + 2x$ find $\lim_{x \to 3} f(x)$

$$\lim_{x \to 3} f(x) = 3^2 + 2(3)$$

$$= 9 + 6$$

$$= 15$$

Theorem : 1

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

Then,

i) $\lim_{x \to a} [f(x) + g(x)] = L + M$

ii) $\lim_{x \to a} [f(x) - g(x)] = L - M$

iii) $\lim_{x \to a} [f(x) \cdot g(x)] = LM$

iv) $\lim_{x \to a} [f(x)/g(x)] = \frac{L}{M}$

Here $M \neq 0$

Proof:

Given that

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

i) $\lim_{x \to a} [f(x) + g(x)] = L + M$

Given, $\epsilon > 0$ there exist $\delta > 0$ such that

$|f(x) - L| < \epsilon/2$ and

$|g(x) - M| < \epsilon/2$,

for $0 < |x - a| < \delta$.

$|f(x)+g(x)\cancel{2}-(L+M)| = |(f(x)-L)+(g(x)-M)|$

2

$$\le |f(x)-L| + |g(x)-M|$$

$$< \epsilon/2 + \epsilon/2 = \frac{\cancel{2}\epsilon}{\cancel{2}}$$

$|(f(x)+g(x))-(L+M)| < \epsilon$; for $0<|x-a|<\delta$

hence $\lim\limits_{x\to a} [f(x)+g(x)] = L+M$.

ii] $\lim\limits_{x\to a} [f(x)-g(x)] = L-M$

Given $\epsilon>0$ there exist $\delta>0$ such that

$|(f(x)-g(x))-(L-M)| = |(f(x)-L)\cancel{+}(M-g(x))|$

$$\le |(f(x)-L)| + |(g(x)-M)|$$

$$< \epsilon/2 + \epsilon/2, = 2\epsilon/2.$$

$|(f(x)-g(x))-(L-M)| \le \epsilon$ for $x<|x-a|<\delta$.

hence $\lim\limits_{x\to a} [f(x)-g(x)] = L-M$

iii] $\lim\limits_{x\to a} [f(x)\cdot g(x)] = LM$

since $\lim\limits_{x\to a} g(x) = M$

for some $\delta_1 > 0$.

$\Rightarrow |g(x) - M| < 1 \quad (0 < |x-a| < \delta_1)$

$\Rightarrow |g(x)| = |g(x) - M + N|$

$\leq |g(x) - M| + |N|$

$\leq 1 + |N| = Q \quad (\text{say})$

$|g(x)| < Q \quad (0 < |x-a| < \delta_1)$

Now,

$|f(x) g(x) - LM| = |f(x) g(x) - L g(x) + L g(x) - LM|$

$= |g(x)[f(x) - L] + L[g(x) - M]|$

$|f(x) g(x) - LM| < |g(x)| \, |f(x) - L| + |L| \, |g(x) - M|$

$|f(x) g(x) - LM| \leq Q \, |f(x) - L| + |L| \, |g(x) - M| \rightarrow (1).$

if $0 < |x-a| < d$

Given $\epsilon > 0$, there exist $\delta_2 > 0$ such that

$Q \, |f(x) - L| < \epsilon/2 \rightarrow (2) \quad 0 < |x-a| < \delta_2$

and also there exist $\delta_3 > 0$ such that

$|L| \, |g(x) - M| < \epsilon/2 \rightarrow (3).$

if $0 < |x-a| < \delta_3$

Let, $\delta = \min(\delta_1, \delta_2, \delta_3)$

sub ② and ③ in ①.

(4)

$$① \Rightarrow |f(x)g(x) - LM| \le \epsilon/2 + \epsilon/2$$
$$\le 2\epsilon/2.$$

$|f(x)g(x) - LM| < \epsilon$ for $0 < |x-a| < \delta$

$$\lim_{x \to a} [f(x)g(x)] = LM.$$

iv] $\lim_{x \to a} [f(x)/g(x)] = L/M$

since $\lim_{x \to a} g(x) = M \ne 0.$

Let $\epsilon = \frac{|M|}{2}.$

we can find $\delta_1 > 0$ such that

$$|g(x) - M| < \frac{|M|}{2} \rightarrow ①.$$

for $0 < |x-a| < \delta_1$

Now,

$$|M| = |M - g(x) + g(x)|$$
$$\le |M - g(x)| + |g(x)|$$

$$|M| \le \frac{|M|}{2} + |g(x)|$$

$$\Rightarrow |g(x)| > \frac{|M|}{2} \rightarrow ②.$$

for $0 < |x-a| < \delta_1$

Consider

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| = \left| \frac{Mf(x) - Lg(x)}{g(x)M} \right|$$

It $\exists$ # x choose $\epsilon < |L - a|$

$$= \left| \frac{M f(x) - LM + LM - L g(x)}{g(x) M} \right|$$ (5)

$$\leq \left| \frac{M (f(x) - L)}{g(x) M} \right| + \left| \frac{L (g(x) - M)}{g(x) M} \right|$$

$$< \frac{2 |f(x) - L|}{|M|} + \frac{2 |L|}{M^2} |g(x) - M| \quad (\text{by } ①)$$

$$\left| \frac{f(x)}{g(x)} - \frac{L}{m} \right| \leq 2 \frac{|f(x) - L|}{|M|} + \frac{2 |L|}{m^2} |g(x) - M| \to ③.$$

when ever $0 < |x - a| < \delta_1$

let $\epsilon > 0$ be given since

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

we can find $\delta_2, \delta_3 > 0$ such that

$$|f(x) - L| < \frac{\epsilon |M|}{4} \to ④ \quad \text{if } 0 < |x - a| < \delta_2.$$

and

$$|g(x) - M| < \frac{\epsilon |M|^2}{4 |L|} \to ⑤ \quad \text{if } 0 < |x - a| < \delta_3.$$

Let us choose $\delta = \min(\delta_1, \delta_2, \delta_3)$

sub ④ and ⑤ in ③.

$$③ \Rightarrow \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| \leq \frac{2 |f(x) - L|}{|M|} + \frac{2 |L|}{|M|^2} |g(x) - M|$$

$$< \frac{2}{|M|} \left(\frac{\epsilon |M|}{4} \right) + \frac{2 |L|}{|M|^2} \left(\frac{\epsilon |M|^2}{4 |L|} \right)$$

$$< \epsilon/2 + \epsilon/2 = 2\epsilon/2.$$

$$\therefore \left| \frac{f(x)}{g(x)} - \frac{L}{m} \right| < \epsilon$$

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{m} \quad ; 0 < |x - a| < \delta$$

Definition :-

we say that $f(x)$ approaches 'L' as 'x' approaches infinity if given $\epsilon > 0$ $\exists$: $M \in R$ $\ni$:

$$|f(x) - L| < \epsilon \quad (x > M)$$

In this case, we write:

$$\lim_{x \to \infty} f(x) = L \quad (\text{or}) \quad f(x) \to L \text{ as } x \to \infty$$

Definition :-

[Right hand limit of 'f' at 'a']

we say that $f(x)$ approaches 'L' as 'x' approaches 'a' from the right. if given $\epsilon > 0$, $\exists$: $\delta > 0$ $\ni$:

$$|f(x) - L| < \epsilon \quad (a < x < a + \delta)$$

In this case :

we write $\lim_{x \to a^+} f(x) = L$

The number 'L' is called the Right hand limit of 'f' at 'a'

It $z \neq x$ choose $\epsilon < |z - \dots$

(7)

Definition :-

[Left hand limit of 'f' at 'a']

we say that $f(x)$ approaches 'L' as 'x' approaches 'a' from the left. if given $\epsilon > 0$,

$\exists : \delta > 0 \ni :$

$$|f(x) - L| < \epsilon \qquad (a-\delta < x < a)$$

In this case

we write $\lim_{x \to a-} f(x) = L$

The number L is called the left hand limit of 'f' at a.

Definition :- [non-decreasing function] (or) [Increasing Function] :

If 'f' is a real valued function, on an interval $J \subset R$ we say that 'f' is increasing on J. If,

$$f(x) < f(y) \cdot \quad (x < y \; ; \; x, y \in J)$$

Definition :-

[decreasing function :-]

(or)

[Non-increasing Function]

If 'f' is a real valued function on an interval $J \subset R$ we say that 'f' is non-increasing on J. If,

$$f(x) \geq f(y) \cdot (x > y \,;\, x, y \in J)$$

Definition :-

[Monotone Function]

we say that, 'f' is monotone function if, 'f' is either non-decreasing (or) non-increasing

Theorem : 2

let 'f' be a non-decreasing function on the bounded open interval (a, b). If 'f' is bounded above on (a, b) Then

$\lim_{x \to b-} f(x)$ exist. Also if 'f' is bounded below on (a, b) Then $\lim_{x \to a+} f(x)$ exist

It $z \neq x$ choose ...

Given that

'f' is bounded above and 'f' is non-decreasing function - on (a, b)

Let $A = \{ f(x) \mid x \in (a, b) \}$.

A is bounded above

let, $M = $ L.u.b of 'A'

by the defintion :

$$f(x) \leq M \quad \forall\, x \in (a, b) \rightarrow (1).$$

If $\epsilon > 0$,

$\exists$: a no. $(M - \epsilon)$ is not an upper bound for 'A'

hence,

$\exists$: $y \in (a, b) \ni$:

$$f(y) > (M - \epsilon) \rightarrow (2).$$

let, $\delta = b - y \Rightarrow \boxed{y = b - \delta}$

$(2) \Rightarrow f(b - \delta) > (M - \epsilon)$

$\Rightarrow \quad f(y) > (M-\epsilon)$

since, 'f' is non - decreasing

$\therefore \quad f(x) > (M-\epsilon) \quad$ if $b-\delta < x < b \quad \rightarrow (3).$

From (1) and (3).

$\Rightarrow M-\epsilon < f(x) \le M \quad$ if $(b-\delta < x < b)$

$\Rightarrow M-\epsilon < f(x) < M+\epsilon$

$|f(x) - M| < \epsilon$

ie) $\lim\limits_{x \to b-} f(x) = M \quad ; \quad (b-\delta < x < b)$

next

also given that

'f' is bounded below and

f is an non - decreasing fun - on (a,b)

Let $a = \{ f(x) \mid x \in (a,b) \}$

a is an bounded below.

let

M = greatest lower bound of A

by the definition

$f(x) \ge M \quad \forall x \in (a,b)$

$\hookrightarrow (1).$

if $\epsilon > 0$,

$\exists$; a number $(m+\epsilon)$ is not an lower bound for 'A'

hence,

$\exists$: $y \in (a, b) \ni$:

$$f(y) < (M+\epsilon) \rightarrow (2).$$

let, $\delta = -a+y \Rightarrow y = a+\delta.$

$$(2) \Rightarrow f(a+\delta) < (M+\epsilon)$$

$$\Rightarrow f(y) < (m+\epsilon)$$

since, 'f' is non-decreasing

$f(x) < (M+\epsilon) \rightarrow (3)$ if $a < x < a+\delta.$

(1) and (3)

$$M \leq f(x) < M+\epsilon$$

$$\Rightarrow M-\epsilon < f(x) < M+\epsilon$$

$$\Rightarrow |f(x) - M| < \epsilon$$

$a < x < a+\delta.$

$$\lim_{x \to a^+} f(x) = M.$$

Theorem : 3

Let 'f' be a non-increasing function on the bounded open interval (a,b) and if 'f' is bounded above on (a,b) then $\lim_{x \to b-} f(x)$ exists and if 'f' is bounded below then $\lim_{x \to a+} f(x)$ exists.

Proof :

we can apply the last theorem to the function $(-f)$ which is non-decreasing

Theorem :- 4

If 'f' is a monotone function on the open interval (a,b) and if $c \in (a,b)$ then $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ both exist

Proof:

suppose that 'f' is non-decreasing function let us choose, $\epsilon > 0$ $\ni$: $(c-\delta, c+\delta)$ containd a in (a, b)

The the values of 'f' on the open interval $(c-\delta, c)$ are bounded above by $f(c)$

by the lenocon theorem

"f is non-decreasing function on (a, b) and bounded above on (a, b) Then

$$\lim_{x \to b-} f(x) \text{ exist"}$$

$$\Rightarrow \lim_{x \to c-} f(x) \text{ is exist "}$$

next we prove.

suppose that

let 'f' is non- ~~decreasing~~ In function.

let us choose, $\epsilon > 0$ $\ni$: $(c-\delta, c+\delta)$ contained in (a, b) then the values of 'f' on the open interval $(c, c+\delta)$ are bounded below by $f(c)$

by the known theorem,

"f is non-decreasing function on (a,b) and bounded below on (a,b)

Then $\lim_{x \to a+} f(x)$ exist

$\Rightarrow \lim_{x \to c+} f(x)$ is exist, if $(c < x < c+\delta)$

Definition : [strictly increasing function]

The real valued function 'f' on $J \subset R$ is called strictly increasing

if $f(x) < f(y)$ $(\forall x, y \in J \subset R)$

Definition [strictly decreasing function]

The real valued function 'f' on $J \subset R$ is called strictly decreasing

if $f(x) > f(y)$ $(\forall x, y \in J \subset R)$

$\boxed{e(s_n, L) < \epsilon}$ $n \geq N$

In this case we write,

$\lim_{n \to \infty} s_n = L$ (or) $s_n \to L$ as $n \to \infty$. and say that s_n is convergent in M. to the point L.

Definition:

Let $\langle M, e \rangle$ be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in 'M' we say that s_n is an cauchy sequence. If given $\epsilon > 0$ there exist $N \in I$ such that

$$e(s_n, s_m) < \epsilon \quad (m, n \geq N)$$

Theorem:

Let $\langle M, E \rangle$ be a metric space If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points 'M'. Then 's_n' is cauchy sequence.

Proof:

Let $\{s_n\}_1^{\infty}$ be a convergent sequence of points in 'M' and.

$$\lim_{n \to \infty} s_n = L \quad (L \in M)$$

Then, given $\epsilon > 0$ $\exists$: $N \in I$ $\ni$:

$$e(s_n, L) < \epsilon/2 \quad (n \geq N)$$

hence, if $m, n \geq N$

$$e(s_m, s_n) \leq e(s_n, L) + e(L, s_m)$$

$\leq e(s_n, L) + e(s_m, L)$

$\leq \epsilon/2 + \epsilon/2$ (by symmetry)

$= \cancel{2}\epsilon/\cancel{2}$

$e(s_m, s_n) < \epsilon$

hence $\{s_n\}$ is a cauchy sequence.

Note:

The converse of the above theorem is not true.

ie) For some metric space there are cauchy sequence which are not converges.

UNIT - $\bar{V}$

Continous Function on Metric spaces

Functions continous at a Points on a real line :.

Definition: [contionus]

we say that, The function 'f' is contionus at $a \in R^1$

$$\text{if, } \lim_{x \to a} f(x) = f(a)$$

Note:.

The metric space $e(x,y) = |x-y|$, we denote the resulting metric spaces $<R, e>$ by R^1.

Ex:. [Or]

we say that 'f' is contionus at $x=a$ if for every $\epsilon > 0 \; \exists \; \delta > 0 \ni$:

$$|f(x) - f(a)| < \epsilon \; ; \; 0 < |x-a| < \delta.$$

Eg:.

The question of continuity does not arise if the function is not defined at the point

Let : $f(x) = \frac{\sin x}{x}$; $x \in R^1$; $x \neq 0$.

The funtion is not defined at $x = 0$, and hen

is not contionus at $x = 0$

But

$\Rightarrow \quad g(x) = \frac{\sin x}{x}$

$= \frac{\cos x}{1}$

$g(x) = \cos x$

$\lim_{x \to 0} g(x) = \cos(0)$

$= 1$

Then 'g' is contionus at $x = 0$.

as $\lim_{x \to 0} g(x) = g(0)$

Theorem : 1

The real valued functions f and g are contion at $a \in R^1$. Then so are $(f+g)$, $(f-g)$, (fg). If $g(a) \neq 0$ Then $\left(\frac{f}{g}\right)$ is also contionus at 'a'.

Proof :.

i] since,

f and g are contionus at a

$\Rightarrow \lim_{x \to a} f(x) = f(a)$ and

$\lim_{x \to a} g(x) = g(a)$

It $\Sigma \neq \Sigma$ choose

$\Rightarrow \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$= f(a) + g(a)$

ie $\lim_{x \to a} [(f+g)(x) = (f+g)(a)]$ This proves that

$(f+g)$ is contionus at 'a'

ii]

$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

$= f(a) - g(a)$

ie $\lim_{x \to a} [(f-g)(x) = (f-g)(a)]$ This proves that

$(f-g)$ is contionus at 'a'

iii]

$\lim_{x \to a} [f(x) \cdot g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

$= f(a)\, g(a)$

ie $\lim_{x \to a} [fg](x) = [fg](a)$ This proves

that (fg) is contionus at a.

iv] if $g(a) \neq 0$.

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

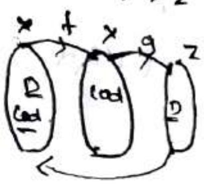
$$= \frac{f(a)}{g(a)}$$

$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(a)$ This proves that (f/g) is continous at a.

Theorem : 8.

If f and g are real valued functions 1e f is contionus at 'a' and 'g' is contionus at $f(a)$ then the composite function $g \circ f$ is also contionus at 'a'

$f : x \to y$
$g : y \to z$



Proof:

let

$b = f(a)$

since 'g' is contionus at 'b'   ∴ [illegible]

for a given $\epsilon > 0$ there exist $\delta > 0$. such that

$$|y - b| < \delta \Rightarrow \boxed{|g(y) - g(b)| < \epsilon} \to (1).$$

again since, 'f' is contionus at a.

corresponding to 'δ' (taking 'ϵ' to be δ in the case)

we can find $\eta > 0$ such that,

$|x-a| < \eta \Rightarrow |f(x) - f(a)| < \delta$.

(or)

$|x-a| < \eta \Rightarrow \boxed{|f(x) - b| < \delta.} \rightarrow$ (2)

(2) $\Rightarrow$ Shows that if, $|x-a| < \eta$

$\Rightarrow$ Then, $f(x)$ lies in the interval $(b-\delta, b+\delta)$

and so we may sub $\underline{f(x) = y}$ in (1).

hence we get from (1) and (2).

$|x-a| < \eta \Rightarrow |g(f(x)) - g(f(a))| < \epsilon$.

$g\circ f$ is contionus at 'a'

Reformulation :-

Definition : [Reformulation of defnition of Continuity]

By the definition of continuity of 'f' at 'a'

we get for any $\epsilon > 0$ There exist a $\delta > 0$ such that

$\boxed{|f(x) - f(a)| < \epsilon}$ when ever $0 < |x-a| < \delta$ the inequality

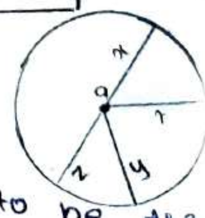
$|f(x) - f(a)| < \epsilon$ is true for $x = a$ also. Thus it is

enough we write $|x-a| < \delta$ instead of $0 < |x-a| < \delta$.

Note :-

The real valued function 'f' is contionus at $a \in R^1$

iff for any given $\epsilon > 0$ $\exists 1 : \delta > 0 \ni :$ $\boxed{|f(x) - f(a)| < \epsilon}$ $(|x-a| < \delta)$



Definition : [open ball]

If $a \in R^1$ and $r > 0$ we define $B[a; r]$ to be the

$B[a;r]$ the open ball of radius r about a.

Theorem : 3

The real valued function 'f' is contionus at $a \in R^1$ $\Leftrightarrow$ the inverse image under 'f' of any open ball $B[f(a);\epsilon]$ about $f(a)$ contains an open ball $B[a;\delta]$ about a.
[ie) given $\epsilon > 0$. $\exists\, \delta > 0 \ni f^{-1}[B(f(a);\epsilon)] \supset B[a;\delta]$]

Proof:

we can prove that,

"f is contionus $\Leftrightarrow$ if for given $\epsilon > 0$. There exist $\delta > 0$ such that: $f^{-1}(B[f(a);\epsilon]) \supset B[a,\delta]$

we know that

'f' is contionus at a $\Leftrightarrow$ if for $\epsilon > 0$ there exist $\delta > 0$ such that

$\Leftrightarrow |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

$\Leftrightarrow$ hence $x \in B[a;\delta] \Rightarrow f(x) \in B[f(a);\epsilon]$

$\Leftrightarrow x \in f^{-1}(B[f(a);\epsilon])$

hence,

'f' is contionus at 'a' $\Leftrightarrow B[a;\delta] \subset f^{-1}(B[f(a);\epsilon])$

Definition: [continuity of convergence :.]

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to 'a' $\Leftrightarrow$ if given $\epsilon > 0$ there exist $N \in I$ such that $x_n \in B[a;\epsilon]$ $\forall\, n \geq N$

Theorem : 4

Statement:

The real valued function 'f' is contionus at $a \in R^1$ if and only if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to 'a'. Then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ ie) 'f' is contionus at "a" $\Leftrightarrow \lim_{n\to\infty} x_n = a \Rightarrow$ $\lim_{n\to\infty} f(x_n) = f(a) \rightarrow (1)$.

suppose that 'f' is contionus at 'a' the we will prove that equation (1) is true

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of real numbes converging to "a" then $f(x_n)$ will be defind for sufficiently large "n"

we must show that $\lim_{n\to\infty} f(x) = f(a)$:

ie, given $\epsilon > 0$ we must find $N \in I$ such that

$\boxed{f(x_n) \in B[f(x); \epsilon] \ \forall \ n \geq N}$ → (2).

since,

'f' is contionus at 'a" there exist $\delta > 0$ such that

$\boxed{f(x) \in B[f(a); \epsilon] \ ; \ x \in B[a; \delta]}$ → (3).

further, since,

$\lim_{n\to\infty} \{x_n\} = a$ there exist $N \in I$ such that

$\boxed{x_n \in B[a, \delta] \ \forall n \geq N}$ → (4).

for this 'N'

condition (2) follows from (3) and (4).

(3) and (4) ⇒ $f(x) \in B[f(a); \epsilon] \ ; x \in B[a, \delta]$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$

conversely, suppose that (1) is true

we must prove that f is contionus at a.

Let us assume that the contraduction.

by the known theorem

" The real valued function 'f' is contionus at $a \in R'$ ⇔ the inverse image under 'f' of any open ball $B[f(a); \epsilon]$ about f(a) contains an open ball $B[a; \delta]$ about a".

Theorem : 5

The function 'f' is contionus at $a \in M_1$ ~~is~~ any one (hence all) of the following conditions holds

i] Given $\epsilon > 0$ $\exists : \delta > 0 \ni : e_2[f(x) : f(a)] < \epsilon \ (e_1(x,a) < \delta)$

ii] The inverse image under 'f' of any open ball $B[f(a) ; \epsilon]$ about $f(a)$ contains an open ball $B[a ; \delta]$ about 'a'

iii] whenever $\{x_n\}$ is a sequence of points in M_1 converging to 'a' Then the sequence $\{f(x_n)\}_1^{\infty}$ of points of M_2 converges to $f(a)$

Proof:

Let us prove [iii]

let $f : M_1 \to M_2$ be contionus at 'a'

let $\{x_n\}_1^{\infty}$ be a sequence of points in M_1 converges to 'a'

since 'f' is contionus at 'a'

Given $\epsilon > 0$ $\exists : \delta > 0 \ni :$

$e_2(f(x), f(a)) < \epsilon \to (1)$ whenever $e_1(x, a) < \delta$.

since, $x_n \to a$ we can find the integer $N \ni :$

$e_1(x_n, a) < \delta \quad \forall n \geq N$

by (1) it follows that,

$e_2[f(x_n), f(a)] < \epsilon \quad \forall n \geq N$

$\Rightarrow f(x_n) \to f(a) \quad (n \to \infty)$

Conversely,

suppose every sequence $\{x_n\}$ in M_1 converges to 'a' the sequence $\{f(x_n)\}$ converges to $f(a)$ in M_2 . ... contionus at 'a' if 'f' is not contionus

Then for $\epsilon > 0$. and for every $\delta > 0$.

$\exists$ a points $x \in M_1 \ni$:

$e_2(f(x), f(a)) \geq \epsilon$ $(e_1(x, a) < \delta)$

let $\delta = 1$

Then $\exists$ a points $x_1 \in M_1$,

$\ni$: $e_2(f(x_1), f(a)) \geq \epsilon$ $(e_1(x_1, a) < 1)$ and

let $\delta = 1/2$

Then $\exists$: $x_2 \in M_1 \ni$:

$e_2(f(x_2), f(a)) \geq \epsilon$ $(e_1(x_2, a) < 1/2)$

in gen $\delta = 1/n$

$\exists$: $x_n \in M_1 \ni$:

$e_2(f(x_n), f(a)) \geq \epsilon$ $(e_1(x_n, a) < 1/n)$

Thus we get a sequence.

$\{x_n\} \in M_1 \ni$:

$e_2(f(x_n), f(a)) \geq \epsilon$ $(e_1(x_n, a) < 1/n)$

$\Rightarrow x_n \to a$ as $n \to \infty$

but $\{f(x_n)\}$ does not converges to $f(a)$

which is $\Rightarrow \Leftarrow$

'f' is must be contionus at 'a'

Theorem: 6

Let (M_1, e_1), (M_2, e_2) and (M_3, e_3) be three metric spaces and Let $f: M_1 \to M_2$; $g: M_2 \to M_3$ if 'f' is contionus at 'a' $\in M_1$ and 'g' is contionus at $f(a) \in M_2$ Then $g \circ f$ is contionus at 'a'

Proof:

Let $h = g \circ f$ from $M_3 \to M_1$ cont.

we have to prove that 'h' is continuous at 'a'

let $f(a) = b$; given $\epsilon > 0$ $\exists\, \delta > 0 \ni$: ~~$e_2(f(x), f(a)) < \epsilon$ $(e_1(x,a) < \delta)$~~

'g' is continuous at 'b' $\Rightarrow e_3(g(y), g(b)) < \epsilon \quad (e_2(y,b) < \delta) \rightarrow (1)$.

also, 'f' is continuous at 'a'

given $\epsilon > 0$ $\exists\, \delta_1 > 0 \ni$:

$e_2(f(x), f(a)) < \epsilon \quad (e_1(x,a) < \delta_1) \rightarrow (2)$.

take, $y = f(x)$

$\Rightarrow e_1(x,a) < \delta_1 \Rightarrow e_2(y,b) < \delta$.

$\Rightarrow e_3(g(y), g(b)) < \epsilon$ (by (1))

$\Rightarrow e_3(g(f(x)), g(f(a))) < \epsilon \rightarrow (3)$.

ie) $h = g \circ f$ is continuous at 'a'

Theorem: 7

Let 'M' be a metric spaces and let f and g are real valued functions which are continuous at $a \in M$ then $(f+g)$ $(f-g)$ (fg) and (f/g) are also continuous at 'a'

Proof:

given, f and g are continuous at 'a',

now, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$

given $\epsilon > 0$ $\exists\, \delta_1 > 0 \ni$:

$0 < e_1(x,a) < \delta_1$

$\Rightarrow \boxed{e_2(f(x), f(a)) < \epsilon/2}$ and given $\epsilon > 0$,

$\exists\, \delta_2 > 0 \ni$:

$0 < e_1(x, a) < \delta_2$

$\boxed{\Rightarrow e_2(g(x), g(a)) < \epsilon/2}$

let $\delta = \min(\delta_1, \delta_2)$

claim (1) $\lim_{x \to a} (f+g)(x) = (f+g)(a)$

given, $\epsilon > 0$ $\exists \cdot \delta > 0$ $\ni$:

$e_2[(f+g)x, (f+g)(a)] < \epsilon$

$[0 < e_1(x, a) < \delta]$

$\Rightarrow e_2[(f+g)(x), (f+g)(a)] \leq e_2(f(x), f(a)) + e_2(g(x), g(a))$

$< \epsilon/2 + \epsilon/2 = \cancel{2}\epsilon/\cancel{2}$.

$\Rightarrow \boxed{e_2[(f+g)(x), (f+g)(a)] < \epsilon}$

$\lim_{x \to a} (f+g)(x) = (f+g)(a)$

claim (2) $\lim_{x \to a} (f-g)(x) = (f-g)(a)$

given, $\epsilon > 0$ $\exists$: $\delta > 0$ $\ni$:

$e_2((f-g)x, (f-g)(a)) < \epsilon$ $(0 < e_1(x, a) < \delta)$

$\Rightarrow e_2((f-g)(x), (f-g)(a)) \leq e_2(f(x), f(a)) + e_2(g(x), g(a))$

$< \epsilon/2 + \epsilon/2 = 2\epsilon/2$.

$\Rightarrow \boxed{e_2((f-g)(x), (f-g)(a)) < \epsilon}$

$\lim_{x \to a} (f-g)(x) = (f-g)(a)$

claim (3) $\lim_{x \to a} (fg)(x) = f(g)(a)$

$f(x) \to f(a)$; $g(x) \to g(a)$ as $x \to a$

$\Rightarrow [f(x) + g(x)] \to [f(a) + g(a)]$ and $[f(x) - g(x)] \to [f(a) - g(a)]$

$\Rightarrow [f(x)+g(x)]^2 \to [f(a)+g(a)]^2$ and $[f(x)-g(x)]^2 \to [f(a)-g(a)]^2$ as $x \to a$

$\Rightarrow 4 f(x) g(x) \to 4 f(a) g(a)$ as $x \to a$.

$\to \frac{1}{4} [4 f(x) g(x)] \to \frac{1}{4} [\cancel{4}[f(a) g(a)]]$ as $x \to a$

$f(x) g(x) \to f(a) g(a)$ as $x \to a$

hence,

$$\boxed{\lim_{x\to a} (fg)(x) = (fg)(a)}$$

claim (A) $\lim_{x\to a} (f/g)(x) = (f/g)(a)$

$0 < e_1(x,a) < \delta \Rightarrow e_2(f(x), f(a)) < \epsilon$ and,

$0 < e_1(x,a) < \delta \Rightarrow e_2(g(x), g(a)) < \epsilon$

by theorem:

$0 < e_1(x,a) < \delta \Rightarrow e_2\left(1/g(x), \frac{1}{g(a)}\right) < \epsilon$

now, $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a} \left[f(x) \cdot \frac{1}{g(x)}\right] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} \frac{1}{g(x)}\right)$

$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = f(a) \cdot \frac{1}{g(a)}$

hence, $\boxed{\lim_{x\to a} (f/g)(x) = (f/g)(a)}$

Definition:

Let M_1 and M_2 be two metric spaces we say that 'f' is contionus function from M_1 into M_2 if 'f' is contionus at each points in M_1

Theorem: 8

If f and g are contionus function from a metric space 'M_1' into a metric space 'M_2' then ... (f/g), here $g \neq 0$

Proof:

Answer is previous theorem.

Definition: [open sets]

Let M be a metric space. Let U be a subset of M. We say that U is an open subset of M (or) [U is open]. If $\forall x \in U$. There exist a number $r>0$ such that the entire open ball $B[a;r]$ is contained in 'U'

Theorem: 9

In any metric space (M, d) both M and the empty set ϕ are open set

Proof:

If $x \in M$

Then by the definition of open ball $B[a;r]$ every open ball is contained in M.

hence M is open

also,

ϕ is open

since, there is no 'x' in 'ϕ'

and hence every $x \in \phi$

It ... satisfies the condition in the definition for an open

Theorem : 10.

If $\{U_i \mid i \in I\}$ is a family of open sets in a metric space 'M' then $\bigcup_{i \in I} U_i$ is also an open set in 'M'

Proof:

Let, $U = \bigcup_{i \in I} U_i$

if $U = \phi$

Then by the known theorem

" In any metric space 'M' both M and ϕ are open"

$\Rightarrow$ U is open

Assume that, $U \neq \phi$

let $x \in U$

To prove that $\exists$: open ball $B[a;r] \subset U$

$\therefore x \in U = \bigcup_{i \in I} U_i$

$\Rightarrow x \in U_i \ \forall i$

but U_i is open

$\therefore$ there is an open ball

$B(x;r) \subset U_i \subset \bigcup_{i \in I} U_i = U$

ie) $B(x;r) \subset U$

and hence 'U' is open.

Theorem : 11

every subset's of R_d is open.

Proof:

[Note:- $R_d = \begin{cases} d(x,x) = 0 ; x \in R \\ d(x,y) = 1 ; x,y \in R, x \neq y \end{cases}$]

For if

$a \in R_d$

Then $B(a;1) = \{ x / d(a,x) < 1 \} = \{a\}$

But every open ball is an open set thus all single points in R_d are open

Also, any subset 'S' of R_d is a union of single point set

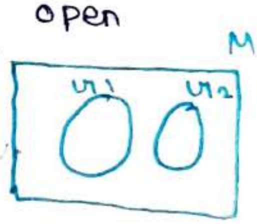
$\therefore$ 'S' is open

Theorem : 12

If U_1 and U_2 are open subsets of M Then $U_1 \cap U_2$ is also open.

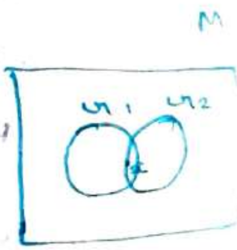
Proof:

if $U_1 \cap U_2 = \phi$, Then it is open



let, $U_1 \cap U_2 \neq \phi$

let $x \in U_1 \cap U_2$ be arbitrary



Then, $x \in U_1$ and $x \in U_2$

but U_1 and U_2 are open in M

hence, $\exists$ a +ve number r_1 and r_2 $\ni$:

$B(x, r_1) \subset U_1$ and $B(x, r_2) \subset U_2$

Let $r = \min(r_1, r_2)$

Then $r > 0$ and

$B(x, r) \subset B(x, r_1) \subset U_1$

$B(x, r) \subset B(x, r_2) \subset U_2$

$\therefore \forall\, x \in U_1 \cap U_2$, $\exists$ a open ball $\ni$:

$B(x, r) \subset U_1 \cap U_2$

$U_1 \cap U_2$ is open

Theorem: 13

Every open subset U of R^1 can be expressed as a union of countable number of mutually disjoint open intervals.

Proof:

let $x \in U$

since, U is open

$\exists$ an open ball $B(x, r) = (x-r, x+r) \subset U$

let $I = (x-r, x+r)$

$\exists\, I \ni : x \in I ; I \subset U$

let I_x be the largest open interval

$\ni : x \in I_x$ and $I_x \subset U$

then $U = \bigcup_{x \in a} I_x$

now, if $x, y \in U$ then

$(1,4), (4,3)$

$I_a \leq y$

either $I_x = I_y$ (or) $I_x \cap I_y = \phi$.

$I_x \cap I_y \neq \phi$

suppose $I_x \cap I_y \neq \phi$

$(1,2), (2,4)$

$I_a \cap I_y = \phi$

we show that $I_x = I_y$

I_x and I_y are open intervals

$\ni$: $x \in I_x$; $I_x \subset G$

$y \in I_y$; $I_y \subset G$

Also, $I_x \cap I_y \neq \phi \Rightarrow I_x \cup I_y$ is an open interval

$I_x \cup I_y$ is an open interval containing

$x \ni$: $I_x \cup I_y \subset G$

but, I_x is the largest open interval containing $x \ni$: $I_x \subset G$

$$I_x \cup I_y = I_x \Rightarrow I_y \subset I_x$$

$|||^{ly}$,

$$I_x \subset I_y$$

$$\boxed{I_x = I_y}$$

let,

$$S = \{ I_x \mid x \in G \}$$

Then 'S' is a family of disjoint open intervals I_x

$\forall I_x \in S$ choose a rational number $q_x \in I_x$

Define a maping $f : S \to Q$

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by $f(Ix) = qx$

Then $Ix \neq Iy \Rightarrow q.x \neq qy$

f is 1-1

Thus 's' is equivalent to the subset of 'Q' but 'Q' is countable hence 's' is also countable. Thus 'U' is a union of countable no. of mutually disjoint open intervals

Note :-

Theorem :- 14

Let $f : M_1 \to M_2$ then f is continuous on $M_1 \Leftrightarrow$ $f^{-1}(U)$ is open in M_1 whenever 'U' is open in M_2

(or)

f is continuous $\Leftrightarrow$ the inverse image of every open set is open.

Proof:

[Theorem :-

we prove that

'f' is continuous $\Leftrightarrow$ if for given $\epsilon > 0$ there exist $\delta > 0$ such that $f^{-1}(B[f(a) ; \epsilon]) \supset B[a ; \delta]$

we know that

'f' is continous at a $\Leftrightarrow$ if for $\epsilon > 0$ there exis[ts]

$\delta > 0$ such that

$\Leftrightarrow |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

$\Leftrightarrow$ There hence $x \in B[a ; \delta] \Rightarrow f(x) \in B[f(a) ; \epsilon]$

$\Leftrightarrow x \in f^{-1}[B[f(a) ; \epsilon)]$

hence,

f is continous at a $\Leftrightarrow B[a;\delta] \subset f^{-1}(B[f(a);$ [ε)]

Definition :.

Let E be a subset of M a point $x \in M$ is called limit point of E if $\exists$ a sequence $\{x_n\}_1^\infty$ of point E which converges to 'x' The set $\bar{E}$ of all limit points of E is called the closure of 'E'.

Theorem: 15

Let (M, e) be a metric spaces and $E \subset M$ a point $x \in M$ is a limit point of E $\Leftrightarrow$ every open ball centerd at x. contains atleast one point of E.

Proof :.

Let 'x' be a limit point of E and $B(x, r)$ be an open ball about 'x' then $\exists$: a sequen[ce]

$\{x_n\}$ in E $\ni$: $\{x_n\}$ converges to 'x'

so, $\exists$ n_0 $\ni$: $\rho(x_n, x) < r$ for $n \geq n_0$.

ie) $\rho(x_{n_0}, x) < R$

$\Rightarrow x_{n_0} \in B(x; r)$

$B[x; r]$ contains a point of E

Conversely :.

Suppose every open ball centered at 'x' contains a point of 'E'. The each open ball $B(x; 1/n)$.

contains a point x_n of E

$\rho(x_n, x) < 1/n \to 0$ as $n \to \infty$

ie) $x_n \to x$ as $n \to \infty$.

Definition [Closed]

Let E be a subset of metric spaces 'M'

we say that E is a closed subset of 'M'

if $\bar{E} = E$

Note :.

A set is closed $\Leftrightarrow$ it contains all its limit points

5m x Theorem: 16.

If E is a any subset of metric space M

5m (x) then $\bar{E} = \bar{\bar{E}}$ (ie) $\bar{E}$ is a closed subset.

Proof:

since, $\bar{E} \subset \bar{\bar{E}}$

we prove $\bar{E} \supset \bar{\bar{E}}$

Let $x \in \bar{\bar{E}}$

To show that $x \in \bar{E}$:

it is enough to prove that any open ball $B(x;r)$ contains a points of $\bar{E}'$.

since, $x \in \bar{\bar{E}}$

The Ball contains a points $y \in \bar{E}$

Let, $s = \rho(x,y)$ and let $t > 0$

with $t < r - s$

since, $y \in \bar{E}$

The ball $B(y,t)$ contains a point $z \in E$

But, $\rho(x,y) = s$

$$\rho(x,z) \le \rho(x,y) + \rho(y,z)$$
$$\le s + t$$
$$< \cancel{s} + r - \cancel{s}$$
$$= r$$

hence , $z \in B(B; r)$

Thus $B(x; r)$ contains a points of E'

Theorem: 17

In any metric space (M, e) the sets M and ϕ are both closed.

Proof:

Clearly M contains all its limit points. and that ϕ has no. limit points and hence contains all it's limit points.

Theorem: 18.

If F_1 and F_2 are closed subsets of metric space M then $F_1 \cup F_2$ is also a closed set in M

Proof:

Since F_1 and F_2 are closed

$$F_1 = \overline{F_1} \quad ; \quad F_2 = \overline{F_2}$$

let $x \in \overline{F_1 \cup F_2}$

Then $\exists$ a sequence $\{x_n\}_1^\infty$ of points $F_1 \cup F_2$ which converges. to x

But $\{x_n\}$ must have a subsequence consists of all points . F_1 (or) all points in F_2

since, any subsequence of $\{x_n\}$ mus be a convergent sequence.

$\Rightarrow x \in \bar{F_1} = F_1$ (or) $x \in \bar{F_2} = F_2$

Thus,

$x \in F_1 \cup F_2$

$\Rightarrow F_1 \cup F_2 \supset \overline{F_1 \cup F_2}$

but,

$\overline{F_1 \cup F_2} \subset \overline{F_1 \cup F_2}$

hence,

$\overline{F_1 \cup F_2} = F_1 \cup F_2$

ie) $F_1 \cup F_2$ is closed.

Theorem: 19

If F is any family of closed subset of metric space M Then $\bigcap_{f \in F} F$ is closed.

Proof.

Let $x \in \overline{\cap F}$

Then any $B(x', r)$ contains a point $y \in \cap F$

$\Rightarrow y \in F \ \forall f$

For any F :

The ball contains a point of F (is y)

hence, $x \in \bar{F} = F$

Thus, x lies in every $f \in \mathcal{F}$ $\cap$ - intersection

$\Rightarrow x \in \cap F$

$\Rightarrow \cap F \supset \overline{\cap F}$

but $\Rightarrow \overline{\cap F} \subset \cap F$

$\Rightarrow \cap F = \overline{\cap F}$

$\cap F$ is closed.

Theorem: 20.

Let U be an open subset of metric space M then $U' = M - U$ is closed conversely F is closed subset of M then $F' = M - F$ is open.

Proof:

Let U is open

If $x \in U$ then by the definition of open set $B = B(x; r)$ which lies entirely in U.

Hence, B contains no point of U'

we know that

x canot be a limit point of U'

Thus no point in 'U' is a limit point of U' and so U' contains all it's limit points

Hence, U' is closed

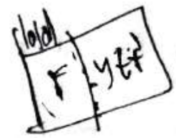
Conversely ;

Let F is closed if $y \in F'$

if $B = B(y;r)$ which contains no point in F (or) y be a limit point $F \Rightarrow y \notin F$

since, F is closed $\Rightarrow \notin y \in F'$

$\forall\, y \in F'$ the ball $B(a;r)$ lying entirely in F'



$\therefore F'$ is open

Theorem : 2)

$(M_1, \rho_1), (M_2, \rho_2)$ be metric spaces $f : M_1 \to M_2$ Then F is contionus on M_1 iff

10m ~~f^{-1} of~~ . $f^{-1}(F)$ is closed subset of $\in M_1$.

whenever F is closed subset of M_2

Proof :

Let f is contionus on M_1

if $F \subset M_2$ is a closed set

we now that

$F' = M_2 - F$ is open

also by a theorem

$f^{-1}(F')$ is open in M_1

Since, $f : A \to B$ & $Y \subset B$

$\Rightarrow f^{-1}(F) \cup f^{-1}(F') = f^{-1}(M_2)$

ie) $f^{-1}(F) \cup f^{-1}(F') = M_1$

hence, $f^{-1}(F)$ is complement (relative to M_1) of $f^{-1}(F')$,

since, $f^{-1}(F')$ is open Then, $f^{-1}(F)$ is closed.

Theorem: 22

Let 'f' be a 1-1 function from a metric spaces M_1 onto M_2 Then if 'f' has any one of the following properties (hence all)

1] both f and f^{-1} are contionus (on M_1 and M_2

2] $U \subset M$ is open $\Leftrightarrow$ its image $f(U) \subset M_2$ is open.

3] $F \subset M$ is closed $\Leftrightarrow$ its image $f(F)$ is closed

Definition: [Homeomorphism]

A 1-1 and onto function f which is also contionus defined from a metric space M_1 to M_2 is called a homemomorphism.

Definition : [Dense set]

Let M be a metric space the subset A of M is said to be dense in M

if $\boxed{\bar{A} = M}$

Discontionus function on R^1

Definition : [F_σ - type]

The sub set D of R^1 is said to be F_σ - type if $D = \bigcup_{n=1}^{\infty} F_n$ where, each F_n is a closed subset of R^1

Definition :-

Let $f : R^1 \to R^1$ if J is any bounded open interal in R^1 $\omega[f ; J]$ (called oscillation of f over as $\omega[f ; J] = \underset{x \in J}{\text{lub}}\, f(x) - \text{glb}\, f(x)$.

Then if $a \in R^1$

define $\omega[f ; a] = \text{glb}\, \omega[f ; J)$ is called oscill-ion of 'f' at a.

Thm 5.6 C, 5.6 D, 5.6 E

Definition :-

The subset A of R^1 is called no where dense in R^1 if $\bar{A}$ contains no (non empty) open intervals

Definition :-

The subset of D of R^1 is called first ... if $D = \bigcup^{\infty} E_n$

Where each E_n is nowhere dense R^1

If 'D' is not of the 1^{st} category then we say

that 'D' is 2^{nd} category.

x. Theorem: 23.

10m

Baire category theorem

10m

x. 5m

Statement:

The set R^1 of the 2^{nd} category

(or)

Prove that the set of all real numbers is 2^{nd} category.

Proof:

suppose R^1 is 1^{st} category

$\Rightarrow R^1 = \cup F_n$

where, each F_n is nowhere dense subset

Assume, F_n are closed.

$\Rightarrow R^1 = \cup \bar{F_n}$

where each $\bar{F_n}$ is closed. & no where dense set

Step-1:

now, F_1 is nowhere dense set

choose $x_1 \notin F_1$

$\exists$ an open interval $I_1 \ni : x_1 \in I_1$ and

$I_1 \cap F_1 = \phi$

Let J_1 be closed interval $\ni$:

$0 < \text{length of } J_1 < 1$, and $J_1 \subseteq I_1$

$\Rightarrow J_1 \cap F_1 = \phi$

$0 <$ length of $J_2 <$ and $J_2 <$

Step - 2

Now, F_2 is nowhere dense set

choose $x_2 \notin F_2$

$\exists$ an open interval $I_2 \ni : x_2 \in I_2$, and $I_2 \cap F_2 = \phi$

Let J_1 be closed interval $\ni$:

$0 <$ length of $J_2 < 1/2$ and $J_2 \subseteq I_2$.

$\Rightarrow J_2 \cap F_2 = \phi$

Step - 3

Continue the above process

Now we get a sequence of non-empty closed sub intervals $J_1, J_2, \ldots \ni : J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$

and $0 <$ length of $J_n < 1/n$

$$J_n \cap F_n = \phi$$

by nested interval theorem:.

$\exists$: only one $y \in R' \ni : y = \cap J_n$

$\Rightarrow y \in J_n \ \forall n \Rightarrow y \notin J_n \ \forall n$

$\Rightarrow y \in R' \Rightarrow y \notin \cup F_n$

which is $\Rightarrow \Leftarrow$

R' is of 2-nd category sets.

Theorem : 24

5m
*.

If A and B are sets of 1-st category then $A \cup B$ is also 1-st category

Proof:

Let A and B are two, ~~Set~~ 1st category sets

$$\Rightarrow A = \bigcup_1^\infty H_n; \quad B = \bigcup_1^\infty F_n$$

where, H_n and F_n are nowhere dense sets

$$\Rightarrow A \cup B = \left(\bigcup_1^\infty H_n\right) \cup \left(\bigcup_1^\infty F_n\right)$$

$$A \cup B = \bigcup_1^\infty (H_n \cup F_n)$$

here $H_n \cup F_n$ is nowhere dense sets

hence $A \cup B$ is 1-st category sets

Theorem : 25

The set of all irrationals is

i] 2nd category set

ii] not of F_σ type set

<u>Proof</u> :

Let $R' = A \cup B$

here, A is irrational and B is rationals

we know that

B is 1st category set

we prove that A is 1st category set

$\Rightarrow R'$ is 1st category set

which $\Rightarrow \Leftarrow$

the set A of irrationals is 2nd

category set

ii]

Let A be irrationals

we prove that A is not F_σ type.

if A is of F_σ type,

$$\boxed{A = \cup F_n}$$

where, each F_n is closed, and nowhere, dense subsets

$\Rightarrow$ A is (set of irrationals) is 1^{st} category set.

which is $\Rightarrow \Leftarrow$

A is not of F_σ type.