BUSINESS STATISTICAL DECISION TECHNIQUES

(COMMON ON B.Com, B.Com (CA) & B.Sc (C/S))

PAPER CODE:

UNIT – I

Matrix: definitions – operations on matrix. Determinant of matrix – inverse of a matrix (ad joint method only) – Application: Solving of linear equations – matrix inverse method: Cramer's rule.

UNIT – II

Sequence and series – Arithmetic progression and geometric progression;

Interpolation: binomial expansion method; Newton's forward and backward method, Lagrange's method.

UNIT – III

Probability: definition – addition and multiplication theorems – conditional and probability – (simple problems only).

UNIT – IV

Linear programming problem – formation of LPP, solution to LPP – Graphical simplex method – BIG-M method.

UNIT – V

Transportation problem – North west corner rule (method) – matrix minima (or) least cost method – Vogel's approximation method – Modi method (Modified distribution method) or U – V method.

Assignment problem – Balanced Hungarian assignment method.

NOTE: Problem 80%; Theory 20%

Text books:

1. P.A. Navanitham – Business statistics

Reference books:

1. Dr.S.P. Gupta; Dr.P.A. Gupta; Dr. Manmohan (Business statistics and operation research)
2. M.R. Vittal – Business Mathematics.
3. Pillai R.S.N & Mrs. Bagavathi , statistics Sultan Chand & sons, New Delhi

UNIT - I

MATRICES

Definition of Matrix: Matrix is an arrangement of elements (numbers) in rows and columns. The numbers are enclosed by parentheses or brackets or double bars.

For example, 1. $\begin{bmatrix} 1 & 5 & 9 \\ 3 & 7 & 6 \\ 5 & 14 & 19 \end{bmatrix}$ 2. $\begin{bmatrix} 0 & 15 \\ -3 & 32 \\ 7 & 41 \end{bmatrix}$ 3. $\begin{Vmatrix} 13 & 16 \\ 23 & 50 \end{Vmatrix}$

Importance and Uses:

- As matrix enables compact presentation and facilitates smooth manipulations, it is used in many fields of study.
- It is convenient for computer operations also.
- The common operations of addition, multiplication, transposition, inversion, etc. are possible and simple in matrix algebra.
- Matrix occupies an important place from ode message to solution of simultaneous equations.
- In statistics (in Design of Experiments, Multivariate Analysis, etc.), in Economics (in Social Accounting, Input-Output Tables, etc.), in Commerce (in Linear Programming, Allocation of Expenses, etc.), etc. the matrix form is the most convenient one.

Order of a Matrix: Order of a matrix indicates the number of rows and the number of columns of the matrix. The general form, given above is of order 'm by n' or '$m \times n$'. The following matrix is of order 2×3.

For e.g., $\begin{bmatrix} 5 & 10 & 19 \\ 41 & 49 & 50 \end{bmatrix}$ In $A = \begin{bmatrix} 5 & 10 & 19 \\ 41 & 49 & 50 \end{bmatrix}$ $a_{11}=5$, $a_{12}=10$, $a_{13}=19$, $a_{21}=41$, $a_{22}=49$, $a_{23}=50$

Types of Matrices:

1. **Square Matrix:** When the number of rows and the number of columns of a matrix are equal, the matrix is called a square matrix.

   For Example, $A=\begin{bmatrix} 3 & 4 & 7 \\ 0 & 1 & 1 \\ 5 & 9 & 10 \end{bmatrix}$ is a square matrix of order 3.

2. **Row Matrix:** If there is only one row in a matrix, it is called a row matrix or a row vector.

   Example: A= [10, 32, 50]. It is of order 1×3.

3. **Column Matrix:** If there is only one column in a matrix, it is called a column matrix or a column vector.

   Example: $A = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. It is of order 2×1.

4. **Zero or Null Matrix:** If all the elements of a matrix are zeros, it is called a zero or a null matrix and is denoted by 0.

Example: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is of order 2 × 2.

5. **Equal Matrices:** Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal (i.e., A=B) if and only if

   (i) they have same order, i.e., $A = (a_{ij})$ m×n and $B = (b_{ij})$ m×n and

   (ii) the elements at the corresponding places are equal, that is, $a_{ij} = b_{ij}$ for every i and j.

   Example: If $A = \begin{bmatrix} 2 & 4 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ 5 & 7 & 9 \end{bmatrix}$, $A = B$

6. **Equivalent Matrices:** Two matrices A and B of the same order are said to be equivalent if one of them can be obtained from the other by elementary transformations.

It is written as A~B and read A equivalent to B.

7. **Diagonal Matrix:** A square matrix all of whose elements expect those in the principal or leading or main diagonal are zeros is called a diagonal matrix.

The matrix $A = \begin{bmatrix} a_{11} & 0 & 0 & . & . & . & 0 \\ 0 & a_{22} & 0 & . & . & . & 0 \\ 0 & 0 & a_{33} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a_{nn} \end{bmatrix}$ a diagonal matrix. It is written also as

$A = \text{diag.}(a_{11}\, a_{22} \ldots. a_{nn})$

Example $\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ i.e., A= diag (5, 6, 10) is a diagonal matrix.

8. **Scalar Matrix:** A diagonal matrix in which all the elements in the diagonal are equal is called a scalar matrix.

The matrix $A = \begin{bmatrix} a & 0 & 0 & . & . & . & 0 \\ 0 & a & 0 & . & . & . & 0 \\ 0 & 0 & a & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a \end{bmatrix}$ is a scalar matrix. It is written as A= diag. (a a a….a).

9. **Unit Matrix or Identity Matrix:** A square matrix whose diagonal elements are 1 (unity) each and non diagonal elements are zeros is called a unit matrix or a identity matrix and is denoted by alphabet I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. **Symmetric Matrix:** A square matrix such that $a_{ij} = a_{ji}$ for all i and j is called a symmetric matrix. i.e., A^1=A where A^1 is the transpose of A and it has been defined under 'Matrix Operations I'.

Example: $A = \begin{bmatrix} 5 & 7 & 9 \\ 7 & 0 & -3 \\ 9 & -3 & -1 \end{bmatrix}$

11. **Skew Symmetric Matrix:** A square matrix such that $a_{ij} = - a_{ji}$ for all I and j is called a skew symmetric matrix. That is, $A^1 = -A$.

Example: $\mathbf{A} = \begin{bmatrix} 0 & 7 & 9 \\ -7 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$

12. **Triangular Matrix:** Triangular matrices are of two kinds, viz., upper triangular matrix and lower triangular matrix. A matrix $A = (a_{ij})_{mxn}$ is an upper triangular matrix is $a_{ij} = 0$ for $i > j$.

Example: $A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 14 & 15 \\ 0 & 0 & 32 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 19 & 23 & 33 & 0 \end{bmatrix}$ are upper and lower triangular matrices respectively. Triangular matrix need not be a square matrix.

13. **Sub-Matrix:** A matrix obtained by deleting one or more rows or one or more columns is a sub-matrix of a given matrix.

For Example: if $A = \begin{bmatrix} 9 & 10 & 15 \\ 6 & 19 & 41 \end{bmatrix}$ is a given matrix.

14. **Orthogonal Matrix:** A square matrix A is said to be an orthogonal matrix if

$$A^1 A = A A^1 = I.$$

15. **Non-Singular Matrix:** A Square matrix A is said to be non-singular if $|A| \neq 0$.

$|A|$ Denotes determinant A and the definition of a determinant is to be seen later.

A square matrix A is said to be singular if $|A|$ =0.

Matrix Operations- I

1.(i) Addition: If $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$, their sum, $A+B=(a_{ij} + b_{ij})_{mxn}$. Two matrices of the same order are said to be **conformable** for addition. That is, two matrices can be added if and only if they are of the same order. Elements at identical positions are to be added.

Example 1: If $A=\begin{bmatrix}4 & 6 & 9\\ 3 & 5 & 10\end{bmatrix}$ and $B=\begin{bmatrix}5 & 0 & -1\\ 4 & -7 & -3\end{bmatrix}$ Find the A + B.

Solution: $A+B=\begin{bmatrix}4+5 & 6+0 & 9-1\\ 3+4 & 5+(-7) & 10+(-3)\end{bmatrix}=\begin{bmatrix}9 & 6 & 8\\ 7 & -2 & 7\end{bmatrix}$

1. (ii) Subtraction: If $A=(a_{ij})_{mxn}$ and $B=(b_{ij})_{mxn}$, their sum, $A-B=(a_{ij}-b_{ij})_{mxn}$. Two matrices of the same order are said to be **conformable** for Subtraction. That is, subtraction is defined if and only matrices are of the same order. Elements at identical positions are to be subtracted.

Example 2: If $A=\begin{bmatrix}4 & 6 & 9\\ 3 & 5 & 10\end{bmatrix}$ and $B=\begin{bmatrix}5 & 0 & -1\\ 4 & -7 & -3\end{bmatrix}$ Find the A - B.

Solution: $A-B=\begin{bmatrix}4-5 & 6-0 & 9-1\\ 3-4 & 5-(-7) & 10-(-3)\end{bmatrix}=\begin{bmatrix}-1 & 6 & 8\\ -1 & 12 & 13\end{bmatrix}$

Note: $B-A=\begin{bmatrix}5-4 & 0-6 & 1-9\\ 4-3 & -7-5 & -3-10\end{bmatrix}=\begin{bmatrix}1 & -6 & -8\\ 1 & -12 & -13\end{bmatrix}\neq A-B$

Example 3: If $A=\begin{bmatrix}4 & -1 & 0\\ -3 & 5 & -6\\ 2 & -7 & 8\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0 & 1\\ 5 & -2 & 2\\ 3 & 4 & 3\end{bmatrix}$ Find the A + B and A - B.

Solution:

$$A+B=\begin{bmatrix}3 & -1 & 1\\ 2 & 3 & -4\\ 5 & -3 & 11\end{bmatrix} \quad \text{and} \quad A-B=\begin{bmatrix}5 & -1 & -1\\ -8 & 7 & -8\\ -1 & -11 & 5\end{bmatrix}$$

2. Scalar Multiplication: Scalar is a real number in the context of matrix operations. To get a scalar multiple of a matrix, every element of the matrix is to be multiplied by the scalar. If K is a scalar and A is a matrix, their product is KA which a scalar multiple of A.

E.g., If $A=\begin{bmatrix}0 & 4\\ 2 & 5\\ -7 & 9\end{bmatrix}$, $3A=\begin{bmatrix}0 & 12\\ 6 & 15\\ -21 & 27\end{bmatrix}$ and $-4A=\begin{bmatrix}0 & -16\\ -8 & -20\\ 28 & -36\end{bmatrix}$

Example 4: If $A=\begin{bmatrix}2 & 3 & 5\\ 4 & 7 & 9\\ 1 & 6 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1 & 2\\ 4 & 2 & 5\\ 6 & -2 & 7\end{bmatrix}$, Show that 5(A+B)=5A+5B.

Solution: From the given matrices A and B,

$$A+B=\begin{bmatrix}5 & 4 & 7\\ 8 & 9 & 14\\ 7 & 4 & 11\end{bmatrix} \quad \text{and} \quad 5(A+B)=\begin{bmatrix}25 & 20 & 35\\ 40 & 45 & 70\\ 35 & 20 & 55\end{bmatrix} \quad ....(1)$$

$$5A = \begin{bmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{bmatrix} \quad 5B = \begin{bmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{bmatrix} \text{ and } \quad 5A + 5B = \begin{bmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{bmatrix} \text{............(2)}$$

Hence, from (1) and (2), 5(A+B) = 5A+5B.

Example 5: If $A = \begin{bmatrix} 3 & 5 \\ 2 & a \end{bmatrix}$, $B = \begin{bmatrix} 4 & b \\ 2 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 26 & a \\ 14 & 45 \end{bmatrix}$ find a and b when 2A+5B = C.

Solution: From the given matrices A, B, and C,

$$2A = \begin{bmatrix} 6 & 10 \\ 4 & 2a \end{bmatrix}, \quad 5B = \begin{bmatrix} 20 & 5b \\ 10 & 45 \end{bmatrix} \quad \text{and } \; 2A + 5B = \begin{bmatrix} 26 & 10 + 5b \\ 14 & 45 + 2a \end{bmatrix}$$

2A+5B = C gives $\begin{bmatrix} 26 & 10 + 5b \\ 14 & 45 + 2a \end{bmatrix} = \begin{bmatrix} 26 & a \\ 14 & 45 \end{bmatrix}$

$\therefore$ 10+5b = a ……….. (1) and 45+2a = 45 …………. (2)

$\therefore$ from (2), a = 0 and so from (1), b = -2.

Example 6: If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find the matrix X such that 3A+5B+2X=0.

Solution: From the given matrices A and B, $3A = \begin{bmatrix} 27 & 3 \\ 12 & 9 \end{bmatrix}$ and $5B = \begin{bmatrix} 5 & 25 \\ 35 & 60 \end{bmatrix}$

Let

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\therefore 2X = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$. $3A + 5B + 2X = 0$ gives $\begin{bmatrix} 32 + 2a & 28 + 2b \\ 47 + 2c & 69 + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ 32+2a=0; 28+2b=0; 47+2c=0; 69+2d=0

$\therefore$ a = -16; b = - 14; c = -23.5; d = -34.5

$$\therefore \quad X = \begin{bmatrix} -16.0 & -14.0 \\ -23.5 & -34.5 \end{bmatrix}$$

3. Multiplication: If A = $(a_{ij})_{mxp}$ and B = $(b_{ij})_{nxp}$, their product AB is a matrix C = $(c_{ij})_{mxn}$ where

$C_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots\ldots + a_{ip} b_{pj}$ for all I and j.

Hence, the product **AB is defined** or A is **conformable** to B for multiplication if and only if the number of columns of A is equal to the number of rows of B.

Example 7: If $A = \begin{pmatrix} 3 & 5 & 6 \end{pmatrix}_{1\times3}$ and $B = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}_{3\times1}$

$$AB = \begin{pmatrix} 3\times4+ & 5\times6+ & 6\times2 \end{pmatrix}_{1\times1} = (29)$$

Example 8: If $A = \begin{pmatrix} 2 & 3 & 5 \end{pmatrix}_{1\times3}$ and $B = \begin{pmatrix} 4 & 7 \\ 0 & 1 \\ -6 & 9 \end{pmatrix}_{3\times2}$

$$AB = \begin{pmatrix} 2\times4 + 3\times0 + 5\times-6 & 2\times7 + 3\times1 + 5\times9 \end{pmatrix} = \begin{pmatrix} -22 & 62 \end{pmatrix}$$

Example 9: Examine whether AB = B and BA = A, given $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

Solution: $AB = \begin{bmatrix} 2\times4+4\times3 & 2\times(-2)+4\times(-1) \\ 3\times4+6\times3 & 3\times(-2)+6\times(-1) \end{bmatrix} = \begin{bmatrix} 20 & -8 \\ 30 & -12 \end{bmatrix} \neq A$

4. **Transpose:** Let A be a matrix of order $m\times n$. The transpose of A is denoted by A^1 or A^t. It is of order $n\times m$. It is obtained by writing the first row of A as the first column, the second row of A as the second column and the last row of A as the last column.

Example 10: If $A = \begin{bmatrix} 1 & 5 & 9 \\ 10 & 14 & 19 \end{bmatrix}$, find the A^1.

Solution: $A^1 = \begin{bmatrix} 1 & 10 \\ 5 & 14 \\ 9 & 19 \end{bmatrix}$

Example 11: Verify that $B^T A^T = (AB)^T$ when $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

Solution: From the given matrices A and B.

$$B^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } B^T A^T = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \longrightarrow (1)$$

$$AB = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \text{ and } (AB)^T = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \longrightarrow (2)$$

From (1) and (2), $B^T A^T = (AB)^T$

5. **Properties:**

(a) Addition of Matrices.

(i) **Commutative.** If A and B matrices of the same order **A+B = B+A**

(ii) **Associative.** If A, B, and C are matrices of the same order, **(A+B)+C = A+(B+C)**

(iii) **Distributive with respect to scalar.** **k(A+B) = kA+kB**

(iv) **Existence of identity.** The null matrix **O** of the order of **A** is the additive identity of A. It exists and **A+O = A = O+A**

(v) **Existence of inverse.** –A is the additive inverse of **A.** **A+(-A) = 0 =(-A)+A**

(vi) **Cancellation law.** If A, B and C are matrices of the same order, **A+C = B+C** implies A = B

(b) Multiplication of Matrices.

(i) **Not Commutative.** For two matrices A, and B, AB and BA might have been defined but they need not be equal. i.e., $\mathbf{AB \neq BA}$ **sometimes.**

(ii) **Associative.** If **ABC** is defined, **ABC = (AB)C = A(BC)**

(iii) **Multiplication is distributive with respect to addition.** If **A,B and C** are matrices of order mxn, nxp and nxp respectively, **A(B+C) = AB+AC**

If **A, B and C** are matrices of order mxn, mxn and nxp respectively, **(A+B)C = AC+BC.**

(iv) **Existence of identity.** If **A** is a square matrix of order **n,** $\mathbf{I_n}$ is the identity matrix.

$$AI_n = A = I_n A$$

For the rectangular matrix **A** whose order is mxn $\mathbf{AI_n = A}$ **and** $\mathbf{I_m A = A}$

(v) **Existence of inverse.** If **A** is a square matrix of order n and **non singular**, there exists a square matrix **B** of order n such that $\mathbf{AB = I_n = BA}$

B is the inverse of **A.** Similarly, **A** is the inverse of **B.**

(vi) **AB = 0** (null matrix) does not imply **A = 0** or **B = 0** or both

(vii) If **A** is of order mxn, the null matrix **O** of order nxm gives $\mathbf{AO = O_{mxm}}$ and $\mathbf{OA = O_{nxn}}$ **AB = AC** does not imply B = C

(c) Transpose.

(i) The transpose of a matrix is the original matrix itself. $\mathbf{(A^1)^1 = A}$

(ii) The transpose of the sum of matrices is the sum of the transpose of the individual matrices. $\mathbf{(A+B)^1 = =A^1+B^1}$

(iii) $\mathbf{(KA)^1 = KA^1}$ where K is a scalar.

(iv) The transpose of the product of matrices conformable for multiplication is equal to the product of the transpose of the individual matrices taken in reverse order.
$\mathbf{(AB)^1 = B^1A^1, (ABC)^1 = C^1B^1A^1,\ldots..}$

DIFFERENCE BETWEEN MATRICES AND DETERMINENTS

MATRICES	DETERMINANTS
• Number of rows and number of columns can be equal or unequal.	• Number of rows and number of columns are equal.
• Elements are enclosed by brackets or parentheses or double bars.	• Elements are enclosed by pair of vertical lines
• A matrix has no numerical value.	• A determinant has a numerical value
• Matrices are arrangements. By interchanging two elements in a matrix a new matrix is obtained.	• Even after interchanging two elements in a determinant, the value may remain the same.

6. Determinants: With each square matrix A we can associate a determinant which is denoted by the symbol $|\mathbf{A}|$ or det. A or Δ**.**

For Example: $A = \begin{bmatrix} 1 & 5 \\ 6 & 9 \end{bmatrix} = |A| = \begin{bmatrix} 1 & 6 \\ 5 & 9 \end{bmatrix} = (-21)$

Example 1:

The value of a first order determinant is the single element itself.

e.g: $|5| = 5;\ |0| = 0;\ |-13| = -13$

Example 2: The value of a second order determinant:

e.g., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example 3: Find the determinant.

$\begin{vmatrix} 6 & 1 \\ 4 & 3 \end{vmatrix} = 6\times3 - 4\times1 = 14$ $\begin{vmatrix} 8 & 2 \\ -3 & -1 \end{vmatrix} = -8 - (-6) = -2$

Example 4: Find the value of a third order determinant: $\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix}$

Solution: Expansion by the first row:

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} - 2(-)\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$$

$= 3(4) - 2(-3) + 1(-1) = 17$

Example 5: Find the value of the determinant B = $\begin{vmatrix} 3 & -1 & 2 \\ 6 & 3 & 0 \\ 1 & 4 & 6 \end{vmatrix}$

Solution: Expansion by the third column.

$$B = 2\begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} - 6\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} = 2(17) - 0 - 6(14) = -50$$

Example 6: Find the value of the determinant A = $\begin{vmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ 7 & 2 & 1 \end{vmatrix}$

Solution:

$$B = 3\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} + 7\begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix} = 3(-5) - 4(-19) + 7(3) = 82$$

Properties of Determinant:

(i) The value of a determinant remains unchanged if rows are changed into columns.

This means $|A| = |A'|$

(ii) If any two adjacent rows (or columns) of a determinant are interchanged, the value of the determinant remains the same but the sign changes.

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(iii) If any two rows (or columns) of a determinant are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0; \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix} = 0$$

(iv) If the elements of a row (or column) are multiplied by a constant K, the value of the determinant gets multiplied by K.

$$\begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} = K\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & Ka_{13} \\ a_{21} & a_{22} & Ka_{23} \\ a_{31} & a_{33} & Ka_{33} \end{vmatrix}$$

(v) If every element of a row (or column) is zero, the value of the determinant is zero.

$$Ex: \begin{vmatrix} a_{11} & a_{12} \\ 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix} = 0$$

Cramer's Rule or Determinant Method:

Cramer's rule [named in honor of Swiss Mathematician] G. Cramer (1704-1752) is based on determinants. It is used for solving a system of linear simultaneous equations.

It matrix form, it is AX = C where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \; C = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

By Cramer's rule,

$$x = \frac{|A_x|}{|A|}, \qquad y = \frac{|A_y|}{|A|}, \qquad z = \frac{|A_z|}{|A|}$$

Example 7: Solve the following equations by Cramer's rule.

$3x + 2y = 8$

$5x - 3y = 7$

Solution: Given equations in matrix form: **AX = C**

Where, $A = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad C = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{vmatrix} 3 & 2 \\ 5 & -3 \end{vmatrix} = 3\times(-3) - 5\times 2 = -19$$

$$|A_x| = \begin{vmatrix} 8 & 2 \\ 7 & -3 \end{vmatrix} = 8\times(-3) - 7\times 2 = -38 \text{ and}$$

$$|A_y| = \begin{vmatrix} 3 & 8 \\ 5 & 7 \end{vmatrix} = 3\times 7 - 5\times 8 = -19$$

By Cramer's rule, $x = \frac{|A_x|}{|A|} = \frac{-38}{-19} = 2$

and $y = \frac{|A_y|}{|A|} = \frac{-19}{-19} = 1$

Example 8: Use determinants and solve.

$$\frac{1}{a} + \frac{2}{b} = 4$$

$$\frac{3}{a} - \frac{1}{b} = 5$$

Solution: Let $x_1 = \frac{1}{a}$ and $x_2 = \frac{1}{b}$

The given equations become,

$x_1 + 2x_2 = 4$

$3x_1 - x_2 = 5$

i.e., AX = C where $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1\times(-1) - 3\times 2 = -7$$

$$|A_x| = \begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix} = 4\times(-1) - 5\times 2 = -14 \quad \text{and}$$

$$|A_y| = \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} = 1\times 5 - 3\times 4 = -7$$

By Cramer's rule,

$$x = \frac{|A_x|}{|A|} = \frac{-14}{-7} = 2$$

and $$y = \frac{|A_y|}{|A|} = \frac{-7}{-7} = 1$$

$$\therefore \qquad \frac{1}{a} = 2 \text{ or } a = \frac{1}{2} \qquad \text{and } \frac{1}{b} = 1 \text{ or } b = 1$$

Example 9: Solve the following system of simultaneous equations by Cramer's Rule:

$2x + 3y + 3z = 22$

$x - y + z = 4$

$4x + 2y - z = 9$

Solution: Given: AX = C where $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $C = \begin{bmatrix} 22 \\ 4 \\ 9 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= 2\times(-1)\times(-1) - 2\times 1 - 3\times 1\times(-1) - 4\times 1 + 3\times 1\times 2 - 4\times(-1) = 31$$

$$|A_x| = \begin{vmatrix} 22 & 3 & 3 \\ 4 & -1 & 1 \\ 9 & 2 & -1 \end{vmatrix} = 68 \quad \text{and}$$

$$|A_y| = \begin{vmatrix} 2 & 22 & 3 \\ 1 & 4 & 1 \\ 4 & 9 & -1 \end{vmatrix} = 63$$

$$|A_z| = \begin{vmatrix} 2 & 3 & 22 \\ 1 & -1 & 4 \\ 4 & 2 & 9 \end{vmatrix} = 119$$

By Cramer's Rule,

$$x = \frac{|A_x|}{|A|} = \frac{68}{31}$$

and $$y = \frac{|A_y|}{|A|} = \frac{63}{31}$$

$$z = \frac{|A_z|}{|A|} = \frac{119}{31}$$

Minor and Cofactor: Minor of an element of a determinant is the value of the lower order determinant obtained by deleting the row and the column which contain the element. The minor of the element a_{ij} is denoted by M_{ij}.

The element of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ together with their minors and cofactors are found below:

Element	**Minor**	**Cofactor**
a_1	$\lvert b_2 \rvert = b_2$	b_2
b_1	$\lvert a_2 \rvert = a_2$	$-a_2$
a_2	$\lvert b_1 \rvert = b_1$	$-b_1$
b_2	$\lvert a_1 \rvert = a_1$	a_1

The minors of the first and the last elements are cofactors also. For others, the sign is changed.

Example 10: Find the minors and Cofactors of all the elements of $\begin{vmatrix} 3 & 2 \\ 5 & 0 \end{vmatrix}$

Solution:

Element	**Minor**	**Cofactor**
3	0	0
2	5	-5
5	2	-2
0	3	3

Example 11: Find the minors and Cofactors of all the elements of $\begin{vmatrix} 1 & 5 \\ -3 & 0 \end{vmatrix}$

Solution:

Element	**Minor**	**Cofactor**
1	0	0
5	-3	3
-3	5	-5
0	1	1

Example 12: Find the minors and Cofactors of all the elements of $\begin{vmatrix} 5 & 6 & 7 \\ 0 & 1 & -3 \\ -2 & 4 & 9 \end{vmatrix}$

Solution:

Element	Minor	Cofactor
5	$\begin{vmatrix} 1 & -3 \\ 4 & 9 \end{vmatrix} = 1\times 9 - 4\times(-3) = 21$	21
6	$\begin{vmatrix} 0 & -3 \\ -2 & 9 \end{vmatrix} = 0\times 9 - (-2)\times(-3) = -6$	6
7	$\begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} = 0\times 4 - (-2)\times 1 = 2$	2
0	$\begin{vmatrix} 6 & 7 \\ 4 & 9 \end{vmatrix} = 6\times 9 - 4\times 7 = 26$	-26
1	$\begin{vmatrix} 5 & 7 \\ -2 & 9 \end{vmatrix} = 5\times 9 - (-2)\times 7 = 59$	59
-3	$\begin{vmatrix} 5 & 6 \\ -2 & 4 \end{vmatrix} = 5\times 4 - (-2)\times 6 = 32$	-32
-2	$\begin{vmatrix} 6 & 7 \\ 1 & -3 \end{vmatrix} = 6\times(-3) - 1\times 7 = -25$	-25
4	$\begin{vmatrix} 5 & 7 \\ 0 & -3 \end{vmatrix} = 5\times(-3) - 0\times 7 = -15$	15
9	$\begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix} = 5\times 1 - 0\times 6 = 5$	5

Ad joint Matrix: The ad joint matrix of a square matrix A is denoted by **adjA.** It is obtained from A after writing the transpose of A (ie.A^1) first and then replacing every element in A^1 by its cofactor. Ad joint of A is the A is the transpose of the cofactor matrix of A also.

$$Adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} When\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Once of the important properties of adj A is

$$\mathbf{A.(adj\,A) = |A|\,I = (adj\,A).A}$$

Example 13: Find the ad joint of $A = \begin{vmatrix} 1 & 5 \\ -3 & 0 \end{vmatrix}$

Solution: $A^1 = \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}$ and $\text{adj}A = \begin{bmatrix} 0 & -5 \\ 3 & 1 \end{bmatrix}$

Example 14: Find the ad joint of $A = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

Solution: $A^1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $\text{adj}A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Example 15: If $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 4 & -1 \\ 1 & -8 & -3 \end{bmatrix}$, show that A. (Adj A) = $|A|\ I_3$.

Solution: $A^1 = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 4 & -8 \\ -1 & -1 & -3 \end{bmatrix}$; $\operatorname{Adj} A = \begin{bmatrix} -20 & 8 & 4 \\ 5 & -5 & 0 \\ -20 & 16 & 8 \end{bmatrix}$

$$|A| = 2\times(-20) + 0\times 5 + (-1)(-20) = -20$$

$$A.(\operatorname{Adj} A) = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 4 & -1 \\ 1 & -8 & -3 \end{bmatrix} \begin{bmatrix} -20 & 8 & 4 \\ 5 & -5 & 0 \\ -20 & 16 & 8 \end{bmatrix} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix}$$

$$= -20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

Example 16: Find the (i) Minor (ii) Cofactor (iii) Adjoint and (iv) determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

(i) Replacing every element by its minor: $\begin{bmatrix} -1 & -1 & -1 \\ -3 & 1 & 5 \\ 7 & 5 & 13 \end{bmatrix}$

(ii) (Minors with appropriate signs) Cof: $\begin{bmatrix} -1 & 1 & -1 \\ 3 & 1 & -5 \\ 7 & -5 & 13 \end{bmatrix}$

Adjoint = $\begin{bmatrix} -1 & 3 & -7 \\ 1 & 1 & -5 \\ -1 & -5 & 13 \end{bmatrix}$

(iii) $|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 3\times 5 - 6 - 1\times 2 - 3 + 2\times 4 - 5$

$$= 3\times(-1) - 1\times(-1) + 2\times(-1) = -4$$

Matrix Operations – II

Inverse of a Matrix: The inverse of a matrix A is denoted by A^{-1}.A^{-1} is unique. A^{-1} exists if and only if A is non – singular. i.e., $|A| \neq 0$. Further, $(A^{-1})^{-1}$ = A. i.e., inverse of the inverse is the original matrix itself.

$$A^{-1} = \frac{1}{|A|}(\text{adj}\,A)$$

Remarks:

- **(i)** A rectangular matrix does not possess an inverse.
- **(ii)** $A^{-1}.A = I = A.A^{-1}$
- **(iii)** $(A.B)^{-1} = B^{-1}.\,A^{-1}$
- **(iv)** $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

Example 17: Find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\therefore\ |A| = ad - bc$

$A^1 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $\text{adj}\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 18: Find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$. $\therefore\ |A| = 2 \times 5 - 3 \times 2 = 4$

$A^1 = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$, $\text{adj}\,A = \begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$ and $A^{-1} = \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1.25 & -0.50 \\ -0.75 & 0.50 \end{bmatrix}$

Example 19: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

Solution: $A^1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -6 \\ -1 & 5 & -7 \end{bmatrix}$

Element	Minor	Cofactor
1	$\begin{vmatrix} 4 & -6 \\ 5 & -7 \end{vmatrix} = 4\times(-7) - 5\times(-6) = 2$	2
3	$\begin{vmatrix} 0 & -6 \\ -1 & -7 \end{vmatrix} = 0\times(-7) - (-1)\times(-6) = -6$	6
0	$\begin{vmatrix} 0 & 4 \\ -1 & 5 \end{vmatrix} = 0\times 5 - (-1)\times 4 = 4$	4
0	$\begin{vmatrix} 3 & 0 \\ 5 & -7 \end{vmatrix} = 3\times(-7) - 5\times 0 = -21$	21
4	$\begin{vmatrix} 1 & 0 \\ -1 & -7 \end{vmatrix} = 1\times(-7) - (-1)\times 0 = -7$	-7
-6	$\begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} = 1\times 5 - (-1)\times 3 = 8$	-8
-1	$\begin{vmatrix} 3 & 0 \\ 4 & -6 \end{vmatrix} = 3\times(-6) - 4\times 0 = -18$	-18
5	$\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = 1\times(-6) - 0\times 0 = -6$	6
-7	$\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 1\times 4 - 0\times 3 = 4$	4

Based on the first row, $|A| = 1\times 2 + 0\times 21 + (-1)(-18) = 20$

$$\text{adj}\,A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}; \qquad A^{-1} = \frac{1}{|A|}(\text{adj}\,A) = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Example 20: If 10A - 50I = 0 and $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ find A^{-1}.

Solution: Consider 10A – 50I = 0. Post multiplying by A^{-1}, $10A\,.A^{-1} - 50I\,.A^{-1} = 0$.

i.e., $10I - 50A^{-1} = 0$ $\qquad \because A.A^{-1} = I;\ I.A^{-1} = A^{-1}$

$\therefore$ $\quad -50A^{-1} = -10I$

$$A^{-1} = 0.2I = 0.2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$A^{-1}.A = I = A.A^{-1}$ and hence A^{-1} is correct.

Example 21: Show that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$ where I is the identity matrix and 0 denotes the zero matrix. Hence find the inverse of A.

Solution: $A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Post multiplying both sides of $A^2 - 4A - 5I = 0$ by A^{-1}

$$A^2. A^{-1} - 4A . A^{-1} - 5I . A^{-1} = 0.A^{-1}$$

$$A - 4I - 5A^{-1} = 0$$

$$-5 A^{-1} = - (A - 4I) \text{ and}$$

$$A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left[\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right]$$

$$= \frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.4 & 0.4 \\ 0.4 & -0.6 & 0.4 \\ 0.4 & 0.4 & -0.6 \end{pmatrix}$$

$A^{-1}. A = I = A.A^{-1}$ and hence A^{-1} is correct.

Example 22: Show that the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 6A^2 + 9A - 4I = 0$. Hence deduce the value of A^{-1}.

Solution: $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = A.A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$6A^2 = 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$9A = 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$4I = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\therefore \quad A^3 - 6A^2 + 9A - 4I =$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 0$$

Post multiplying $A^3 - 6A^2 + 9A - 4I = 0$ by A^{-1},

$$A^3.A^{-1} - 6A^2.A^{-1} + 9A.A^{-1} - 4I.A^{-1} = 0A^{-1}$$

$\therefore \quad A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore \quad -4A^{-1} = -(A^2 - 6A + 9I)$

$$\therefore \quad A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 & -0.25 \\ 0.25 & 0.75 & 0.25 \\ -0.25 & 0.25 & 0.75 \end{pmatrix}$$

$A^{-1}.A = I = A.A^{-1}$ and hence A^{-1} is correct.

Solving Simultaneous Linear Equation by Inverse Matrix:

The given system or equations is to be expressed in the matrix form AX = C where A – coefficients matrix, X – unknowns column vector and and C – constants column vector.

From A, A^{-1} is to be found out. C is to be premultiplied by A^{-1}. Then, from $X = A^{-1}C$, the values of the unknowns are to be determined.

The correctness of the answer can be verified as seen under Cramer's Rule.

Example 23: Solve the equation by matrix method:

$3x+2y = 14$

$3x+3y = 18$

Solution: Given equations in matrix form:

AX = C where the coefficients matrix $A = \begin{pmatrix} 3 & 2 \\ 3 & 3 \end{pmatrix}$, the unknowns column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and the constants column vector, $C = \begin{pmatrix} 14 \\ 18 \end{pmatrix}$

$$\therefore \quad A' = \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix}, \ |A| = 3\times 3 - 3\times 2 = 3, \ \text{adj}\,A = \begin{pmatrix} 3 & -2 \\ -3 & 3 \end{pmatrix} \text{ and}$$

$$A' = \frac{1}{A'}(\text{adj}\,A) = \frac{1}{3}\begin{pmatrix} 3 & -2 \\ -3 & 3 \end{pmatrix}$$

$$X = A^{-1}C \text{ gives } \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & -2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 14 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$\therefore \quad x = 2 \quad \text{and} \quad y = 4$

Example 24: Using matrix inversion method, solve the following system of equation:

$2x - y + 3z = 1$

$x + y + z = 2$

$x - y + z = 4$

Solution: Given equations in matrix form:

$$\therefore \quad A^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}; \quad \text{adj}\,A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$|A| = 2\times 2 + 1\times 0 + 3\times(-2) = -2$$

$$A^{-1} = \frac{1}{|A|}\ (\text{adj}\,A) = \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0.5 & -0.5 \\ 1 & -0.5 & -1.5 \end{bmatrix}$$

$$X = A^{-1}C \quad \text{gives} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0.5 & -0.5 \\ 1 & -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -6 \end{bmatrix}$$

$\therefore$ x = 9, y = -1 and z = -6

UNIT - II

SERIES

Sequence:

A set of numbers arranged in a specific order is a sequence. Each number in the sequence is called a term. Generally sequence is written as;

$u_1, u_2, u_3, \ldots..$

u_1, u_2, and u_3 are the first, second and third terms of the sequence respectively. The following are some of the examples of sequences.

(i) 1, 5, 9, ………………………
(ii) 2, 6, 18, ……………………..
(iii) -7, 49, -343, ………………..

Series:

When the successive terms of a sequence are connected by plus or minus signs, the sequence is called a series.

For examples,

(i) $u_1+ u_2+ u_3+ \ldots\ldots\ldots.$
(ii) 1+ 5+ 9 +………….
(iii) -7+49-343+……….

A series is called a finite series if it contains finite number of terms. It is called an infinite series if it contains infinite number of terms. Two important types of series, namely,

(i) Arithmetic series or Arithmetic Progression (A.P.)
(ii) Geometric series or Geometric Progression (G.P.) and
(iii) Harmonic series or Harmonic Progression (H.P.)

Arithmetic Progression (A.P):

If the successive terms increase and decrease by a constant (quantity), the series is called Arithmetic Progression. That constant quantity is called the common difference.

e.g. 2, 7, 12, ……is an A.P. 2 is the first term while 5(7-2=12-7) is the common difference. The 'a' is the first term and 'd' is the common difference respectively.

a+ a+d, a+2d, …… is the standard form of an A.P.

When d is positive, the series is an increasing A.P. The series is a decreasing A.P. when d is negative.

In this, n^{th} term, denoted by t_n, is the general term is defined by,

$$t_n = a + (n-1)\,d$$

Example 1: The fourth and seventh terms of an A.P. are 3 and 36. Find the A.P. and its 15th terms.

Solution:

4^{th} term $a + 3d = 3 \longrightarrow (1)$

7^{th} term $a + 6d = 36 \longrightarrow (2)$

(-) (-) (-)

$-3d = -33$

$d = \frac{-33}{-3}$

$d = 11$

By substituting in (1),

$a + 3d = 3$

$a + 3(11) = 3$

$a + 33 = 3$

$a = 3 - 33$

$a = -30$

Hence, the A.P. is -30, -30+11, -30+2(11)

That is, -30, -19, -8, …….. is the A.P.

The 15th term, $t_{15} = a + 14d$

$= -30 + 14(11)$

$= 124$

Example 2: In an Arithmetic series, the seventh and the ninth terms are respectively 16 and 20. Find the nth term.

Solution:

t_7: $a + 6d = 16 \longrightarrow (1)$

t_9: $a + 8d = 20 \longrightarrow (2)$

(-) (-) (-)

$-2d = -4$

$d = \frac{-4}{-2}$

$d = 2$

By substituting in (1), $a + 6d = 16$

$a + 6(2) = 16$

$a + 12 = 16$

$a = 16 - 12$

$\boxed{a = 4}$

Hence, the n^{th} term, $t_n = a + (n-1)d$

$= 4 + (n-1)2$

$= 4 + 2n - 2$

$\boxed{t_n = 2 + 2n}$

Example 3: The sum of three numbers in Arithmetic Progression is 24 and their product is 440. Find the numbers.

Solution: Let the three numbers be a-d, a, a+d.

Given : $(a-d) + a + (a+d) = 24$

$a - \not{d} + a + a + \not{d} = 24$

$3a = 24$

$\therefore \quad a = 24/3$

$\boxed{a = 8}$

Also given : $(a-d).\ a.\ (a+d) = 440$

By (1), this becomes, $(8-d).\ 8.\ (8+d) = 440$

$(8-d)(8+d) = 440/8$

$(8-d)(8+d) = 55$

$64 - d^2 = 55$

$d^2 = 64 - 55$

$d^2 = 9$

$d = \pm\sqrt{9}$

$d = \pm 3$

Case 1: a=8 and d=3. The three numbers are 5, 8, and 11.

Case 2: a=8 and d= -3. The three numbers are 11, 8, and 5.

Example 4: Find three numbers in A.P. whose sum is 12 and the sum of whose cubes is 408.

Solution: Let the three numbers be a-d, a and a+d

Given : $(a-d) + a + (a+d) = 12$

$a - \not{d} + a + a + \not{d} = 12$

$3a = 12$

$\therefore \quad a = 12/3$

$\boxed{a = 4}$

Also given $(a-d)^3 + a^3 + (a+d)^3 = 408$

By (1), this becomes, $(4-d)^3.\ 4^3.\ (4+d)^3 = 408$

$$\therefore \quad (a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$4^3-3.\cancel{(4)^2}d+3.4d^2-\cancel{d^3}+4^3+4^3+3.\cancel{(4)^2}d+3.4d^2+\cancel{d^3} = 408$$

$$64+12d^2+64+64+12d^2 = 408$$

$$192+24d^2 = 408$$

$$24d^2 = 408-192$$

$$24d^2 = 216$$

$$d^2 = 216/24$$

$$d^2 = 9$$

$$d = \pm\sqrt{9}$$

$$d = \pm 3$$

Case 1: a=4 and d=3. The three numbers are 1, 4, and 7.
Case 2: a=4 and d= -3. The three numbers are 7, 4, and 1.

Example 5:

If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. Prove that a^2, b^2, c^2 are also in A.P.

Solution: If a^2, b^2, c^2 are to be in A.P., $b^2-a^2 = c^2-b^2$ is to be true.

Given: $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

i.e., $$\frac{(b+c)-(c+a)}{(c+a)(b+c)}=\frac{(c+a)-(a+b)}{(a+b)(c+a)}$$

$$\frac{b+c-c-a}{(c+a)(b+c)}=\frac{c+a-a-b}{(a+b)(c+a)}$$

i.e., $$\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$$

Multiplying both sides by (c+a) (b+c) (a+b)

$$=\frac{b-a}{(c+a)(b+c)}\times(c+a)\ (b+c)\ (a+b)=\frac{c-b}{(a+b)(c+a)}\times(c+a)\ (b+c)\ (a+b)$$

= (b-a) (a+b) = (c-b) (b+c

Hence, $b^2 - a^2 = c^2 - b^2$ are in A.P.

Formula for the sum of the first n terms of an A.P.:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \forall n \in N$$

Example 6: Find the sum of the following series.

i) 8+13+18+….. Up to 23 terms.

ii) $3\frac{1}{4} + 5\frac{1}{2} + 7\frac{3}{4} + \cdots 23\frac{1}{2}$

iii) 40+36+32+……+0

Solution: (i) The given series is an A.P. where, common difference, d= 13-8 = 5

a is first term, a = 8, n=23

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$= \frac{23}{2}\left[2 \times 8 + (23-1)5\right]$$

= 11.5 [16+ (22) 5]

= 11.5 [16+110]

= 11.5 [126]

= 1449

(ii) The given series is an A.P. where, common difference, $d = 5\frac{1}{2} - 3\frac{1}{4} = 7\frac{3}{4} - 5\frac{1}{2} = 2\frac{1}{4} = 2.25$

a is first term, $a = 3\frac{1}{4} = 3.25$ Assuming $23\frac{1}{2}$ as nth term,

t_n= a+ (n-1) d = 23.5

i.e., 3.25 + (n-1) 2.25 = 23.5

(n -1) 2.25 = 23.5 – 3.25 = 20.25

(n - 1) = 20.25 / 2.25 = 9

i.e., n = 9+1 = 10

Required sum is given by $S_n = \frac{n}{2}(a+1)$

$$= \frac{10}{2}(3.25 + 23.5) = 133.75$$

(iii) The given series is an A.P. where, common difference, d= 36-40 =32 – 36 = -4 a is first term, a = 40. Assuming 0 as nth term,

$$t_n = a + (n-1)d = 0$$

i.e., $40 + (n-1)(-4) = 0$

$(n-1)(-4) = 0 - 40 = -40$

$(n-1) = -40 / -4 = 10$

ie., $n = 10 + 1 = 11$

Required sum is given by $S_n = \frac{n}{2}(a+1)$

$$= \frac{11}{2}(40+0) = 220$$

Example 7: If S_1, S_2, and S_3 be respectively the sum of the first n, 2n and 3n terms of an A.P., Prove that $S_3 = 3(S_2 - S_1)$.

Solution: By substituting the values of n in

$$S_n = \frac{n}{2}[2a + (n-1)d], \text{ we get}$$

$$S_3 = \frac{3n}{2}[2a + (3n-1)d]$$

$$= 3n\left[\frac{2a}{2} + \left(\frac{3}{2}n - \frac{1}{2}\right)d\right]$$

$$= 3na + \frac{9}{2}n^2 d - \frac{3}{2}nd \qquad \text{…………… (1)}$$

$$S_2 = \frac{2n}{2}[2a + (2n-1)d]$$

$$= 2n\left[\frac{2a}{2} + \left(\frac{2}{2}n - \frac{1}{2}\right)d\right]$$

$$= 2na + \frac{4}{2}n^2 d - \frac{2}{2}nd$$

$$= 2na + 2n^2 d - nd \qquad \text{……………..…. (2)}$$

$$S_1 = \frac{1n}{2}[2a + (1n-1)d]$$

$$=1n\left[\frac{2a}{2}+\left(\frac{1}{2}n-\frac{1}{2}\right)d\right]$$

$$=na+\frac{1}{2}n^2d-\frac{1}{2}nd \qquad \ldots\ldots\ldots\ldots\ldots\ (3)$$

(2) – (3) is mutipiled by 3

$$3\ (S_2\text{-}S_1) = 3\ (\,2na+2n^2d-nd - na+\frac{1}{2}n^2d-\frac{1}{2}nd\,)$$

$$= 3\ (na+\frac{3}{2}n^2d-\frac{1}{2}nd\,)$$

$$= 3na + \frac{9}{2}n^2d-\frac{3}{2}nd$$

S_3 by (1).

Example 8: The first and the last terms of an A.P. are -4 and 146 and the sum of the A.P. is 7171. Find the number of terms in the A.P. and the common difference.

Solution: We know that $S_n = \frac{n}{2}(a+1)$

Given: a = -4, 1 = 146 and S_n = 7171

i.e., $\frac{n}{2}(-4+146) = 7171$

$$n\left[\frac{-4}{2}+\frac{146}{2}\right] = 7171$$

n[-2+73] = 7171

71n = 7171

n = 7171 / 71

∴ No of terms in the A.P. n = 101

By substituting the known values in t_n = a+ (n-1) d

146 = -4 + (101-1) d

100d = -4 – 146

100d = 150

d = 150 / 100

d = 1.5

Example 9: Sum of the first n terms of a series is $3n^2 + 6n$. Show that it is an A.P. Which term of the series is 105?

Solution: Given: $S_n = 3n^2 + 6n$

$\therefore \quad S_{n-1} = 3(n-1)^2 + 6n$

$= 3(n^2 - 2n-+1) + 6n - 6$

$= 3n^2 - \not{6}n + 3 + \not{6}n - 6$

$= 3n^2 - 3$

$t_n = S_n - S_{n-1}$

$= \not{3}n^2 + 6n - \not{3}n^2 - 3$

$= 6n - 3$

By substituting n=1, 2, and 3, we get $t_1 = 9$, $t_2 = 15$ and $t_3 = 21$. As $t_2 - t_1 = t_3 - t_2$, the series is an A.P.

Further, a= 9 and d = 6

Let $t_n = 105$

i.e., $a + (n-1)\,6 = 105$

$9 + (n-1)\,6 = 105$

$(n-1)\,6 = 105 - 9 = 96$

$(n-1) = 96 / 6 = 16$

$n = 16 + 1 = 17$

Hence, 17th term of the given A.P. is 105.

Example 10: Mr. X arranges to pay off a debt of Rs.9,600.00 in 48 monthly instalments which from an A.P. When 40 of these instalments are paid, Mr. X becomes insolvent and his creditor finds that Rs. 2,400.00 still remains unpaid. Find the values of the first three instalments paid by Mr. X. Ignore interest.

Solution: As given $S_{48} = 9600$

i.e., $\frac{48}{2}[2a + 47d] = 9600$

$24\,[2a + 47d] = 9600$

$2a + 47d = 9600 / 24$

$2a + 47d = 400 \quad \ldots\ldots\ldots\ldots (1)$

Also given, $S_{40} = 9600 - 2400 = 7200$

i.e., $\frac{40}{2}[2a+39d] = 7200$

$20\,[2a + 39d] = 7200$

$2a + 39d = 7200 / 20$

$2a + 39d = 360$ ……………. (2)

By subtract (1) – (2) $\cancel{2a}+47d - \cancel{2a}+39d = 400 - 360$

$8d = 40$

$d = 40 / 8$

$d = 5$

By Substituting in (2), $2a + 39d = 360$

$2a + 39 \times 5 = 360$

$2a + 195 = 360$

$2a = 360 – 195 = 165$

$a = 165 / 2$

$a = 82.5$

The first three instalments are a, a+d, 2a+d. i.e., Rs. 82.50, Rs. 87.50 and Rs. 92.50.

Example 11: A man borrows Rs.1200 at the total interest of Rs. 168. He repays the entire amount in 12 instalments each instalment being less than the preceding one by Rs.20. Find the first instalment.

Solution: Given: $S_{12} = 1200 + 168 = 1368$; $n = 12$ and $d = -20$

To find a consider $S_n = \frac{n}{2}[2a+(n-1)d]$

i.e., $1368 = \frac{12}{2}[2a+(12-1)\text{ - }20]$

$1368 = \frac{12}{2}[2a+11\times(-20)]$

$1368 = 6[2a - 220]$

$\therefore$ $2a – 220 = 1368 / 6 = 228$

$2a = 228 + 220 = 448 / 2$

$a = 224$ Hence, the first instalment = Rs.224.

Example 12: A person buys National Savings Certificates of value exceeding the last year's purchase by Rs.200. After 10 years, he finds that the total value of the certificates purchased by him is Rs.10, 500. Find the value of the certificates purchased by him (i) in the first year (ii) in the seventh year.

Solution: Given: d = 200; n= 10 and S_n =10500

$$S_n = 10500$$

i.e., $$\frac{n}{2}\left[2a+(n-1)d\right]=10500$$

$$\frac{10}{2}\left[2a+(10-1)200\right]=10500$$

$$\frac{10}{2}\left[2a+(9)200\right]=10500$$

$$5\left[2a+1800\right]=10500$$

$$2a+1800=10500 / 5$$

$$2a+1800=2100$$

$$2a=2100 - 1800$$

$$a=300 / 2$$

$$a=150$$

Seventh year, $t_7 = a+(n-1)\,d$

$$=150+ (7-1)\,200 =150+6\,(200)$$

$$t_7=150+1200 = 1350$$

(i) The value of the certificates purchased in the first year, a= Rs. 150.
(ii) The value of the certificates purchased in the seventh year, = Rs. 1,350.

Example 13: The first term of an Arithmetic Series is 5. The number of terms is 15 and their sum is 390. Find the common difference and the middle term.

Solution: Given: a=15, n=15 and S_n=390. To find d and t_8, consider

$$\frac{n}{2}\left[2a+(n-1)d\right]= S_n$$

$$\frac{15}{2}\left[2(5)+(15-1)d\right]=390$$

$$\frac{15}{2}\left[10+(14)d\right]=390$$

$$[10+14d]=390\times\frac{2}{15}$$

$$[10+14d]=390\times\frac{2}{15}$$

$$[10+14d]=\frac{780}{15}$$

$$[10+14d]=52$$

$$14d=52 - 10 = 42$$

$$d=42/14$$

$\therefore$ Common difference $d= 3$

The middle term, $t_n = a+ (n-1)\ d$

$t_8 = 5+ (8-1)\ 3$

$= 5+7\ (3)$

$= 5+21$

$t_8 = 5+21$

$t_8 = 26$

Example 14: Two posts are offered to a person. The first carries a starting salary of Rs.1000 per month and an annual increment of Rs.40. The second carries a starting salary of Rs.800 and an annual increment of Rs.50. Assuming that he works for 25 years, which of the two is more profitable?

Solution: For post I, first year salary (a) is 1000x12 = 12000, total increment for one year (d) is 40x12=480 and n=25 (years).

Total salary, $S_n = \frac{n}{2}[2a+(n-1)d]$

$$= \frac{25}{2}[2(12000)+(25-1)480]$$

$$= \frac{25}{2}[24000+24\ (480)]$$

$$= \frac{25}{2}[24000+11520]$$

$$= \frac{25}{2}[35520] = \text{Rs. } 4{,}44{,}000$$

For post II, first year salary (a) is 800x12 = 9600, total increment for one year (d) is 50x12=600 and n=25 (years).

$$\text{Total salary, } S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{25}{2}\left[2(9600)+(25-1)600\right]$$

$$= \frac{25}{2}\left[19200+24\,(600)\right]$$

$$= \frac{25}{2}\left[19200+14400\right]$$

$$= \frac{25}{2}\left[33600\right] = \text{Rs. } 4{,}20{,}000$$

The first post is more profitable as the total salary is more for that post.

FORMULA FOR THE SUM OF NATURAL NUMBERS

1,2,3,… are called the natural numbers. 1,2,3,….n are the first n natural numbers. They are in A.P. with a=1, d=1, 1=n and n=n.

$$S_n = \frac{n(n+1)}{2}$$

Example 15: Find the sum of the first 100 natural numbers.

Solution:

By substituting n=100 in the formula, the sum of the first 100 natural number $= \frac{100\times 101}{2} = 5{,}050$

Example 16: Find the sum of all natural numbers between 100 and 1000 which are divisible by 13.

Solution: Between 100 and 1000, 104 and 988 are the first and last numbers divisible by 13. Hence the required sum is 104+117+130+…+988.

Here, a=104, d=13 and 1=988,

To find the n, consider t_n=988

i.e., a+ (n-1)d =988

104+ (n-1) 13 =988

$$(n-1)13 = 988 - 104 = 884$$

$$n - 1 = 884 / 13 = 68$$

$$n - 1 = 68$$

$$n = 68+1 = 69$$

$$\therefore \quad 104+117+130+\ldots..+988 = \frac{n}{2}(a+1)$$

$$= \frac{69}{2}(104+988)$$

$$= \frac{69}{2}(1092) = 37,674$$

Example 17: Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Solution: Sum of integers from 1 to 100 that are divisible by 2 is 2+4+6+…+100. As a=2, 1=100 and n=50, their sum,

$$2+4+6+\ldots+100 = \frac{n}{2}(a+1)$$

$$= \frac{50}{2}(2+100) = 2550 \qquad \ldots\ldots\ldots (1)$$

Sum of integers from 1 to 100 that are divisible by 5 is 5+10+15+…+100. As a=5, 1=100 and n=20, their sum,

$$5+10+15+\ldots+100 = \frac{n}{2}(a+1)$$

$$= \frac{20}{2}(5+100) = 1050 \qquad \ldots\ldots\ldots (2)$$

The numbers which are divisible by both 2 and 5 (i.e., 10) have been added in both the series. Their sum is to be subtracted once.

$$10+20+30+\ldots+100 = \frac{n}{2}(a+1)$$

$$= \frac{10}{2}(10+100) = 550 \qquad \ldots\ldots\ldots (3)$$

(1) + (2) - (3) gives the sum of integers from 1 to 100 that are divisible by 2 or 5 as

2550+1050-550 = 3050.

Arithmetic Means (A.M)

If a series of n terms is in A.P., the first and the last terms are called the extremes and the intermediate terms are called arithmetic means.

To insert are A.M $(x_1, x_2, ..., x_n)$ between two given numbers

$$A = \left(\frac{a+b}{2}\right) \qquad d = \left(\frac{b-a}{n+1}\right)$$

$$\text{The first A.M., } x_1 = a + \left(\frac{b-a}{n+1}\right)$$

$$\text{The second A.M., } x_1 = a + \left(\frac{b-a}{n+1}\right)$$

$$\vdots \qquad\qquad \vdots$$

$$\text{The } n^{th} (\text{last}) \text{A.M.}, x_n = a + n\left(\frac{b-a}{n+1}\right)$$

Example 18: Insert one A.M. between 70 and 50

Solution: Given: a = 70 and b = 50

Required to find $A = \frac{a+b}{2} = \frac{70+50}{2} = 60$

Example 19: Insert 5 A.M.'s between 23 and 47

Solution: Given: n = 5, a = 23 and b = 47

$\therefore$ Common difference, $d = \frac{b-a}{n+1} = \frac{47-23}{5+1} = 4$

The five A.M's between 23 and 47 are

27, 31, 35, 39 and 43.

Geometric Progression (G.P.)

If the successive terms increase or decrease by a constant factor, the series is called Geometric Progression (G.P). That constant factor is called the common ratio and is denoted by r.

e.g. 7, 42, 252, is a G.P. 7 is called the first term while $\frac{42}{7} = \frac{252}{42} = 6$ is called the common ratio (constant factor). The alphabets a and r are used to denote the first term and the common ratio respectively.

a, ar, ar^2,….. is the standard form of a G.P.

When $r > 1$, the series is an increasing G.P. The series is a decreasing G.P. when $r < 1$.

In this series, n^{th} term, denoted by $t_n = ar^{n-1}$

Example 20: If the third and the seventh terms of a G.P. are 2 and 1/8, find the G.P. and its tenth term.

Solution: Given: t_3, $ar^2 = 2$ …….. (1)

t_7, $ar^6 = 1/8$ …….. (2)

$\therefore$ $(2) \div (1)$, $r^4 = 1/16 = (1/2)^4$

$\therefore$ $r = \pm 1/2$

Case I: By substituting $r = 1/2$ in (1)

$a(1/2)^2 = 2$

$a = 8$

The G.P. is 8, 4, 2,…. and 10^{th} term, $t_{10} = ar^9 = 8\ (1/2)^9 = 1/64$.

Case II: By substituting $r = -1/2$ in (1),

$a\ (-1/2)^2 = 2$

$a = 8$

The G.P. is 8, -4, 2,…. and 10^{th} term, $t_{10} = ar^9 = 8\ (-1/2)^9 = -1/64$.

Example 21: Find the number of terms in geometric series 0.03+0.06+0.12+ …. + 1.92.

Solution: We find a = -0.03; $r = \frac{0.06}{0.03} = \frac{0.12}{0.06} = 2$. Let 1.92 be the n^{th} term.

$\therefore$ $t_n = 1.92$

i.e., $ar^{n-1} = 1.92$

$(-0.03)\ (2^{n-1}) = 1.92$

$2^{n-1} = 1.92 / 0.03 = 64 = 2^6$

Equating the powers, $n-1=6$

$n = 7$

Number of terms in the geometric series = 7.

Example 22: the sum of 3 numbers in G.P. is 35 and their product is 1000. Find the numbers.

Solution: Let the three numbers be $\frac{a}{r}$, a and ar.

Given: $\frac{a}{r} + a + ar = 3$ ……………….. (1)

$\frac{a}{r} . a . ar = 3$ ……………..… (2)

$\therefore$ $a^3 = 1000 = 10^3$

$\therefore$ $a = 10$

Substituting in (1), $\frac{10}{r} + 10 + 10r = 35$

Multiplying by r, $10 + 10r + 10r^2 = 35$

Transposing 35r, $10 – 25r + 10r^2 = 0$

i.e., $10 – 20r – 5r + 10r^2 = 0$

$10 (1-2r) – 5r (1-2r) = 0$

$(1-2r) (10-5r) = 0$

$1 – 5r$ or $10 – 5r = 0$

$\therefore$ r = ½ or r = 2

Case I: a = 10 and r = 1/2 -

The three numbers are 20, 10, 5

Case I: a = 10 and r = 2 -

The three numbers are 5, 10, 20.

Example 23: Find the four numbers forming a geometric progression if the first number exceeds the second by 36 and the third number is greater than the fourth by 4.

Solution: Let the four numbers be a, b, c, d.

Given: a = b+36; c = d+4.

$\therefore$ The numbers becomes b+36, b, d+4, d.

As they are in G.P., $\frac{b}{b+36} = \frac{d+4}{b}$

By cross multiplying, $b^2 = (b+36)(d+4)$

$= bd+4b+36d+144$ ……….. (1)

As they are in G.P., $\frac{b}{b+36} = \frac{d}{d+4}$

By cross multiplying, b (d+4) = d (b+36)

i.e., $bd + 4b = bd + 36d$

$bd + 4b - bd - 36d = 0$

$4b = 36d$

$b = 36d / 4 = 9d$

By substituting this in (1),

$(9d)^2 = (9d)d+4(9d)+36d+144$

i.e., $81d^2 = 9d^2 + 36d + 36d + 144$

i.e., $72d^2 - 72d - 144 \quad = 0$

Dividing by 72, $d^2 - d - 2 = 0$

If $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

lllly, $d = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$

$= \frac{1+3}{2}$

d = 2 or -1

When d=2, b=18 and the numbers are 54, 18, 6, 2.

When d=-1, b=9 and the numbers are 27, -9, 3, 1.

Example 24: Three numbers form an increasing G.P. If the third number is decreased by 16, we get A.P. If then the second number is decreased by 2, we again get a G.P. Find the three numbers.

Solution: Let the three numbers forming an increasing G.P. (r >1) be a, b and c.

$\therefore \quad \frac{b}{a} = \frac{c}{b}$ or $b^2 = ac$ …………. (1)

It is given that a, b, c-16 form an A.P.

$\therefore \quad b = \frac{a + c - 16}{2}$

It is further given that a, b-2, c-16 form a G.P.

$\therefore \quad (b-2)^2 = a\,(c-16)$

By substituting for the b form (2),

$$\left(\frac{a+c-16}{2}-2\right)^2 = a(c-16)$$

i.e., $(a+c-16-4)^2 = 4\,(ac-16a)$

$a^2 + c^2 + 400 + 2ac - 10a - 40c = 0$ ………… (3)

By substituting for b form (2), (1) becomes $\left(\frac{a+c-16}{2}\right)^2 = ac$

i.e., $a^2+c^2+256+2ac-32a-32c = 4ac$

$a^2+c^{2-}2ac-32a-32c+256 = 0$

(3) – (4), $56a-8c+144 = 0$

i.e., $56a+144 = 8c$

$\therefore$ dividing by 8, $7a+18 = c$ ………. (5)

By substituting this value of c in (3),

$a^2\,(7a+18)^2 - 2a\,(7a+18) + 24a - 40\,(7a + 18) + 400 = 0$

i.e., $a^2 + 49a^2 + 252a + 324 - 14a^2 - 36a + 24a - 280a - 720 + 400 = 0$

i.e., $36a^2 - 40a + 4 = 0$

Dividing by 4, $9a^2 - 10a + 1 = 0$

$9a^2 - 9a - a + 1 = 0$

$9a\,(a-1) - 1\,(a-1) = 0$

$(a-1)\,(9a-1) = 0$

$a = 1;\ 1/9$

$\therefore$ From 5, $c = 25;\ 169/9$

$\therefore$ From 2, $b = 5;\ 13/9$

$\therefore$ the three numbers are 1, 5, 25 or 1/9, 13/9, 169/9.

Example 25: If a, b, c from an A.P AND b, c, a form a G.P. show that $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ form an A.P.

Solution: Required to show that $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ form an A.P.

That is $\frac{1}{a}-\frac{1}{c}=\frac{1}{b}-\frac{1}{a}$

Multiplying both sides by abc,

$$\frac{1}{a}\times abc-\frac{1}{c}\times abc=\frac{1}{b}\times abc-\frac{1}{a}\times abc$$

$$bc - ab = ac - bc$$

$$2bc = ab+ac$$

$$2bc = a\,(b+c) \qquad \ldots\ldots\ldots\ldots (1)$$

As a, b, c form an A.P., $2b = a+c$ ……….. (2)

As a, b, c form a G.P., $c^2 = ab$ ……….. (3)

Multiplying both sides of (2) by c, $2bc = (a+c)\,c$

$= ac+c^2$

$= ac + ab$ by (3)

$= a\,(b+c)$ Q.E.D. by (1)

Hence $\frac{1}{c},\frac{1}{a},\frac{1}{b}$ form an A.P.

Formula for the Sum of the First N Terms of a G.P.

Derive the formula to find the sum of n terms in Geometric Progression.

$$\therefore \qquad S_n=\frac{a\left(r^n-1\right)}{r-1}$$

Note1: Usually the first form is used when r < 1 and then second, when r > 1.

1. when $r < 1$, r^n becomes smaller and smaller as becomes larger and larger, when $n\to\infty, r^n\to 0.$

$\therefore$ The formula becomes $S_\infty=\frac{a}{1-r}=a\left(1-r\right)^{-1}$

Example 26: The first three terms of a G.P. are x, x+3, and x+9. Find the value of x and the sum of the first eight terms.

Solution: As x, x+3 and x+9 are in G.P.,

$$(x+3)^2 = x\,(x+9)$$

i.e., $x^2+6x+9 = x^2+9x$

$\therefore \quad \cancel{x^2}+6x+9 \; - \cancel{x^2} - 9x$

$\therefore \quad -3x = -9$

$x = -9 / -3 = 3$

As the first three terms are 3,6, and 12, a= 3 and r= 6/3=12/6 = 2

The sum of the first 8 terms, $S_n = \frac{a\left(r^n - 1\right)}{r - 1}$

$$S_8 = \frac{3\left(2^8 - 1\right)}{2 - 1} = 765$$

Example 27: A person is entitled to receive an annual payment which for each is less by one tenth of what it was for the year before. If the first payment is Rs.100, show that he cannot receive more than Rs.1000 however long he may live.

Solution: First payment, a =100.

Second payment = $\frac{9}{10} \times 100$ $\quad \because$ less by one tenth.

Third payment $= \frac{9}{10}\left(\frac{9}{10} \times 100\right) = \left(\frac{9}{10}\right)^2 100$

$\vdots \qquad \vdots$

Number of payments is unlimited or ∞.

The person is entitled to receive

$$100 + \frac{9}{10}(100) + \left(\frac{9}{10}\right)^2 100 + ...\infty$$

$$= \frac{100}{1 - \frac{9}{10}} \quad \because S_\infty = \frac{a}{1 - r} \text{ and } a = 100; \; r = \frac{9}{10}$$

$$= \frac{100}{\frac{1}{10}} = 100 \times 10 = \text{Rs.}1000$$

Hence, however long he may live, he cannot receive more than Rs.1000.

Example 28: The first term of a G.P. is 4 while its sum to infinity is 5. Find the sum to 8 terms.

Solution: Given: a = 4; $\quad S_\infty = 5$

Consider $S_\infty = \dfrac{a}{1-r}$

By substituting the given values,

$$5 = \frac{4}{1-r}$$

i.e., $1 - r = \dfrac{4}{5} = 0.8$

∴ $-r = 0.8 - 1 = -0.2$ or $r = 0.2$

Required sum, $S_8 = \dfrac{a\left(1-r^n\right)}{1-r}$

$$= \frac{4\left[1-(0.2)^8\right]}{1-0.2} = 4.9999872$$

Example 29: Find the sum of n terms of the following series:

(i) 7+77+777+…..
(ii) .7+.77+.777+…..

Solution: (i) Required sum, S_n = 7+77+777+….. to n terms

$$= \frac{7}{9}\left[9\left(1+11+111+....+\text{to n terms}\right)\right]$$

$$= \frac{7}{9}\left[9+99+999+....+\text{to n terms}\right]$$

$$= \frac{7}{9}\left[(10-1)+(100-1)+(1000-1)+......+\text{to n terms}\right]$$

$$= \frac{7}{9}\left[10+10^2+10^3+.....+\text{to n terms}\right]$$

$$= \frac{7}{9}\left[\frac{10\left(10^n-1\right)}{10-1}-n\right] \qquad \because a=10;\ r=10;\ n=n$$

(ii) Required sum, S_n = 0.7+0.77+0.777+… to n terms

= 7 (0.1+0.11+0.111+….. to n terms)

$$= \frac{7}{9}\left[0.9+0.99+0.999+....+\text{to n terms}\right]$$

$$= \frac{7}{9}\left[(1-0.1)+(1-0.01)+(1-0.001)+....+\text{to n terms}\right]$$

$$S_n = \frac{7}{9}\left[1+1+1+....+\text{to n terms} - (0.1+0.01+0.001+.... \text{ to n terms})\right]$$

$$= \frac{7}{9}\left[n - \frac{0.1\{1-(0.1)^n\}}{1-0.1}\right] \qquad \because a = 0.1;\ r = 0.1;\ n = n$$

$$= \frac{7}{9}\left[n - \frac{1-(0.1)^n}{9}\right] \qquad \because \frac{0.1}{0.9} = \frac{1}{9}$$

$$= \frac{7}{81}\left[9n - \{1-(0.1)^n\}\right]$$

Example 30: Find the sum of the series 0.7+0.07+0.007+….. to ∞.

Solution: Required sum, $S_\infty = 0.7+0.07+0.007+......\text{to } \infty$.

$$= \frac{0.7}{1-0.1} \qquad \because S_\infty = \frac{a}{1-r} \quad \text{and} \quad a = 0.7;\ r = 0.1$$

$$= \frac{0.7}{0.9} = \frac{7}{9}$$

Geometric Means (G.M.): If a series of n terms is in G.P., the first and the last terms are called the extremes and the intermediate terms are called geometric means.

∴ the last term, $(n+2)^{th}$, $b = ar^{n+2-1}$

$r = b / a$

∴ Common ratio, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Hence the first G.M., $G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

the second G.M., $G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

the n^{th} (last) G.M., $G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Example 31: Insert one G.M., between 6 and 294.

Solution: Required G.M. $\sqrt{6\times 294}=42$

Example 32: Insert 4 G.M's., between 2 and 6250.

Solution: Given: n=4, a=2, and b=6250.

$$\therefore \text{Common ratio,} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{6250}{2}\right)^{\frac{1}{4+1}} = 5$$

$\therefore$ the 4 G.M.'s G_1, G_2, G_3 and G_4 are respectively ar, ar^2, ar^3, and ar^4. i.e., 10, 50, 250, and 1250.

Example 32: Three numbers are in G.P. Their sum is 70. If the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. Find the numbers.

Solution: Let a, ar, and ar^2 be the three numbers.

Given: $a+ar+ar^2 = 70$ ……….. (1)

and 4a, 5ar, $4ar^2$ are in A.P.

$$\therefore \quad 5ar = \frac{4a+4ar^2}{2} = 2a+2ar^2$$

$$\therefore \quad 2.5r = a+ar^2$$

By substituting in (1), ar+2.5ar = 70

i.e., 3.5ar = 70 and ar=20

$$a = \frac{20}{r} \quad \text{……….. (2)}$$

By substituting in (1), $\frac{20}{r}+20+20r=70$

Multiplying by r, $20+20r+20r^2 = 70r$

i.e., $20r^2-5r+2 = 0$

Dividing by 10, $2r^2-5r+2 = 0$

i.e., $2r^2-4r-r+2 = 0$

$2r(r-2) – 1(r-2) = 0$

$(r-2)(2r-1) = 0$

$r–2 = 0;\ 2r–1 = 0$

$r = 2;\ 1/2$

From (2) a = 10 when r = 2

a = 40 when r = ½

Case 1: a=10 and r=2. The three numbers in G.P. are 10, 20 and 40.

Case 2: a=40 and r=1/2. The three numbers in G.P. are 40, 20 and 10.

INTERPOLATIONS:

Interpolation is an estimation of a value within two known values in a sequence of values. Polynomial interpolation is a method of estimating values between known data points.

(OR)

A few values of an independent variable (arguments) and the corresponding values of a dependent variable (entries), called function, finding the values of the function. Corresponding to an intermediate value of the argument is the problem of interpolation.

EXTRAPOLATION:

Extrapolation is an estimation of a value based on extending a known sequence of values (or) facts beyond the area that is certainly known.

MEHTODS:

1. Graphic method
2. Algebraic method
   (i) Binomial expansion method
   (ii) Newton's method of forward differences.
   (iii) Lagrange's method
   (iv) Parabolic curve method.

Graphic method:

Arguments are represented in X axis and entries in Y axis. Corresponding to each pair of argument and entry, one point is marked on a graph sheet.
Graphic method does not involve calculations. It is the simplest method.

Algebraic method:

i) Binomial expansion method

Binomial expansion method is suitable when the entries corresponding to one (or) more arguments are missing and all the arguments including those corresponding to which the entries are to be found out. It is the simplest algebraic method. It enquires the least time for finding the missing value algebraically. It does not involve many calculations. One equation is considered and the known values are substituted. By solving the equations, an estimate of the missing value is get,

Its limitations is that it can be used only for finding one (or) more values of y_0, $y_1,y_2,\ldots y_n$ if and only if x_0, $x_1,x_2,\ldots x_n$. have equal differences.

When there are two missing values, two equations are considered and solved simultaneously.

One missing method

EXAMPLE 1:

Find the figure of sales for the year 1990 from the data given below. (Use Binomial expansion method)

Year:	1988	1989	1990	1991	1992
Sales(000):	100	107	-	157	212

Solution:

4 pairs of values are given. Hence, $Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$ is to be solved. It is seen that,

Year:	$X_0 = 1988$	$X_1 = 1989$	$X_2 = 1990$	$X_3 = 1991$	$X_4 = 1992$
Sales(000):	$Y_0 = 100$	$Y_1 = 107$	$Y_2 = ?$	$Y_3 = 157$	$Y_4 = 212$

By the substituting the given values in the above equation.

$$Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$$
$$212 - 4(157) + 6Y_2 - 4(107) + 100 = 0$$
$$212 - 628 + 6Y_2 - 428 + 100 = 0$$
$$212 - 628 + 6Y_2 - 328 = 0$$
$$6Y_2 - 744 = 0 \Rightarrow 6Y_2 = 744$$
$$Y_2 = \frac{744}{6} = 124$$
$$\therefore Y_2 = 124$$

$\therefore$ The sales in 1990 = 124(000) units.

EXAMPLE 2:

Interpolate Y when X = 32 from the following

X:	30	34	36	38	40
Y:	340	353	358	364	369

Solution:

After inclusion of 32, the values of X have equal differences. Hence, by Binomial method.

$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0$ is to be solved as 5 pairs of X and Y are given. It is seen that,

$X_0 = 30$	$X_1 = 32$	$X_2 = 34$	$X_3 = 36$	$X_4 = 38$	$X_5 = 40$
$Y_0 = 340$	$Y_1 = ?$	$Y_2 = 353$	$Y_3 = 358$	$Y_4 = 364$	$Y_5 = 369$

By substituting the given values in the above equation,

$$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$$
$$369 - 5(364) + 10(358) - 10(353) + 5Y_1 - 340 = 0$$
$$369 - 1820 + 3580 - 3580 + 5Y_1 - 340 = 0$$
$$5Y_1 - 1741 = 0$$
$$5Y_1 = 1741$$
$$Y_1 = \frac{1741}{5} = 348.2$$
$$\therefore Y_1 = 348.2$$

When X = 32, Y = 348.2

EXAMPLE 3:

Interpolate the missing term in the following

Year:	1987	1988	1989	1990	1991
Production:	20	30	50	?	80

Solution:

4 pairs of values are given. Hence

4 pairs of values are given. Hence, $Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$ is to be solved. It is seen that,

Year:	X_0 = 1987	X_1 = 1988	X_2 = 1989	X_3 = 1990	X_4 = 1991
Sales(000):	Y_0 = 20	Y_1 = 30	Y_2 = 50	Y_3 = ?	Y_4 = 80

By the substituting the given values in the above equation.

$$Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$$
$$80 - 4Y_3 + 6(52) - 4(30) + 20 = 0$$
$$80 - 4Y_3 + 300 - 120 + 20 = 0$$
$$-4Y_3 + 400 - 120 = 0$$
$$-4Y_3 + 280 = 0$$
$$-4Y_3 = -280$$
$$Y_3 = \frac{280}{4} = 70$$
$$\therefore Y_3 = 70$$

$\therefore$ X = 1990 & Y_3=70

EXAMPLE 4:

By Binomial method, estimate the EPS of 1998 – 1999 .

Year:	1997 – 1998	1998 – 1999	1999 – 2000	2000 – 2001	2001 – 2002
EPS(Rs):	21.39	?	13.74	12.67	13.97

Solution:

4 pairs of values are given. Hence, $Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$ is to be solved. It is seen that,

Year:	X_0 =1997 – 1998	X_1 = 1998 – 1999	X_2 = 1999 – 2000	X_3 = 2000 – 2001	X_4 = 2001 – 2002
Sales(000):	Y_0 = 21.39	Y_1 = ?	Y_2 = 13.74	Y_3 = 12.67	Y_4 = 13.97

By the substituting the given values in the above equation.

$$Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$$
$$13.97 - 4(12.67) + 6(13.74) - 4Y_1 + 21.39 = 0$$
$$117.8 - 50.68 - 4Y_1 = 0$$
$$67.12 - 4Y_1 = 0$$
$$-4Y_1 = -67.12$$
$$Y_1 = \frac{67.12}{4} = 16.78$$
$$\therefore Y_1 = 16.78$$

$\therefore\ Y_1 = 16.78$ & $X_1 = 1998 - 1999$.

EXAMPLE 5:

Using an appropriate formula for interpolation, estimate the average. No.of children born per month for mothers aged 30 – 34 .

Age of months (in years)	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39	40 – 45
Average no.of children born	0.7	2.1	3.1	?	5.7	5.8

Solution:

5 pairs of values are given. Hence

$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0$ is to be solved. It is seen that,

$X_0 = 15 - 19$	$X_1 = 20 - 24$	$X_2 = 25 - 29$	$X_3 = 30 - 34$	$X_4 = 35 - 39$	$X_5 = 40 - 45$
$Y_0 = 0.7$	$Y_1 = 2.1$	$Y_2 = 3.1$	$Y_3 = ?$	$Y_4 = 5.7$	$Y_5 = 5.8$

By substituting the given values in the above equation,

$$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$$
$$5.8 - 5(5.7) + 10Y_3 - 10(3.1) + 5(2.1) - 0.7 = 0$$
$$5.8 - 28.5 + 10Y_3 - 31 + 10.5 - 0.7 = 0$$
$$16.3 - 60.2 + 10Y_3 = 0$$
$$-43.9 + 10Y_3 = 0$$
$$10Y_3 = 43.9$$
$$Y_3 = \frac{43.9}{10} = 4.39$$
$$\therefore Y_3 = 4.39 \,\&\, X_3 = 30 - 34$$

EXAMPLE 6:

Find the figures of sales for the year 2001 from the data given below using Binomial expansion method of extrapolation.

Year:	1995	1996	1997	1998	1999	2000
Sales(000)units:	120	150	160	180	200	225

Solution:

6 pairs of values are given. Hence, the sales forecast for 2001 is obtained such that, $Y_6 - 6Y_5 + 15Y_4 - 20Y_3 + 15Y_2 - 6Y_1 + Y_0 = 0$ is to be solved. It is seen that,

Year:	X_0 =1995	X_1 = 1996	X_2 = 1997	X_3 = 1998	X_4 = 1999	X_5 = 2000
Sales(000):	Y_0 = 120	Y_1 = 150	Y_2 = 60	Y_3 = 180	Y_4 = 200	Y_5 = 225

By the substituting the given values in the above equation.

$$Y_6 - 6Y_5 + 15Y_4 - 20Y_3 + 15Y_2 - 6Y_1 + Y_0 = 0$$
$$Y_6 - 6(225) + 15(200) - 20(180) + 15(160) - 6(150) + 120 = 0$$
$$Y_6 - 1350 + 3000 - 3600 + 2400 - 900 + 120 = 0$$
$$Y_6 + 5520 - 4950 - 900 = 0$$
$$Y_6 + 5520 - 5850 = 0$$
$$Y_6 - 330 = 0$$
$$\therefore Y_6 = 330$$

Sales in 2001 = 330 (000) units.

TWO MISSING VALUES

EXAMPLE 1:

Obtain estimates of the missing figures in the following table:

X:	2.0	2.1	2.2	2.3	2.4	2.5	2.6
U_x:	0.135	?	0.111	0.100	?	0.082	0.074

Solution:

5 pairs of values are given. Two values (Y_1 & Y_4) are missing. Hence, $\Delta^5 Y_0 = 0$ and are to be solved.

$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$ *and*

$Y_6 - 6Y_5 + 15Y_4 - 20Y_3 + 15Y_2 - 6Y_1 + Y_0 = 0$

are to be solved simultaneously. It is seen that,

$X_0 = 2.0$	$X_1 = 2.1$	$X_2 = 2.2$	$X_3 = 2.3$	$X_4 = 2.4$	$X_5 = 2.5$	$X_6 = 2.6$
$Y_0 = 0.135$	$Y_1 = ?$	$Y_2 = 0.111$	$Y_3 = 0.100$	$Y_4 = ?$	$Y_5 = 0.082$	$Y_6 = 0.074$

By substituting the given values in the above equations,

$$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$$

$$0.082 - 5Y_4 + 10(0.100) - 10(0.111) + 5Y_1 - 0.135 = 0..........(1)$$

$$Y_6 - 6Y_5 + 15Y_4 - 20Y_3 + 15Y_2 - 6Y_1 + Y_0 = 0$$

$$0.074 - 5(0.082) + 10Y_4 - 10(0.100) + 5(0.111) - Y_1 = 0........(2)$$

$$-5Y_4 + 5Y_1 = 0.163.....(1)$$

$$10Y_4 - Y_1 = 0.781.....(2)$$

$(1) \times 2, \quad -10Y_4 + 10Y_1 = 0.326......(3)$

$(2) + (3), \quad 9Y_1 = 1.107$

$\therefore \quad Y_1 = \frac{1.107}{9} = 0.1230$

$From\,(2), \quad 10Y_4 - Y_1 = 0.781$

$10Y_4 - 0.123 = 0.781$

$\therefore \quad 10Y_4 = 0.781 + 0.123$

$10Y_4 = 0.904$

$$Y_4 = \frac{0.904}{10} = 0.0904$$

$\therefore U_{2.1} = 0.1230 \, and \, U_{2.4} = 0.0904$

EXAMPLE 2:

Use Binomial expansion method to interpolate the missing values from the data given below.

X:	5	8	11	14	17	20
Y = f(x)	52	?	376	?	988	1402

Solution:

$X_0 = 5$	$X_1 = 8$	$X_2 = 11$	$X_3 = 14$	$X_4 = 17$	$X_5 = 20$
$Y_0 = 52$	$Y_1 = ?$	$Y_2 = 376$	$Y_3 = ?$	$Y_4 = 988$	$Y_5 = 1402$

$Y_4 - 4Y_3 + 6Y_2 - 4Y_1 + Y_0 = 0$ *and*

$Y_5 - 5Y_4 + 10Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$

$$988-4Y_3+6(376)-4Y_1+52=0$$
$$1402-5(988)+10Y_3-10(376)+5Y_1-52=0$$
$$-4Y_3-4Y_1=-3296........(1)$$
$$10Y_3+5Y_1=7350........(2)$$
$$(1)\times 10 \Rightarrow \quad -40Y_3-40Y_1=-32960$$
$$(2)\times 4 \Rightarrow \quad 40Y_3+20Y_1=29400$$
$$-20Y_1=-3560$$
$$Y_1=\frac{3560}{20}$$
$$Y_1=178$$

$From\,2,$

$$10Y_3+5Y_1=7350$$
$$10Y_3+5(178)=7350$$
$$10Y_3+890=7350$$
$$10Y_3=7350-890$$
$$Y_3=\frac{6460}{10}$$
$$\therefore Y_3=646$$

EXAMPLE 3:

Obtain the estimate of the missing figures in the following table.

X:	3.0	3.1	3.2	3.3	3.4	3.5	3.6
Y = f(x)	0.270	-	-0.222	0.200	-	0.164	0.148

Solution:

5 pairs of values are given. Two values are (Y_1 & Y_2) are given.Hence, $\Delta^5 Y_0=0, \Delta^5 Y_1=0$ and to be solved

$$Y_5-5Y_4+4Y_3-10Y_2+5Y_1-Y_0=0 \qquad and$$
$$Y_6-5Y_5+10Y_4-10Y_3+5Y_2-Y_1=0$$

are to be solved simultaneuously. It is seen that

$X_0=3.0$	$X_1=3.1$	$X_2=3.2$	$X_3=3.3$	$X_4=3.4$	$X_5=3.5$	$X_6=3.6$
$Y_0=0.270$	$Y_1=?$	$Y_2=0.222$	$Y_3=0.200$	$Y_4=?$	$Y_5=0.164$	$Y_6=0.148$

$Y_5 - 5Y_4 + 4Y_3 - 10Y_2 + 5Y_1 - Y_0 = 0$ *and*

$Y_6 - 5Y_5 + 10Y_4 - 10Y_3 + 5Y_2 - Y_1 = 0$

$\Rightarrow 0.164 - 5Y_4 + 10(0.200) - 10(0.222) + 5Y_1 - 0.270 = 0$

$\Rightarrow 0.148 - 5(0.164) + 10Y_4 - 10(0.200) + 5(0.222) - Y_1 = 0$

$1 \times 2 \Rightarrow -10Y_4 + 10Y_1 = 0.652$

$\Rightarrow \quad 10Y_4 - Y_1 = 1.562$

$9Y_1 = 2.214$

$Y_1 = \frac{2.214}{9}$

$\therefore Y_1 = 0.246$

NEWTON'S METHOD OF FORWARD DIFFERENCES

SUITABILITY:

The method is suitable when the given arguments have equal differences and the argument corresponding to which the entry is to be found out is at the first half.

It is considered to be easier than Lagrange's method. It is not as simple and easy as Binomial expansion method. It is used in a kind of problem in which Binomial expansion method cannot be used. Similarly it cannot be used in problems for which Binomial expansion method is suitable.

FORMULA:

$$Y = Y_0 + \frac{u}{1}\Delta Y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 Y_0 + \frac{u(u-1)}{1\times 2\times 3}\Delta^3 Y_0 + ...$$

$where\, u = \frac{X - X_0}{h}$ his the common difference between the arguments, X_0 is the first arguments given and X is the argument corresponding to which y is to be found. It may be noted that Y_0 is the first entry given and $\Delta Y_0, \Delta^2 Y_0, \Delta^3 Y_0, ...$ are the first values in the columns of the difference table.

EXAMPLE 1:

From the following series, obtain the missing value for 12th year using Newton's method.

year	5	10	12	15	20
price	4	14	?	24	34

Solution:

Arguments 5, 10, 15 and 20 have equal differences. Common difference, h = 5. X_0 = 5. x = 12 is in the first half.

$$u = \frac{x - x_0}{h} = \frac{12-5}{5} = 1.4$$

X	Y	Δ_y	$\Delta^2 y$	$\Delta^3 y$
5	4			
		10		
10	14		0	
		10		0
15	24		0	
		10		
20	34			

By Newton's forward difference formula,

$$
\begin{aligned}
Y &= Y_0 + \frac{u}{1}\Delta Y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 Y_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 Y_0 \\
&= 4 + \frac{1.4}{1}\times 10 + \frac{1.4\times 0.4}{1\times 2}\times 0 + \frac{1.4\times 0.4\times(-0.6)}{1\times 2\times 3}\times 0 \\
&= 4 + 14 + 0 + 0 \\
&= 18
\end{aligned}
$$

Price in 12^{th} year = 18.

EXAMPLE 2:

Find premium payable at the age of 26 by using Newton's forward difference formula:

Age (in years)	20	25	30	35	40
Premium(Rs.)	230	260	300	350	420

Solution:

Arguments 20, 25, 30, 35 and 40 have equal differences. Common difference, h = 5. x_0 = 20.

x = 26 is in the first half.

$$u = \frac{x - x_0}{h} = \frac{26-20}{5} = 1.2$$

X	Y	Δ_y	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
20	230				
		30			
25	260		10		
		40		0	
30	300		10		10
		50		10	
35	350		20		
		70			
40	420				

By Newton's forward difference formula,

$$Y = Y_0 + \frac{u}{1}\Delta Y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 Y_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 Y_0 + \frac{u(u-1)(u-2)(u-3)}{1\times 2\times 3\times 4}\Delta^4 Y_0$$

$$= 230 + \frac{1.2}{1}\times 30 + \frac{1.2\times 0.2}{1\times 2}\times 10 + \frac{1.2\times 0.2\times(-0.8)}{1\times 2\times 3}\times 0 + \frac{1.2\times 0.2\times -0.8\times -1.8}{1\times 2\times 3\times 4}\times 10$$

$$= 230 + 36 + 1.2 + 0 + 0.144$$

$$= 267.344$$

the premium payable at the age of 26 = Rs. 267.34

EXAMPLE 3:

Using the Newton's formula for interpolation, estimate the number of students who obtained less than 45 marks from the following:

Marks:	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No.of students:	31	42	51	35	31

Solution:

Class intervals cannot be dealt with directly. Hence, the following differences columns are based on cumulative frequencies.

Marks	*No.of students*	*Marks*(x)	*No.of students*	*Differences* Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
		less than 30	0					
				31				
$30-40$	31	*less than* 40	31		11			
				42		-2		
$40-50$	42	*less than* 50	73		9		-23	
				51		-25		60
$50-60$	51	*less than* 60	124		-16		37	
				35		12		
$60-70$	35	*less than* 70	159		-4			
				31				
$70-80$	31	*less than* 80	190					

$x_0 = 30 : h = 10; x = 45.\ u = \frac{x - x_0}{h} = \frac{45-30}{10} = 1.5$

By Newton's forward difference formula,

$$Y = Y_0 + \frac{u}{1}\Delta Y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 Y_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 Y_0 + \frac{u(u-1)(u-2)(u-3)}{1\times 2\times 3\times 4}\Delta^4 Y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{1\times 2\times 3\times 4\times 5}\Delta^5 Y_0$$

$$= 0 + \frac{1.5}{1}\times 31 + \frac{1.5\times 0.5}{1\times 2}\times 11 + \frac{1.5\times 0.5\times(-0.5)}{1\times 2\times 3}\times -2 + \frac{1.5\times 0.5\times -0.5\times -1.5}{1\times 2\times 3\times 4}\times -23 + \frac{1.5\times 0.5\times -0.5\times -1.5\times -2.5}{1\times 2\times 3\times 4\times 5}\times 60$$

$$= 0 + 46.5000 + 4.1250 + 0.1250 - 0.5391 - 0.7031$$

$$= 49.5078 \approx 50$$

50 students have obtained marks less than 45.

NEWTON'S METHOD OF BACKWARD DIFFERENCES

SUITABILITY:

The method is suitable when the given arguments have equal differences and the arguments have equal differences and the argument corresponding to which the entry is to be found out is at the second half.

It is similar to Newton's method of forward differences. It is considered to be easier than lagranges's method. It is not as simple and easy as Binomial expansion method. It is used in a kind of problem in which neither Newton's forward difference formula nor Binomial expansion method can be used. Similarly, it cannot be used in problems for which they are suitable.

$$Y = Y_n + \frac{u}{1}\Delta Y_{n-1} + \frac{u(u+1)}{1\times 2}\Delta^2 Y_{n-2} + \frac{u(u+1)(u+2)}{1\times 2\times 3}\Delta^3 Y_{n-3} + ...$$

$Where\, u = \frac{x - x_n}{h}$. h is the common difference between the arguments, x_n is the last argument given and x is the argument corresponding to which y is to be found out. It may be noted that $y_n, \Delta y_{n-1}, \Delta^2 y_{n-2}, \Delta^3 y_{n-3}, ...$ are the last values in y and every subsequent column.

EXAMPLE 1:

Age:	20	25	30	35	40
Premium:	33.0	29.8	26.6	23.5	20.5

From the above data, calculate the premium to be paid at age 32.

Solution:

Arguments 20, 25, 30, 35 and 40 have equal differences. Common difference, h = 5. x_n =40. x = 32 is at the second half. $u = \frac{x - x_n}{h} = \frac{32 - 40}{5} = 1.6$

X	Y	Δ_y	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
20	33.0				
		-3.2			
25	29.8		0		
		-3.2		0.1	
30	26.6		0.1		-0.1
		-3.2		0	
35	23.5		0.1		
		-3.0			
40	20.5				

By Newton's backward difference formula,

$$Y = Y_n + \frac{u}{1}\Delta Y_{n-1} + \frac{u(u+1)}{1\times 2}\Delta^2 Y_{n-2} + \frac{u(u+1)(u+2)}{1\times 2\times 3}\Delta^3 Y_{n-3} + \frac{u(u+1)(u+2)(u+3)}{1\times 2\times 3\times 4}\Delta^4 Y_{n-4}$$

$$= 20.5 + \frac{-1.6}{1}\times -3.0 + \frac{-1.6\times -0.6}{1\times 2}\times 0.1 + \frac{-1.6\times -0.6\times 0.4}{1\times 2\times 3}\times 0 + \frac{-1.6\times -0.6\times 0.4\times 1.4}{1\times 2\times 3\times 4}\times -0.1$$

$$= 20.50000 + 4.80000 + 0.04800 + 0 - 0.00224$$

$$= 25.34576 \approx 25.35$$

$\therefore$ premium to be paid at age 32 is 25.35

EXAMPLE 2:

From the following data, find the number of rural students whose I.Q exceeds 102.

I.Q:	70 – 80	80 – 90	90 – 100	100 – 110	110 – 120
No.of Rural students	100	200	100	80	20

Solution:

Class intervals cannot be dealt with directly. Hence, the cumulative frequencies are found out from the given class frequencies and are used for forming the differences columns.

I.Q	*No.of Rural students*	*I.Q* x	*No.of Rural students*(y)	Differences Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
		less than 70	0					
				100				
$70-80$	100	less than 80	100		100			
				200		-200		
$80-90$	200	less than 90	300		-100		-280	
				100		-80		-400
$90-100$	100	less than 100	400		-20		-120	
				80		-40		
$100-110$	80	less than 110	480		-60			
				20				
$110-120$	20	less than 120	500					

$$x_n = 120;\ y_n = 500;\ h = 10;\ x = 102.\ u = \frac{x - x_n}{h} = \frac{102-120}{10} = -1.8$$

By Newton’s forward difference formula,

$$y = y_n + \frac{u}{1}\Delta y_{n-1} + \frac{u(u+1)}{1\times 2}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{1\times 2\times 3}\Delta^3 y_{n-3} + \frac{u(u+1)(u+2)(u+3)}{1\times 2\times 3\times 4}\Delta^4 y_{n-4} + \frac{u(u+1)(u+2)(u+3)(u+4)}{1\times 2\times 3\times 4\times 5}\Delta^5 y_{n-5}$$

$$= 500 + \frac{-1.8}{1}\times 20 + \frac{-1.8\times -0.8}{1\times 2}\times -60 + \frac{-1.8\times -0.8\times 0.2}{1\times 2\times 3}\times -40 + \frac{-1.8\times -0.8\times 0.2\times 1.2}{1\times 2\times 3\times 4}\times -120 + \frac{-1.8\times -0.8\times 0.2\times 1.2\times 2.2}{1\times 2\times 3\times 4\times 5}\times -400$$

$$= 500 - 36 - 43.2 - 1.92 - 1.728 - 2.5344$$

$$= 414.6176 \approx 415$$

Total number of students = 500

Number of students whose I.Q. is less than 102 = 415

Number of students whose I.Q. is exceeds 102 = 500 – 415 = 85

LAGRANGE’S METHOD

It was given by the famous French mathematician lagrange.

SUITABILITY:

Although Lagrange’s formula can be used for all the problems of interpolation, extrapolation and inverse interpolation, it is used in the problems for which no other method is suitable. In other words, it is used when the given arguments do not have equal differences. The given arguments together with the argument corresponding to which the entry is to be found out also do not have equal differences. That is, $x_0, x_1, x_2, ..., x_n$ do not have equal differences.

FORMULA:

When (n+1) pairs of values are known.

$$y = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}y_1 + ... + \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})}y_n$$

If any given problem, the number (n + 1) is to be identified and the formula is to be written accordingly.

EXAMPLE 1:

By using Lagrange’s formula, determine the percentage of criminals under 35 years of age:

Age	Percentage of criminals
Under 25 years	52.0
Under 30 years	67.3
Under 40 years	84.1
Under 50 years	94.4

Solution: 4 pairs of values are given:

$x_0 = 25 \quad x_1 = 30 \quad x_2 = 40 \quad x_3 = 50$

$y_0 = 52.0 \quad y_1 = 67.3 \quad y_2 = 84.1 \quad y_3 = 94.4$

Required to find y when x = 35. x_0, x_1, x_2 *and* x_3 do not have equal differences. x_0, x_1, x, x_2 *and* x_3 also do not have equal differences.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(35-30)(35-40)(35-50)}{(25-30)(25-40)(25-50)} \times 52.0 + \frac{(35-25)(35-40)(35-50)}{(30-25)(30-40)(30-50)} \times 67.3 + \frac{(35-25)(35-30)(35-50)}{(40-25)(40-30)(40-50)} \times 84.1 + \frac{(35-25)(35-30)(35-40)}{(50-25)(50-30)(50-40)} \times 94.4$$

$$= \frac{(5)(-5)(-15)}{(-5)(-15)(-25)} \times 52.0 + \frac{(10)(-5)(-15)}{(5)(-10)(-20)} \times 67.3 + \frac{(10)(5)(-15)}{(1)(10)(-10)} \times 84.1 + \frac{(10)(5)(-5)}{(25)(20)(10)} \times 94.4$$

$$= -10.400 + 50.475 + 42.050 - 4.720$$

$$= 77.405 \approx 77.41$$

Percentage of criminals under 35 years of age = 77.41

EXAMPLE 2:

The following table gives the normal weight of a baby during the first six months of life.

Age(months)	0	2	3	5	6
Weight(Ibs)	5	7	8	10	12

Estimate the weight of the baby at the age of 4 months.

Solution: 5 pairs of values are given

$x_0 = 0 \quad x_1 = 2 \quad x_2 = 3 \quad x_3 = 5 \quad x_4 = 6$

$y_0 = 5 \quad y_1 = 7 \quad y_2 = 8 \quad y_3 = 10 \quad y_4 = 12$

Required to find y when x = 4. x_0, x_1, x_2, x_3 *and* x_4 do not have equal differences. x_0, x_1, x_2, x, x_3*and* x_4 also do not have equal differences. Hence, Lagrange's formula is used.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$= \frac{(4-2)(4-3)(4-5)(4-6)}{(0-2)(0-3)(0-5)(0-6)} \times 5 + \frac{(4-0)(4-3)(4-5)(4-6)}{(2-0)(2-3)(2-5)(2-6)} \times 7 + \frac{(4-0)(4-2)(4-5)(4-6)}{(3-0)(3-2)(3-5)(3-6)} \times 8$$

$$+ \frac{(4-0)(4-2)(4-3)(4-6)}{(5-0)(5-2)(5-3)(5-6)} \times 10 + \frac{(4-0)(4-2)(4-3)(4-5)}{(6-0)(6-2)(6-3)(6-5)} \times 12$$

$$= \frac{2 \times 1 \times -1 \times -2}{-2 \times -3 \times -5 \times -6} \times 5 + \frac{4 \times 1 \times -1 \times -2}{2 \times -1 \times -3 \times -4} \times 7 + \frac{4 \times 2 \times -1 \times -2}{3 \times 1 \times -2 \times -3} \times 8 + \frac{4 \times 2 \times 1 \times -2}{5 \times 3 \times 2 \times -1} \times 10 + \frac{4 \times 2 \times 1 \times -1}{6 \times 4 \times 3 \times 1} \times 12$$

$$= 0.1111 - 2.3333 + 7.1111 + 5.3333 - 1.3333$$

$$= 8.8889 \approx 8.89$$

The weight of the baby at the age of 4 months = 8.9 Ibs.

UNIT – III

PROBABILITY

Chance of an event is expressed numerically. An Italian mathematician, Galileo (1564 - 1642), was the first person who attempted quantitative measure of probability. There are three different approaches – classical approach, Relative frequency approach and axiomatic approach. Classical approach is commonly used in the beginning stage.

Today the theory of probability has been developed to a great extent. It is applied in all the disciplines. It is extensively used in business, economic problems etc,.

<u>MEANING:</u>

It is difficult to give a clear (or) generally accepted meaning of probability. In our day – to – day life, we come across sentences like, for example.

1. “When a coin is tossed, it must fall down”.
2. “It is impossible to live without oxygen”.
3. “Probably, I win the Match”.

RANDOM EXPERIMENT (OR) TRIAL:

An experiment is called a random experiment when it has several possible outcomes and each outcomes has a certain possibility.

Example 1:

1. Tossing a coin head & tail.
2. Throwing a die 1, 2, 3, 4, 5 & 6.

Trial: conducting a random experiment once is called a trial.

(or)

If an experiment (or) trial can be repeated under the same conditions, any number of times and it is possible to count the total no of outcomes, but individual result.,(ie) Individual outcome is not predictable.

Suppose we toss a coin. It is not possible to predict exactly the outcomes. The outcome may be either head. Up (or) tail up. Thus, an action (or) an operation which can produce any result (or) outcome is called a random experiment (or) a trial.

EXAMPLE:

Tossing a coin is an experiment (or) trial. When you toss, it falls head up (or) trial up.

EVENT:

Any possible outcome of a random experiment is called an event. Performing an experiment is called trial & outcomes are termed as events (or) one outcome itself may be an event such events are called elementary events, more than one outcome constitute other events.

In the random experiment of throwing a die, for the event, ‘1 point’ only one outcome is there. For the event ‘odd no of points’, three outcomes 1, 3 and 5 are there,

EXAMPLE:

i) Tossing a coin is a random experiment (or) trial and getting a head (or) a tail is an event.

ii) Drawing a ball from an urn containing red and white balls is a trial and getting a red (or) white ball is an event.

MUTUALLY EXCLUSIVE OUTCOMES:

Outcomes are said to be mutually exclusive if they cannot happen simultaneously in a single trail. In the random experiment of tossing a coin head and tail are mutually exclusive outcomes.

In the random experiment of throwing a die. The points 1, 2, 3, 4, 5 and 6 are mutually exclusive outcomes.

EQUALLY LIKELY OUTCOMES:

Outcomes which have equal chance of happening are said to be equally likely outcomes.

In the random experiment of tossing an unbiased coin, head and tail are equally likely outcomes in the random experiment of throwing a fair die.

1, 2, 3, 4, 5 & 6 are equally likely outcomes.

Ex: Head and tail are equally likely event in tossing an unbiased coin.

EXHAUSTIVE OUTCOMES:

Outcomes are said to be exhaustive when no other outcome is possible from the random experiment.

The two outcomes – head and tail, are exhaustive in the random experiment of tossing a coin. The total no of possible outcomes of a random experiment is called exhaustive events. The group of events is exhaustive, as there is no other possible outcome. Thus tossing a coin, the possible outcome are head (or) tail; exhaustive events are two.

Similarly, throwing a die, the outcomes are 1, 2, 3, 4, 5 & 6. In case of two coins, the possible number of outcomes are 4. (ie) HH, HT, TH & TT.

In case of 3 coins, the possible outcomes are $2^3 = 8$ and so on. Thus, in throw of n coin, the exhaustive no of case is 2^n.

INDEPENDENT EVENTS:

Two events which do not affect the chances of each other are said to be independent events.

When two coins are tossed together, head in the first coin and head in the second coin are independent.

When two cards are drawn one after another without replacement from a pack of well shuffled playing cards, the chance of getting king in the second draw is affected by king in the first draw. Hence, king in the second draw is not independent of king in the first draw.

DEPENDENT EVENTS:

Two events are said to be dependent, if the occurrence (or) non – occurrence of one event in any trial affects the probability of the other subsequent trial, if the occurrence of one event affects the happening of the other events, then they are said to be dependent events.

Example:

The probability of drawing a king from a pack of 52 cards is 4/52; the card is not put pack; then the probability of drawing a king again is 3/51. Thus the outcome of the first event affects the outcome of the second event and they are dependent. But if the card is put pack, then the probability of drawing a king is 4/52 and is an independent event.

SIMPLE AND COMPUND EVENTS:

When a single event take place, the probability of its happening (or) not happening is known as simple event.

When two or more events take place simultaneously their occurrence is known as compound event (compound probability) for instance, throwing a die.

COMPLEMENTARY EVENTS:

The complement of an event A, means non – occurrence of A and is denoted by $\bar{A}$. $\bar{A}$ contains those points of the sample space which do not belong to A for instance. Let there be two events of B and vice versa, If A and B are mutually exclusive & exhaustive.

FAVOURABLE CASES:

The number of outcomes which result in the happening of a desired event are called favourable cases to the event.

Example:

In drawing a card from a pack of cards, the cases favorable to “getting a diamond” are 13 and to “getting an ace of spade” is only one. Take another ex: in a single throw of a dice the no of favorable cases of getting an odd number are three, -1, 3 & 5.

MEASUREMENT OF PROBABILITY:

The origin and development of the theory of probability dates back to the 17th century. Ordinarily speaking the probability of an event denotes the likelihood of its happening. A value of the probability is a number range between 0 and 1. Different schools of thought have defined the term probability differently. The various schools of thought which have defined probability are discussed briefly.

MATHEMATICAL (OR) CLASSICAL (OR) [PRIORI PROBABILITY]

If a trial has ‘n’ exhaustive, mutually exclusive and equally likely outcomes among which ‘m’ are favorable to the occurrence of an event A. the probability of occurrence of A is denoted by P(A) (or) p and

$$p = P(A) = \frac{Number\ of\ favorable\ outcomes}{total\ no.of\ outcomes} = \frac{m}{n}.$$

EXAMPLE 1:

An unbiased coin is tossed once. What is the probability of getting head?

SOLUTION:

Possible outcomes are head and tail.

∴ total number of outcomes, n = 2

Number of favorable outcomes, m = 1

∴ Required probability $p = \frac{m}{n} = \frac{1}{2}$.

EXAMPLE 2:

Two unbiased coins are tossed together. Find the probability of getting tail in both the coins.

SOLUTION:

A – Tail in both the coins

$n = 2 \times 2 = 4$

$m = 1$

∴required probability

$P(A) = \frac{m}{n} = \frac{1}{4}$

Possible outcomes	
Coin I	**Coin II**
1. Head	Head
2. Head	Tail
3. Tail	Head
4. Tail	Tail

EXAMPLE 3:

Two coins are tossed simultaneously. What is the probability of getting a head and a tail?

SOLUTION:

Two possible combinations of the two coins Turing up with head {H} or tail {T} are HH, HT, TH, TT. The favourable ways are two out of these four possible ways and all these are equally likely to happen. Hence the probability of getting head and a tail is $\frac{2}{4} = \frac{1}{2}$.

EXAMPLE 4:

Two cards are drawn from pack of cards at random. What is the probability that it will be

a) A diamond and a heart
b) A king and a queen
c) Two kings?

SOLUTION:

a) The number of ways of drawing 2 cards from out of 52 cards.
   = $52_{C_2} = \frac{52 \times 51}{1 \times 2} = 26 \times 51$`
   The number of ways of drawing a diamond and a heart.
   = 13 x 13
   The required probability,
   = $\frac{13 \times 13}{26 \times 51} = \frac{13}{102}$
b) The number of ways of drawing a king and a queen.
   = $\frac{4 \times 4}{26 \times 51} = \frac{8}{663}$
c) Two kings can be drawn out of 4 kings in

   = 4_{C_2} = 6 ways.

   The probability of drawing 2 kings

   = $\frac{6}{26 \times 51} = \frac{1}{221}$

EXAMPLE 5:

A bag contains 7 red, 12 white and 4 green balls. What is the probability that

a) 3 balls drawn are all white and
b) 3 balls drawn are one of each color?

SOLUTION:

a) Total number of balls
   = 7 + 12 + 4 = 23

Number of possible ways of drawing 3 out of 12 white balls

= 12_{C_3}.

Total number of, possible ways of drawing 3 out of 23 balls.

$= 23_{C_3}$

Therefore, probability of drawing 3 white balls.

$$= \frac{12_{C_3}}{23_{C_3}} = \frac{220}{1771} = 0.1242$$

b) Number of possible way of drawing 1 out of
Red = 7_{C_1}

Number of possible ways of drawing 1out of

12 white = 12_{C_1}

Number of possible ways of drawing 1 out of 4 green = 4_{C_1}. Therefore the probability of drawing balls of different colours,

$$= \frac{7_{C_1} \times 12_{C_1} \times 4_{C_1}}{23_{C_3}} = \frac{7 \times 12 \times 4}{1771} = 0.1897$$

EXAMPLE 6:

A symmetrical die is thrown. Find the probability for

(i) 6 (ii) not 6 (iii) 7 (iv) less than 7

SOLUTION:

Total number of outcomes, n = 6

(i) Number of favorable outcomes, m = 1
∴ the probability for 6 $= \frac{m}{n} = \frac{1}{6}$

(ii) Number of favorable outcomes, m = 6 – 1 = 5
∴ the probability for not 6 = $\frac{m}{n} = \frac{5}{6}$

(iii) Number of favorable outcomes, m = 0
∴ the probability for 7 $= \frac{m}{n} = \frac{0}{6} = 0$

(iv) Number of favorable outcomes, m = 6
∴ the probability for less than 7 = $\frac{m}{n} = \frac{6}{6} = 1$

LIMITATIONS OF CLASSICAL APPROACH:

1. This definition is confined to the problems of games of chance only and cannot explain the problem other than the games of chance.
2. We cannot apply this method, when the total no of cases cannot be calculated.
3. When the outcomes of a random experiment are not equally likely, this method cannot be applied.
4. It is difficult to subdivided the possible outcome of experiment into mutually exclusive, exhaustive and equally likely in most cases.

TYPES OF PROBABILITY:

There are two types

i) Mathematical probability
ii) Priori probability

STATISTICAL (OR EMPERICAL) PROBABILITY:

If n trails and an event E happens 'm' times then the probability of is having that event E is defined by P(E) = $\lim_{n\to\infty} \frac{m}{n}$

AXIOMS OF PROBABILITY:

A limiting value as desired in relative frequency approach may not exist. Even if it exists finding it may be difficult.

A Russian mathematician, kolmogorov, gave a new approach in his book entitled 'foundations of probability'. It was published in 1933

He gave the three axioms,

i) $0 \leq P \leq 1$.
ii) P (S) = 1. Where S is the sample space and
iii) P($A_1 \cup A_2 \cup \ldots \cup A_n$) = P($A_1$) + P($A_2$) + …P($A_n$)

Where $A_1, A_2, \ldots A_n$ are disjoint (mutually exclusive) events. He has developed the method of find the probability of an event on the basis of the axioms.

THEOREM 1:

STATEMENT:

Probability of an impossible event is 0 ie). $P(\emptyset) = 0$

PROOF:

An impossible event contains no sample points

∴ A sample space S and impossible event $\emptyset$ are mutually exclusive.

$S \cup \emptyset = S$

Taking probability on both sides,

P($S \cup \emptyset) = P(S)$ $\qquad$ $\therefore P(S) = 1$

P(S) + P($\emptyset$) = P(S)

P($\emptyset$) = P(S) - P(S) = 1 – 1 = 0.

THEOREM 2:

STATEMENT:

Probability of the complementary event $\bar{A}$ of A is given by P($\bar{A}$) = 1 – P(A).

PROOF:

We know that, A and $\bar{A}$ are mutually exclusive.

$$\therefore (A \cup \bar{A}) = S$$

Taking probability on both sides,

$P(A \cup \bar{A}) = P(S)$

$P(A) + P(\bar{A}) = P(S)$

$P(\bar{A}) = P(S) - P(A)$

$P(\bar{A}) = 1 - P(A)$

RESULT:

1. $P(\bar{A} \cap B) = P(B) - P(A \cap B)$
2. $P(\bar{A} \cap B) = P(A) - P(A \cap B)$

$A = (A \cap B) \cup (A \cap \bar{B})$

$P(A) = P(A \cap B) + P(A \cap \bar{B})$

$P(A) - P(A \cap B) = P(A \cap \bar{B})$

ADDITION THEOREM OF PROBABILITY (TWO EVENTS OCCUR AT A TIME):

If A and B are (not mutually exclusive) two events then P(A) + P(B) – P(A ∩ B).

i.e)., P(A ∪ B) = P(A) + P(B) - P(A ∩ B).

PROOF:

From this figure A and $\bar{A} \cap B$ are disjoint.

∴ It's ∪ is $A \cup B = A \cup (\bar{A} \cap B)$.

$\therefore P(A \cup B) = P[A \cup (\bar{A} \cap B)]$

$\Rightarrow P(A) + P(\bar{A} \cap B)$ (by axioms 3)

$\Rightarrow P(A) + [P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B)]$

∴ P(A ∪ B) = P(A) + P(B) - P(A ∩ B)

Hence the proof.

ADDITION THEOREMS OF THREE EVENTS:

If A, B & C are any three events then

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

PROOF:

$$P(A\cup B\cup C)=P[(A\cup B)\cup C].$$

$\because P(A\cup B)=P(A)+P(B)-P(A\cap B).$

$=P(A\cup B)+P(C)-P(A\cup B\cap C)$

$=P(A)+P(B)-P(A\cap B)+P(C)-P[(A\cup B)\cap C].$

$=P(A)+P(B)-P(A\cap B)+P(C)-P[(A\cap C)\cup(B\cap C)]$

$=P(A)+P(B)-P(A\cap B)+P(C)-P(A\cap C)+P(B\cap C)+P[(A\cap C)\cap(B\cap C)].$

$=P(A)+P(B)-P(A\cap B)+P(C)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

$=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$

THEOREM 3:

If A and B are mutually exclusive are two events then $P(A\cup B)=P(A)+P(B).$ (or) A and B are two events cannot occurs at a time.

PROOF:

Let A happens m_1 times of n exhaustive cases.

$$P(A\cup B)=P(A)+P(B)$$

Where $P(A)=\frac{m_1}{n};P(B)=\frac{m_2}{n}.$

Let be events either A (or)B. happens m_1+m_2 times out of ‘n’ exhaustive cases.

By the definition of probability.

$$P(AorB)=\frac{m_1+m_2}{n}=\frac{m_1}{n}+\frac{m_2}{n}$$

$$P(A\cup B)=P(A)+P(B).$$

Hence the proof.

COMBINATIONS:

Combinations are groups. The members of each group are selected from the available members.

Number of combinations of n objects taken. r (≤) at a time is denoted by the symbol nC_r. In other words, number of ways in which r objects can be chosen or selected from n objects is denoted by nC_r.

$$nC_r = \frac{n(n-1)(n-2)...(n-\overline{r-1})}{1\times 2\times 3\times ...\times r}$$

$$= \frac{n!}{r!(n-r)!}$$

It may be denoted that.

i) $nC_r = nC_{n-r}$

ii) $nC_n = 1$

iii) $nC_1 = 1$

iv) $nC_0 = 1$

v) $nC_r = \frac{nP_r}{r!}$

> In particular, $3C_2 = \frac{3\times 2}{1\times 2} = 3$
>
> $5C_4 = \frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4} = 5$
>
> (or)
>
> $5C_4 = 5C_{5-4} = 5C_1 = 5$

PROBLEM 1:

Number of ways of selecting 2 balls from 1 blue and 3 pink balls = $4C_2 = \frac{4\times 3}{1\times 2} = 6.$

PROBLEM 2:

8 bowlers and 10 batsmen are available how many cricket teams consisting of 5 bowlers can be formed?

SOLUTION:

Number of teams $= 8C_5 \times 10C_6$

PROBLEM 3:

A bag contains 6 white balls, 4 red balls and 10 black balls. Two balls are drawn at random. Find the probability that both of them are black.

SOLUTION:

$$n = 20C_2 = \frac{20\times 19}{1\times 2} = 190$$

$$m = 10C_2 = \frac{10\times 9}{1\times 2} = 45$$

$$\therefore \text{Re}\,quired\ \ probability = \frac{m}{n} = \frac{45}{190} = \frac{9}{38}$$

PROBLEM 4:

In tossing a coin what is the probability a head & tail.

SOLUTION:

Here, S = {H, T}, n(S) = 2.

Let A denotes be an event a head.

To get $P(A) = \frac{n(A)}{n(S)} = \frac{1}{2}$.

Let B denotes be an event to get tail

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{2}.$$

PROBLEM 5:

Two coins are toss simultaneously. Find the probability of

i) At least one head
ii) Exactly one head

SOLUTION:

S = {H, T}= {HH, HT, TH, TT}

n(S) = 4.

i) Let A be an event to get atleast one head

$$P(A) = \frac{3}{4}.$$

ii) Let B be an event to get exactly one head.

$$P(B) = \frac{2}{4} = \frac{1}{2}.$$

EXAMPLE 6:

A contains 7 white balls, 6 red balls and 5 black balls. 2 balls are drawn at random. Find the probability that they will be white ball.

SOLUTION:

Here total no.of balls = 18

Out of this 18 balls 2 white balls are drawn at $18C_2$ ways.

$$\therefore 18C_2 = \frac{18 \times 17}{1 \times 2} = 153$$

Here total no.of white balls = 7 out of this 7 white balls, 2 balls are drawn.

$$7C_2 = \frac{7 \times 6}{1 \times 2} = 21$$

$\therefore$ Exhaustive no.of cases = 153

$\therefore$ Favorable no.of cases = 21

$$P = \frac{F}{T} = \frac{21}{153} = \frac{1}{51}$$

EXAMPLE 7:

From a bag of 52 cards 3are drawn at random. Find the chance that they are a king, queen and an aces.

SOLTUION:

Out of 52 cards 3 cards can be drawn. $52C_3$ ways.

$$\therefore 52C_3 = \frac{52\times 51\times 50}{1\times 2\times 3} = 22100$$

$\therefore$ Exhaustive no.of cases = 22100.

Out of 4 kings can be drawn $4C_1$ ways.

$$\therefore 4C_1 = 4$$

Out of 4 queens 1 queen can be drawn

$$\therefore 4C_1 = 4$$

Out of 4 aces, 1 ace can be drawn $4C_1$ ways

$$\therefore 4C_1 = 4$$

$\therefore$ Favorable no.of cases = $4\times 4\times 4 = 64$.

$\therefore$ Required probability = $\frac{F}{T} = \frac{64}{22100}$

EXAMPLE 8:

There are 2 red, 3 green and 4 black balls of identical size an urn. 3 balls are drawn at random. Find the probability that,

i) They are of different colours.

ii) 2 are green and 1 is black.

iii) 2 are red and

iv) At least 1 is black

SOLUTION:

$$n = 9C_3 = \frac{9\times 8\times 7}{1\times 2\times 3} = 84$$

$$i)\, m = 2C_1 \times 3C_1 \times 4C_1$$

$$= 2\times 3\times 4 = 24$$

$$\therefore \text{Re}\,quired\ probability = \frac{m}{n} = \frac{24}{84} = \frac{2}{7}$$

$$ii)\, m = 3C_2 \times 4C_1$$

$$= \frac{3\times 2}{1\times 2}\times 4 = 12$$

$iii)\, 2\, are\, red\, means\, 2\, are\, red\, and\, 1\, is\, green\left(or\right) black.$

$$\therefore m = 2C_2 \times 7C_1$$

$$= \frac{2\times 1}{1\times 2} = 1\times 7 = 7.$$

$$\therefore \text{Re}quired\ probability = \frac{m}{n} = \frac{7}{84} = \frac{1}{12}.$$

$iv)\, Atleast\, 1\, is\, black\, means\, 1\, is\, black\, and\, 2\, are\ from\, red\, and\, green\left(or\right) 2\, are\, black\, and\, 1\, is\, red\, or\, green\left(or\right) 3\, black.$

$$\therefore m = 4C_1 \times 5C_2 + 4C_2 \times 5C_1 + 4C_3$$

$$= 4\times 10 + 6\times 5 + 4 = 74$$

$$\therefore \text{Re}quired\ probability = \frac{m}{n} = \frac{74}{84} = \frac{37}{42}.$$

MULTIPLICATION THEOREM OF PROBABILITY:(When two events are independent).

STATEMENT:

If A and B are any two events which are not independent.

$$P(A\cap B) = P(A).P\left(B/A\right); P(A) > 0.$$

$$(Or) \qquad = P(B).P\left(A/B\right); P(B) > 0.$$

Where $P\left(B/A\right)$ represents the conditional probability of the event 'B' when the event 'A'has already happened and $P\left(A/B\right)$ is the conditional probability of the happening of a given that B has the already happened.

PROOF:

Let 'S' be the sample space and let the events 'A' and 'B' are in the sample space then the definition of probability.

$$P(A) = \frac{n(A)}{n(S)} \longrightarrow 1$$

$$P(B) = \frac{n(B)}{n(S)} \longrightarrow 2$$

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)} \longrightarrow 3$$

For the conditional event A given B the favourable outcomes must be one of the sample points of B for the events $\left(A/B\right)$ The sample space is 'B' and out of n(B), sample points $n(A\cap B)$

certain the occurrence of the events 'A'.

$$P\left(A/B\right) = \frac{n\left(A \cap B\right)}{n\left(B\right)} \longrightarrow 4$$

From 3, we get

$$P\left(A \cap B\right) = \frac{n\left(A \cap B\right)}{n(S)} \times \frac{n(B)}{n(B)}$$

$$= \frac{n\left(A \cap B\right)}{n(B)} \times \frac{n(B)}{n(S)}$$

$$\therefore P(A \cap B) = P\left(A/B\right).P(B)$$

using 2 and 4

$$\therefore P(A \cap B) = P(B).P\left(A/B\right).$$

Similarly, for the conditional event B/A. The favorable outcomes must be one of the sample points in A.

Let us for the event B/A the sample points 'A' and out of n(A) sample points $n\left(A \cap B\right)$ certain the occurrence of the event 'B'.

$$Thus\ P\left(B/A\right) = \frac{n\left(A \cap B\right)}{n(A)} \rightarrow 5$$

$$From\ 3\ we get, P\left(A \cap B\right) = \frac{n\left(A \cap B\right)}{n(S)} \times \frac{n(A)}{n(A)}$$

$$= \frac{n\left(A \cap B\right)}{n(A)} \times \frac{n(A)}{n(S)}$$

$$\therefore P(A \cap B) = P(A).P\left(B/A\right)$$

Hence the proof.

PROBLEM 1:

If $P(A) = \frac{1}{3}; P(B) = \frac{1}{2}; P\left(A/B\right) = \frac{1}{6}$. Find $P(A) = \frac{1}{6}\ Find\ P\left(B/A\right) P\left(B/\bar{A}\right)$.

SOLUTION:

$$P\left(A/B\right) = \frac{P(AB)}{P(B)}$$

$$P(AB) = P\left(A/B\right)P(B) = \frac{1}{2}.\frac{1}{6} = \frac{1}{12}$$

$$P\left(B/A\right) = \frac{P(AB)}{P(A)} = \frac{1/12}{1/3} = \frac{3}{12} = \frac{1}{4}$$

$$P(B) = P(AB) + P(\bar{A}B) = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

$$P\left(B/\bar{A}\right) = \frac{P(\bar{A}B)}{P(\bar{A})} = \frac{5/12}{2/3} = \frac{5}{12} \times \frac{3}{2} = \frac{5}{8}$$

PROBLEM 2:

From a bag containing 3 white and 5 red balls of identical size, two balls are drawn at random one after another. Find the probability of getting white balls in both the draws if the draws are,

i) Without replacement and

ii) With replacement

SOLUTION:

Let A – white ball in the first draw

B – white ball in the second draw

Required to find $P(A \cap B)$.

$$i)\, P(A) = \frac{3}{8}, P\left(B/A\right) = \frac{2}{7}.$$

$$\therefore P(A \cap B) = P(A).P\left(B/A\right)$$

$$= \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

$$ii)\, P(A) = \frac{3}{8}, P(B) = \frac{3}{8}.$$

$$\therefore P(A \cap B) = P(A).P(B)$$

$$= \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$$

PROBLEM 3:

If $P(A) = 0.4$; $P(A) = 0.4$ and $P(A \cup B) = 0.6$, Find. $(i)\, P\left(A/B\right)$ and $(ii)\, P\left(B/A\right)$, are A and B independent.

SOLUTION:

$$From\, P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(A \cap B) = 0.4 + 0.3 - 0.6 = 0.1$$

$$i)\, From\, P(A \cap B) = P(B).P\left(A/B\right),$$

$$P\left(A/B\right) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3}.$$

$$ii)\, Similarly, P\left(B/A\right) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.4} = \frac{1}{4}.$$

It can be concluded that A and B are not independent from any one of the following.

$$P(A \cap B) = P(A)\,P(B)\,(Or)\,P\left(A/B\right) \neq P(A)\,(Or)\,P\left(B/A\right) \neq P(B).$$

PROBLEM 4:

A lot contains 10 items of which 3 are defective. Three items are chosen at random from the lot one after another. Find the probability that all the three are defective if the draws are

i) With replacement and ii) without replacement

SOLUTION:

Let A, B and C be defective items in the first, second and third draws. Required find $P(A\cap B\cap C)$.

$$i)\, P(A)=\frac{3}{10};\, P(B)=\frac{3}{10}\, and\, P(C)=\frac{3}{10}.$$

$$\therefore P(A\cap B\cap C)=P(A).P(B).P(C)$$

$$=\frac{3}{10}\times\frac{3}{10}\times\frac{3}{10}=\frac{27}{1000}$$

$$ii)\, P(A)=\frac{3}{10},\, P\left(B/A\right)=\frac{2}{9}\, and\, P\left(C/A\cap B\right)=\frac{1}{8}$$

$$\therefore P(A\cap B\cap C)=P(A)\,P\left(B/A\right).P\left(C/A\cap B\right)$$

$$=\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}=\frac{1}{120}.$$

PROBLEM 5:

If $P(A)=\frac{2}{8}, P(B)=\frac{5}{8}\, and\, P(A\cap B)=\frac{3}{4}$. Then A & B are independent is it true?

SOLUTION:

$$P(A)=\frac{2}{8}, P(B)=\frac{5}{8}$$

$$P(A).P(B)=\frac{2}{8}\times\frac{5}{8}=\frac{10}{64}\; we\, are\, given\, that,\, P(A\cap B)=\frac{3}{4}.$$

$$\Rightarrow P(A\cap B)\neq P(A).P(B)$$

CONDITIONAL PROBABILITY:

If A and B are any two events, then the conditional probability of event A then the B has already happened. This denoted by,

$$P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P\left(B\right)}=\frac{P(A).P(B)}{P(B)}\; where\, P(B)>0.$$

The conditional probability of event B then the event A has already happened. It is denoted by,

$$P\left(B/A\right)=\frac{P\left(A\cap B\right)}{P\left(A\right)}=\frac{P(A).P(B)}{P(A)}\; where\, P(A)>0.$$

PROBLEM 1:

If P(A) = 0.4, P(B) = 0.3 and $P(A\cap B)=0.2$ find $P\left(A/B\right)\, and\, P\left(B/A\right)$.

SOLUTION:

If P(A) = 0.4, P(B) = 0.3, $P(A\cap B)=0.2$

$$P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P\left(B\right)}=\frac{0.2}{0.3}=\frac{2}{3}=0.66$$

$$P\left(B/A\right)=\frac{P\left(A\cap B\right)}{P\left(A\right)}=\frac{0.2}{0.4}=\frac{1}{2}=0.50$$

PROBLEM 2:

$$If\ P(A)=\frac{2}{5}, P(B)=\frac{1}{3}, P(A\cup B)=\frac{1}{2}. Find\ P\left(A/B\right) and\ P\left(B/A\right).$$

SOLUTION:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{1}{2}=\frac{2}{5}+\frac{1}{3}-P(A\cap B)$$

$$P(A\cap B)=\frac{2}{5}+\frac{1}{3}-\frac{1}{2}$$

$$P(A\cap B)=\frac{12+10-15}{30}=\frac{7}{30}$$

$$\therefore P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P(B)}=\frac{7/30}{1/3}=\frac{7}{30}\times\frac{3}{1}=\frac{7}{10}$$

$$\therefore P\left(B/A\right)=\frac{P\left(A\cap B\right)}{P(A)}=\frac{7/30}{2/5}=\frac{7}{30}\times\frac{5}{2}=\frac{7}{12}$$

PROBLEM 3:

A die is thrown twice and the sum of the appearing is observed to be. What is the conditional probability then the no.4 appeared at least once.

SOLUTION:

Let S be a sample space. Given a die is thrown twice.

$$\therefore S=\{(1,1),(1,2),......,(6,6)\}=36$$
$$n(S)=36$$

Let A be an event of getting sum of the numbers 6.

$$A=\{(1,5),(2,4),(3,3),(4,2),(5,2)\}$$
$$n(A)=5$$
$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{5}{36}$$

Let B be an event of getting the take the no.of 4 appeared at least once.

$$B=\begin{Bmatrix}(1,4)\,(2,4)\,(3,3)\,(4,2)\,(5,1)\\(4,1)\,(4,2)\,(4,4)\,(4,5)\,(4,6)\end{Bmatrix}=11$$

$$n(B) = 11, P(B) = \frac{11}{36}.$$

$$Now, P\left(B/A\right) = \frac{P(A \cap B)}{P(A)}$$

$$n(A \cap B) = \{(2,4)(4,2)\} = 2$$

$$n(A \cap B) = 2$$

$$\therefore P(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

$$\therefore P\left(B/A\right) = \frac{1/18}{5/36} \Rightarrow \frac{1}{18} \times \frac{36}{5} \Rightarrow \frac{2}{5}$$

EXAMPLE 4:

$$If\ P(A) = \frac{3}{4}, P(B) = \frac{4}{5}, P(A \cup B) = \frac{5}{6}. Find\ P\left(A/B\right) and\ P\left(B/A\right).$$

SOLUTION:

$$P\left(A/B\right) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$\frac{5}{6} = \frac{3}{4} + \frac{4}{5} - P(A \cap B).$$

$$P(A \cap B) = \frac{3}{4} + \frac{4}{5} - \frac{5}{6}$$

$$= \frac{45 + 48 - 50}{60} = \frac{93 - 50}{60} = \frac{43}{60}$$

$$\therefore P\left(A/B\right) = \frac{P(A \cap B)}{P(B)} = \frac{43/60}{4/5} = \frac{43}{60} \times \frac{5}{4} = \frac{43}{60}$$

$$\therefore P\left(B/A\right) = \frac{P(A \cap B)}{P(A)} = \frac{43/60}{3/4} = \frac{43}{60} \times \frac{4}{3} = \frac{43}{45}$$

BAYE'S THEOREM:

THEOREM: If $E_1, E_2, \ldots E_n$ are exhaustive and mutually disjoint events with $P(E_i) \neq 0$ (i = 1, 2,...n) then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that P(A) > 0,

$$P\left(E_i/A\right) = \frac{P(E_i)\,P\left(A/E_i\right)}{\sum_{i=1}^{n} P(E_i)P\left(A/E_i\right)}$$

PROOF:

$$A \subset \bigcup_{i=1}^{n} E_i$$

$$A = A \cap \left[\bigcup_{i=1}^{n} E_i \right]$$

$$= \bigcup_{i=1}^{n} (A \cap E_i)$$

$$\therefore P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i) \right]$$

$$= \sum_{i=1}^{n} P(A \cap E_i) \text{ by addition theorem}$$

$$= \sum_{i=1}^{n} P(E_i) P\left(A/E_i \right) \text{ by multiplication theorem}$$

$$\text{From } P(A \cap E_i) = \frac{P(A \cap E_i)}{P(A)}$$

$$P\left(E_i/A \right) = \frac{P(E_i) P\left(A/E_i \right)}{\sum_{i=1}^{n} P(E_i) P\left(A/E_i \right)} \text{ by multiplication theorem}$$

PROBLEM:

A manufacturing firm produces pipes in two plants, I & II, with daily production of 1500 & 2000 pipes respectively. The fraction of defective pipes produce by the two plants is 0.006 and 0.008 respectively. If a pipe selected at random from the day's production is found to be defective, what is the probability that it has come from plant I & II ?.

SOLUTION:

Let E_1 – A pipe is produced in plant I

E_2 – A pipe is produced in plant II

A – A pipe is defective.

Given		*To be found*
$P(E_i)$	$P\left(A/E_i \right)$	$P(E_i) P\left(A/E_i \right)$
$P(E_1) = \frac{1500}{3500} = 0.4286$	$P\left(A/E_1 \right) = 0.006$	$P(E_1) P\left(A/E_1 \right) = 0.0025716$
$P(E_2) = \frac{2000}{3500} = 0.5714$	$P\left(A/E_2 \right) = 0.008$	$P(E_2) P\left(A/E_2 \right) = 0.0045712$
$\sum_{i=1}^{n} P(E_i) = 1.0000$	$\sum_{i=1}^{n} P\left(A/E_i \right) = 0.014$	$\sum_{i=1}^{n} P(E_i) P\left(A/E_i \right) = 0.0071428$

Probability that it has come from plant I given that a pipe is defective,

$$P\left(E_1/A\right) = \frac{P(E_1)P\left(A/E_1\right)}{\sum_{i=1}^{n} P(E_i)P\left(A/E_i\right)} = \frac{0.0025716}{0.0071428} = 0.3600$$

Probability that it has come from plant II given that a pipe is defective,

$$P\left(E_2/A\right) = \frac{P(E_2)P\left(A/E_2\right)}{\sum_{i=1}^{n} P(E_i)P\left(A/E_i\right)} = \frac{0.0045712}{0.0071428} = 0.6400$$

$$Note: \sum_{i=1}^{n} P\left(E_i/A\right) = P\left(E_1/A\right) + P\left(E_2/A\right) = 1.0000$$

PROBLEM:

In a bolt factory, machines M_1, M_2, and M_3 manufacture respectively 25, 35 & 40 percent of the total defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured in the machine M_3?

SOLUTION:

L et E_1, E_2 & E_3 be the events of a bolt is manufactured by machines M_1, M_2 & M_3 respectively A be a bolt is defective.

Given		*To be found*
$P(E_i)$	$P\left(A/E_i\right)$	$P(E_i)\,P\left(A/E_i\right)$
$P(E_1) = \frac{25}{100} = 0.25$	$P\left(A/E_1\right) = \frac{5}{100} = 0.05$	$P(E_1)\,P\left(A/E_1\right) = 0.0125$
$P(E_2) = \frac{25}{100} = 0.35$	$P\left(A/E_2\right) = \frac{4}{100} = 0.04$	$P(E_2)\,P\left(A/E_2\right) = 0.014$
$P(E_3) = \frac{40}{100} = 0.40$	$P\left(A/E_3\right) = \frac{2}{100} = 0.02$	$P(E_3)\,P\left(A/E_3\right) = 0.0080$
$\sum_{i=1}^{n} P(E_i) = 1.00$	$\sum_{i=1}^{n} P\left(A/E_i\right) = 0.11$	$\sum_{i=1}^{n} P(E_i)\,P\left(A/E_i\right) = 0.0345$

$$\text{Re}\,quired\ probability\ P\left(A/E_3\right) = \frac{P(E_3)\,P\left(A/E_3\right)}{\sum_{i=1}^{n} P(E_i)\,P\left(A/E_i\right)} = \frac{0.0080}{0.0345} = \frac{16}{19}$$

UNIT – IV

Almost all industrial and economic situations are concerned with the problem of planning various activities. Linear programming determines an optimum schedule of these activities which are subject to resource constraints. Linear programming is one of the most widely used and best understood methods in O.R. The linear programming problem (L.P.P) was first developed and made use of in 1947 by the American researchers, George Dantzig and his associates for solving military planning problems of U.S. Air Force. George Dantzig later suggested this method for solving business and industrial problems. He also developed the simplex method to solve the Linear programming problems. Simplex method is the most useful and powerful mathematical method to solve L.P.P as to be seen later.

LINEAR PROGRAMMING PROBLEMS ARE TWO TYPES:

1. Maximization problems such as maximizing the profit.
2. Minimization problems such as minimizing the cost.

Some examples of LPP, their mathematical formulation and their solution by graphical and analytical methods are considered. Simplex method is the analytical method.

Programming means planning. All the relationships between the variables considered in these problems are linear. Hence the name Linear Programming Problem(L.P.P)

The general form of an LPP is as follows.

Maximize (or) Minimize Z = $C_1x_1 + C_2x_2 + ... + C_nx_n \longrightarrow 1$

Subject to the constraints,

$a_{11}x_1+a_{12}x_2+....+a_{1n}x_n (\le, =, \ge)\ b_1$

$a_{21}x_1+a_{22}x_2+....+a_{2n}x_n (\le, =, \ge)\ b_2$

……..

…….. $\longrightarrow 2$

……..

$a_{m1}x_2+a_{m2}x_2+....+a_{mn}x_n (\le, =, \ge)\ b_m$

and $x_1, x_2....x_n \ge 0 \longrightarrow 3$

DEFINITION OF L.P.P(LINEAR PROGRAMMING PROBLEM)

LPP constitutes a set of mathematical methods. Specially designed for the modeling and the solution of certain kinds of constrained optimization problems.

LPP is one of the most widely used and best understood methods in Operation Research.

(OR)

LPP is technique for determining an optimum allocation of limited resources to meet given objectives.

The term linear means that all the variables occurring in the objective function and the constraints of the first degree in the problems under consideration and the term programming means the process of determining a particular course of action.

What is meant by LPP?

LPP deals with the optimization (max (or) min) of a function of decision variables (The variables whose values determine the solutions of problem) known as. Objective function, subject to a set of simultaneous Linear equation (or) inequalities known as constraints the term linear means that all the variables occurring in the objective function and the constrains are of the first degree in the problems under consideration and the term linear programming means the process of determining a particular course of action.

CHARACTERISTICS OF AN L.P.P:

1. Regarding the symbol used in the general model each constraints can take either $\leq$ or $=$ or $\geq$.
2. The decision variables X_j's should take non – negative values only.
3. The values C_j; b_j; a_{ij} (for $i = 1,2,\ldots m$; $j = 1,2,\ldots n$). can be got from the given information. These values are called parameters & assumed to be fixed constant.

MATHEMATICAL FORMULATION OF THE PROBLEM:

If X_j ($j = 1,2,\ldots,n$) are the n decision variable of the problem and if the system is subject to m constraints the general mathematical model can be written in the form;

Optimize $Z = f(x_1,x_2,\ldots,x_n)$.

Subject to g_i $(x_1,x_2,\ldots,x_n) \leq, =; \geq b_i$, $(i = 1,2,\ldots,m)$ and $x_1,x_2,\ldots,x_n \geq 0$,(Called the non – negativity restrictions (or) constraints)

FORMULATION CONSISTS OF THE FOLLOWING FOUR STEPS:

Step 1: Identify the variables (which are also called Decision variables).

Step 2: Write down the objective function to be optimized (maximized (or) minimized) as a Linear function of the decision variables after identifying the cost coefficients.

Step 3 : Note down the constraints on the basis of the conditions specified regarding availability of time, demand for the product, etc. constraints a_{ij}'s and b_i's are involved.

Step 4 : Mention the non – negativity restrictions which imply that the decision variables cannot be negative.

Problem 1:

A company manufactures three types of product which use metals, platinum, gold. Due to shortage of these metals the government neglects the amount that may be used per data. The relevant data with respective supplied, requirements and profits are below.

Product	Platinum required	Gold required	Profit per unit
A	2	3	500
B	4	2	600
C	6	4	1200
	160	**120**	

Daily allotment of platinum &gold is 160 gms & 120 gms respectively. How should the company divide the supply of scale metals? Formulate LPP.

Solution:

Let the company produces

X_1 - units of product A

X_2 - units of product B

X_3 - units of product C

Daily profits are Rs.500,600,1200 per unit of product A,B&C

LPP becomes

Maximize $Z = 500X_1+600X_2+1200X_3$

Subject to the constraints

$2X_1+4X_2+6X_3 \leq 160$

$3X_1+2X_2+4X_3 \leq 120$ and

$X_1,X_2,X_3 \geq 0$

PROBLEM 2:

A company makes three products X,Y,Z which passes through three departments drill, lathe, assembly. The hours available in each department hours required for each product are given below.

Product	Time required in hours			Profit
	Drill	Lathe	Assembly	
X	3	3	8	9
Y	6	5	10	15
Z	7	4	12	20
Hours available	210	240	260	

Solution:

The objective function is

Maximize $Z = 9X_1+15X_2+20X_3$

Subject to

$3X_1+6X_2+7X_3 \leq 210$

$3X_1+5X_2+4X_3 \leq 240$

$8X_1+10X_2+12X_3 \leq 260$ and

$X_1 , X_2 , X_3 \geq 0$

PROBLEMS 3:

A person requires at least 10,12,12 units of the chemicals A,B,C respectively for his garden. A liquid product contain 1,2,4 units of A,B,C respectively per jar. A dry product contains 5,2,1 unit of A,B,C respectively. A liquid product sells for Rs.3 per jar and a dry product sells for Rs.2 per contain. Formulate LPP.

Solution:

Maximize $Z = 3X_1+2X_2$

Subject to

$X_1+5X_2 \geq 10$

$2X_1+2X_2 \geq 12$

$4X_1+X_2 \geq 12$ and

$X_1 , X_2 \geq 0$

GRAPHICAL METHOD :

- LPP involving only two variables can be solved by graphical method.
- It provides the pictorial representation of problems and its solution
- This method is easy to use and simple to understand
- Graphical method is not a powerful tool of LPP

MEANING:

Graphical method gives the solution of an LPP in which they are only two variables. The region in the graph sheet which satisfies all the constraints including the non – negative restrictions is called the solution space.

They are four types of graphical method.

1. No solution
2. Unique solution
3. Infinite number solution
4. Unbounded solution

PROCEDURE FOR GRAPHICAL METHOD:

Given a L.P.P, optimize Z = f(x) subject to the constraints g(x) $\leq$, = , $\geq$ b_j and the non – negativity restrictions $x_i \geq 0$, i = 1,2,…; j = 1,2,…m.

STEP 1: Draw x_1 and x_2 axis (which are mutually perpendicular) on a graph sheet.

STEP 2: Draw a line and identify the region connected with it corresponding to each constraint.

STEP 3: Identify the solution space which is the region that is common to all the constraints including the non – negativity restrictions.

STEP 4: Find the value of Z at each vertex of the solution space.

STEP 5: Identify the optimum solution. The solutions obtained by the graphical method as well as the simplex method are of the following four different kinds.

1.NO SOLUTION:

When there exists no solution space, there is no solution to the given L.P.P. This is when there is no common region corresponding to all the constraints.

2. UNIQUE SOLUTION:

There is only one optimum solution for certain L.P.P.

3.INFINITE NUMBER OF SOLUTIONS:

Certain Linear programming problems have infinite number of optimum solutions. In such cases, the optimum value of Z is the same but the values of the decision variables x_1, x_2,…differ.

4.UNBOUNDED SOLUTION:

In some Linear Programming problems, the maximum value of Z occurs at the point at infinity only.

PROCEDURE FOR GRAPHICAL METHOD:

Step 1: Identify the decision variables, objective and restrictions.

Step 2: set up the mathematical formulation of the problem

Step 3: plot a graph representing all the constraints of the problem and identify the feasible region. The feasible region is the intersection of all the regions represented by constraints of the problem and is restricted to the first quadrant.

Step 4: The feasible region may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step 5: Find out the value of the objective function at each corner (solution) point determined in step 4.

Step 6: select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

PROBLEM 1:

A machine producing either product A or B can produce A by using 2 units of chemicals and one unit of compound and can produce B by using 1 unit of chemicals and 2 units of compound. Only 800 units of chemicals and 1000 units of compound are available. The profit per unit of A&B are respectively Rs.30 and Rs. 20. Formulate LPP and solve by graphically.

Solution:

Maximize $Z = X_1+3X_2$

Subject to the constraints $2X_1+X_2 \leq 20$

$X_1+2X_2 \leq 20$

$X_1, X_2 \geq 0$

$2X_1+X_2 = 20$ …….. 1

$X_1+2X_2 = 20$ ……..2

Let $X_1=0$ in 1

$0 + X_2 = 20$

$X_2= 20$

Let $X_2=0$ in 1

$2X_1+0 = 20$

$X_1 = 10$

X1	X2
0	20
10	0

Let X_1= 0 in eqtn 2

(0) + $2X_2$ = 20

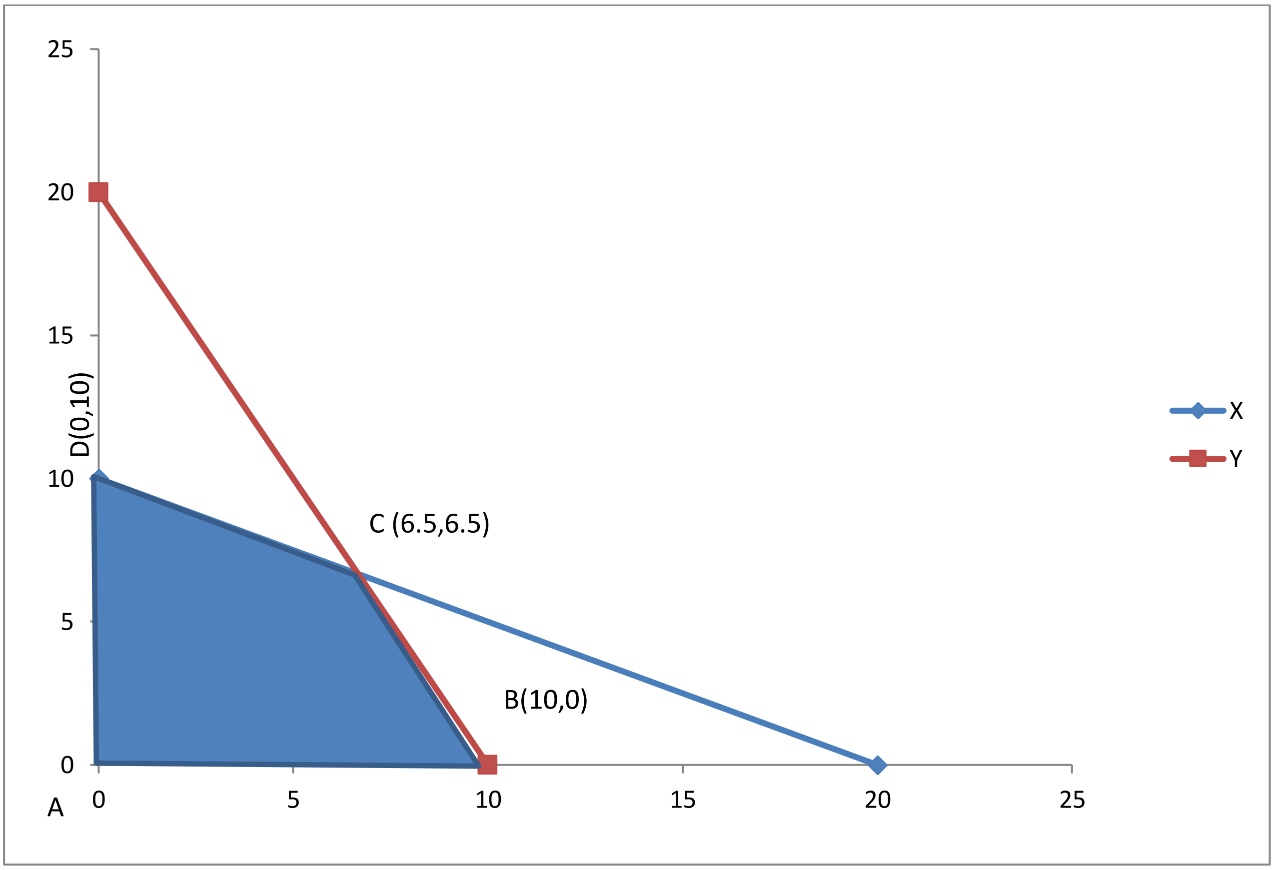
X_2 = 10

Let X_2=0 in eqtn 2

X_1 + 2(0) = 8

X_1= 20

X1	X2
0	10
20	0



Extreme points	Coordinates	Objective fn
A	(0,0)	Z=0+3(0)=0
B	(10,0)	Z=10+3(0)=10
C	(6.5,6.5)	Z=6.5+3(6.5)=26
D	(0,10)	Z=0+3(10)=30

Maximize Z = 30 at X_1 = 0 and X_2 = 10.

PROBLEM 2:

Solve the following LPP by graphical method.

Maximize $Z = 3X_1+4X_2$

Sub to the constraints $4X_1+2X_2\le 80$

$2X_1+5X_2\le 180$

Solution:

$4X_1+2X_2\le 80$ ……….eqn. 1

$2X_1+5X_2\le 180$ ………..eqn. 2

Let $X_1=0$ in eqn. 1

$4(0)+2X_2=80$

$2X_2=80$

$X_2=80/2=40$

Let $X_2=0$ in eqn. 1

$4X_1+2(0)=80$

$4X_1=80$

$X_1=80/4=20$

X1	X2
0	40
20	0

Let $X_1=0$ in eqn. 2

$2(0)+5X_2=180$

$5X_2=180$

$X_2=180/5$

$X_2=36$

X1	X2
0	36
90	0

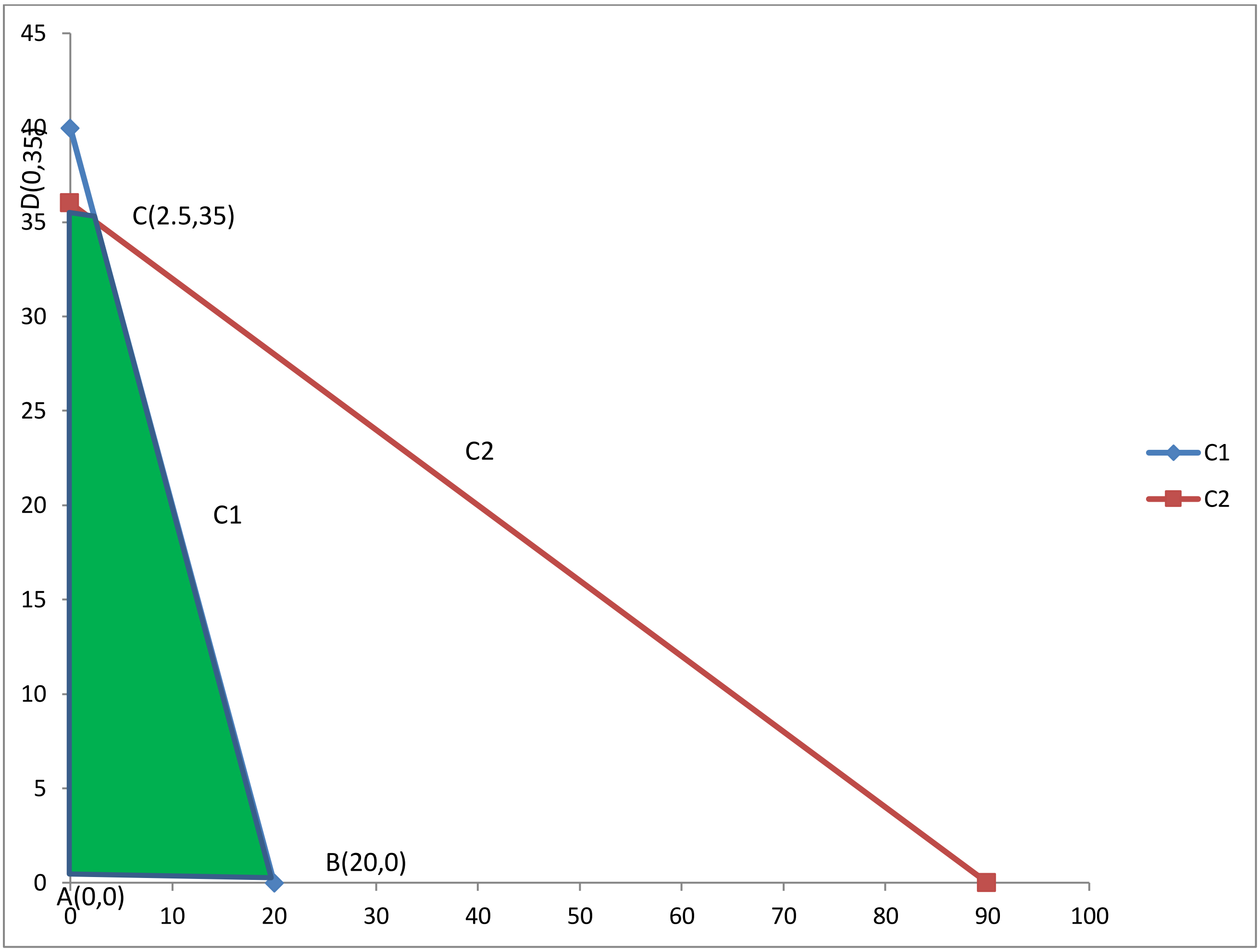
Let $X_2=0$ in eqn. 2

$2X_1+5(0)=180$

$2X_1=180$

$X_1=180/2$

$X_1=90$



Extreme points	Coordinates	Objective fn
A	(0,0)	Z=3(0)+4(0)=0
B	(0,20)	Z=3(0)+4(20)=60
C	(2.5,3.5)	Z=3(2.5)+4(3.5)=147.5
D	(0,35)	Z=3(0)+4(35)=144

Max value = 147.5 when $X_1 = 2.5$, $X_2 = 3.5$

PROBLEM 3:

Solve the LPP by graphical method.

$$\text{Maximize } Z = 3X_1+5X_2$$

Subject to the constraints $X_1+X_2 \leq 2$

$$2X_1+2X_2 \geq 8$$

$$X_1, X_2 \geq 0$$

Solution:

$X_1+X_2 = 2$ …….. eqtn 1

$2X_1+2X_2 = 8$ ……...eqtn 2

Let $X_1=0$ in eqtn 1

$0 + X_2 = 2$

Let $X_2=0$ in eqtn 1

$X_1 = 2$

X1	X2
0	2
2	0

Let $X_1= 0$ in eqtn 2

$2 (0) + 2X_2 = 8$

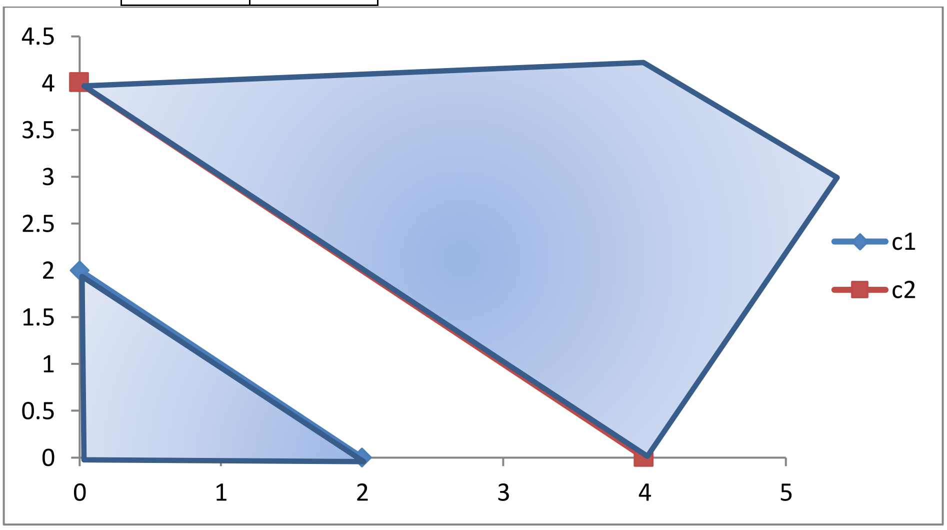
$X_2 = 4$

Let $X_2=0$ in eqtn 2

$2X_1 + 2(0) = 8$

$X_1= 4$

X1	X2
0	4
4	0



There is no common region and no solution to this LPP

SIMPLEX METHOD

INTRODUCTION

- The simplex method is also called simplex technique.
- It was developed by an American mathematician G.B. Dantzing
- The simplex method is a technique of solving linear programming problems by obtaining a feasible solution and by iterative procedure, improving this solution until the optimal solution is reached.
- The simplex method we have to convert the inequality constraints to equality constraints.
- If the inequality is of the form ≤, by adding a positive variable, referred to as slack variables, to the LHS of the inequality to convert into equality.
- If the inequality is of the form ≥, a positive variable is subtracted from the LHS of the constraints to convert into equality.

DEFINE:

The simplex method is also called simple techniques. It was developed by G.B. Dantzing. An American mathematical. It is a powerful iterative method.

DEFINE: SOLUTION

A set of values x_1, $x_2,\ldots,x_n$ which satisfies the constraints of the LPP is called its solution.

FEASIBLE SOLUTION:

A feasible solution to a LPP which satisfies the non – negativity restrictions of the LPP is called its feasible solution.

OPTIMUM (OR) OPTIMAL SOLUTION:

Any feasible solution which optimizes (max (or) mini). The objective function of the LPP is called its optimum solution (or) optimal solution.

BASIC SOLUTION:

Given a system of m linear equations with n variables (m < n). The solution obtained by setting (n – m) variables equal to zero and solving for the remaining m variables is called a basic solution.

BASIC AND NON – BASIC VARIABLES:

- The m variables are called Basic variables and they form the Basic solution.
- The (n – m) variables which are put to zero are called as non – Basic variables.

BASIC FEASIBLE SOLUTION:

A Basic feasible solution which satisfies the non – negativity restrictions (or) constraints is called Basic feasible solution.

A Basic feasible solution of two types.

i) Degenerate solution   ii) Non – degenerate solution

i) <u>Degenerate solution:</u>

A Basic feasible solution is called degenerate if atleast one basic variable possesses zero value.

ii) <u>Non – degenerate solution:</u>

A Basic feasible solution is called non – degenerate if all the basic variables are non – zero & positive.

UNBOUNDED SOLUTION:

A solution which increases (or) decreases the value of the objective function indefinitely is called an unbounded solution.

SLACK VARIABLES:

Let the constraints of a general LPP be.

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i (i = 1, 2, ..., k) \longrightarrow 1$$

Then the non – negative variables s_i which are introduced to convert the inequalities (1) to the equalities.

$\sum_{j=1}^{n} a_{ij} x_j + S_i = b_j (i = 1, 2, ..., k)$ are called slack variables.

SURPLUS VARIABLES:

If the constraints of a general in LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i (i = k, k+1, ...) \longrightarrow 1$$

Then the non – negative variables S_i which are introduced to convert the inequalities (1) to the equalities.

$\sum_{j=1}^{n} a_{ij} x_j - S_i = b_i (i = k, k+1, ...)$ are called surplus variables.

SIMPLEX ALGORITHM:

Step 1: If the objective function is minimization, then convert in to maximization by using

min z = - max(-z).

Step 2: If any b_i is negative, multiply this in equation by -1 so as to get all b_i's are non – negative.

Step 3: Convert all constraints into equations by introducing slack (or) surplus variables in the constraints. Put the cost of these variables to zero and if all $Z_j - C_j$ are the cost of the basic variables are zero. The IBFS is,

$x_1 = 0, x_2 = 0, x_3 = 0, S_1 = 3, S_2 = 2.$

PROCEDURE FOR SOLVING BY SIMPLEX METHOD:

Step 1: check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization by

Minimize Z = - Maximize (-Z)

Step 2: convert the inequality constraints into equality constraints by introducing the slack variables.

step 3: introduce zero co-efficient to the slack variables in the objective function.

i.e.,$4x_{1+}7x_2 + 0s_1 + 0s_2$

step 4: The general form of the simplex table is

$c_j \quad (c_1 \quad c_2 \quad \ldots\ldots\ldots \quad 0)$

c_b	Y_b	X_b	X_1	X_2	X_3	S_1	S_2	S_3	RATIO
			←	Body matrix	→	←	Unit matrix	→	

$$Z_i - C_j$$

$$Z_0 \; Z_{1-}C_1 \; Z_{2-}C_2 \ldots\ldots\ldots\ldots$$

Here $C_j$denotes the co-efficients of the variables in the objective function.

C_b denotes the co-efficients of the basic variables in the objective function.
Y_b denotes the basic variables.

X_b denotes the values of the basic variables.

$(Z_i - C_j)$ denotes the net evaluations (or) index for each column.

Step5 : compute the net evaluations

(i) If all $Z_i - C_j \geq 0$ i.e., positive , then the current basic feasible solution $X_b$is optimal.
(ii) If any one of the $Z_i - C_j \leq 0$ i.e., negative, then go to the next step.

Step 6: (To find the entering variable)

The entering variable is the non-basic variable corresponding to the most negative. The entering variable is known as pivot column (or) key column, which is shown market with an arrow at the bottom.

Step 7: (To find the leaving variable)

Compute the ratio = Min $\left\{\frac{x_b}{\text{entering variable}}\right\}$ i.e., Min +ve

Then find out the minimum +ve value. i.e., the leaving variable row is called pivot row and the intersection of the pivot row and pivot column is called pivot element (or) key element.

Step 8:

Drop the leaving variable and introduce the entering variable. Convert the pivot element to be 1 and other elements to zero by making use of

(i) New pivot equation = $\frac{\text{old pivot equation}}{\text{pivot element}}$

(ii) New equation = old equation – corresponding column of co-eff X new pivot equation

Step 9:

Go to step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

PROBLEM 1:

Solve the following LPP by simplex method.

Max Z= $4X_1+7X_2$

Sub to the constraints

$4X_1+3X_2 \leq 12$

$3X_1+4X_2 \leq 12$

$X_1, X_2 \geq 0$

Solution:

Introducing slack variables we get,

$4X_1+3X_2 +1S_1+0S_2= 12$

$3X_1+4X_2 +0S_1+1S_2 = 12$

Max Z= $4X_1+7X_2+0S_1+0S_2$

Sub to the constraints

$4X_1+3X_2+1S_1+0S_2= 12$

$3X_1+4X_2+0S_1+1S_2 = 12$

C_B Y_B, X_B, C_j, Z_j, ,Z_j-C_j

INITIAL ITERATION:

		C_j	4	7	0	0	Ratio
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	
0	S_1	12	4	3	1	0	12/3=4
0	S_2	12	3	**4**	0	1	12/4=3
	Z_j	0	0	0	0	0	
	Z_j-C_j		-4	-7	0	0	

S_2 leaves the basis and X_2 enter the basis.

New X_2 = Pivot row / pivot element

New S_1 = Old S_1 – 4 * new pivot eqtn

FIRST ITERATION:

		C_j	**4**	**7**	**0**	**0**
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**S_1**	**S_2**
0	**S_1**	3	1.75	0	1	-0.75
7	**X_2**	3	0.75	1	0	0.25
	Z_j	21	5.25	7	0	1.75
	Z_j- C_j		1.25	0	0	1.75

Max Z= $4X_1+7X_2$ $X_1 = 0, X_2 = 3$

Z= 4(0) + 7(3) =21

PROBLEM 2:

Solve the following LPP by simplex method.

Max Z = $X_1+4X_2+5X_3$

Sub to the constraints $3X_1+3X_2 \leq 22$

$X_1+2X_2+3X_3 \leq 14$

$3X_1+2X_2 \leq 14$,

$X_1,X_2,X_3 \geq 0$

Solution:

By introducing slack variables we get,

Max Z = $X_1+4X_2+5X_3+0S_1+0S_2+0S_3$

Subject to the constraints

$3X_1+3X_2+1S_1+0S_2+0S_3= 22$

$X_1+2X_2+3X3+0S_1+1S_2+0S_3 = 14$

$3X_1+2X_2+0S1+0S_2+1S_3 =14$

And $X_1, X_2,X_3 ,S_1,S_2,S_3 \geq 0$

INITIAL ITERATION:

		C_j	**1**	**4**	**5**	**0**	**0**	**0**	**Ratio**
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	
0	**S_1**	22	3	3	0	1	0	0	22/0=∞
0	**S_2**	14	1	2	3	0	1	0	14/3=4.67
0	**S_3**	14	3	2	0	0	0	1	14/0=∞
	Z_j	0	0	0	0	0	0	0	
	Z_j- C_j		-1	-4	-5	0	0	0	

New X_3 = Pivot eqtn/3

FIRST ITERATION:

		C_j	**1**	**4**	**5**	**0**	**0**	**0**	**Ratio**
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	
0	**S_1**	22	3	3	0	1	0	0	22/3=7.33
5	**X_3**	4.67	0.3	0.67	0	0	1	0	4.67/0.67 =6.97
0	**S_3**	14	3	2	0	0	0	1	14/2=7
	Z_j		1.5	3.35	5	0	1.5	0	
	Z_j- C_j		0.5	-0.05	0	0	1.5	0	

New X_2 = Old S_3 / 2

New X_3 = Old X_3 – 0.67(New X_2)

New S_1 = Old S_1 – 3(New X_2)

SECOND ITERATION:

		C_j	1	4	5	0	0	0
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
0	S_1	1	-1.5	0	0	1	0	-1.5
5	X_3	-0.02	0.7	0	1	0	0.3	-0.34
4	X_2	7	1.5	1	0	0	0	0.5
	Z_j	27.9	2.45	4	5	0	1.5	3.3
	Z_j- C_j		1.45	0	0	0	1.5	3.3

Max $Z = X_1+4X_2+5X_3$ $X_1 = 0, X_2 = 7 , X_3 = -0.02$

$Z = 0+4(7)+5(-0.02)$

$Z = 0+28-0.1$

Max Z = 27.9

PROBLEM 3:

Maximize $Z = 21X_1+15X_2$

Subject to $-X_1-2X_2 \leq 6$

$4X_1+3X_2 \leq 12$ and

$X_1 , X_2 \geq 0$

Solution:

By introducing slack variables we get

Maximize $Z = 21X_1+15X_2+0S_1+0S_2$

Subject to $-X_1-2X_2+S_1 = 6$

$4X_1+3X_2 +S_2 = 12$ and

$X_1, X_2, S_1, S_2 \geq 0$

INITIAL ITERATION:

		C_j	21	15	0	0	Ratio
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	
0	S_1	6	1	2	1	0	6
0	S_2	12	4	3	0	1	3
	Z_j	0	0	0	0	0	
	Z_j- C_j		-21	-15	0	0	

S_2 leaves the basis and X_1 enters the basis.

New X_1 = Old S_2/ 4

New S_1 = Old S_1 - New X_1

FIRST ITERATION:

		C_j	21	15	0	0
C_B	Y_B	X_B	X_1	X_2	S_1	S_2
0	S_1	3	0	1.25	1	-0.25
21	X_1	3	1	0.75	0	0.25
	Z_j	63	21	15.75	0	5.25
	Z_j- C_j		0	0.75	0	5.25

Maximize Z = = $21X_1+15X_2$ where X_1 = 3, X_2

Z = 21(3) + 15(0) = 63

PROBLEM 4:

Solve the following LPP by simplex method

Maximize $Z=X_1+2X_2+X_3$

Subject to the constraints

$2X_1+X_2-X_3 \leq 2$

$-2X_1+X_2-5X_3 \leq 6$

$4X_1+X_2+X_3 \leq 6$ and

$X_1 , X_2 , X_3 \geq 0$

Solution:

By introducing the slack variables we get

Maximize $Z=X_1+2X_2+X_3+0S_1+0S_2+0S_3$

Subject to the constraints

$2X_1+X_2-X_3 +S_1 = 2$

$-2X_1+X_2-5X_3+S_2 = 6$

$4X_1+X_2+X_3+S_3 = 6$ and

$X_1 , X_2 , X_3 , S_1, S_2, S_3 \geq 0$

INITIAL ITERATION:

		C_j	1	2	1	0	0	0	Ratio
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	2	2	1	-1	1	0	0	2/1=2
0	S_2	6	2	-1	5	0	1	0	6/1=6
0	S_3	6	4	1	1	0	0	1	6/1=6
	Z_j	0	0	0	0	0	0	0	
	Z_j- C_j		-1	-2	-1	0	0	0	

S_1 leaves the basis and X_2 enters the basis.

New S_2 = Old S_2 + New X_2

New S_3 = Old S_3 – New X_2

FIRST ITERATION:

		C_j	1	2	1	0	0	0	Ratio
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
2	X_2	2	2	1	-1	1	0	0	-2
0	S_2	8	4	0	4	1	1	0	2
0	S_3	4	2	0	2	-1	0	1	2
	Z_j		4	2	-2	2	0	0	
	Z_j- C_j		3	0	-3	2	0	0	

S_2 laves the basis and X_3 enters the basis.
New X_3 = Old S_2 / 4
New X_2 = Old X_2 + New X_3
New S_3 = old S_3 – 2(New X_3)
Second Iteration:

		C_j	1	2	1	0	0	0
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
2	X_2	4	3	1	0	1.25	0.25	0
1	X_3	2	1	0	1	0.25	0.25	0
0	S_3	4	2	0	0	-1	0	1
	Z_j		7	2	1	2.75	0.75	0
	Z_j- C_j		6	0	0	2.75	0.75	0

Since all Z_j- C_j are positive, $X_2 = 4$, $X_3 = 2$

Maximize $Z = X_1+2X_2+X_3$

$Z = 0+2(4)+2$

Maximize Z = 10.

BIG-M METHOD (Method of penalties)

The Big-M method is an alternative method of solving a LPP involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iterative procedure of the algorithm is given below:

Step 1: Write the given LPP into its standard form.

Step 2: Add artificial variable to the left side of each equation. Assign a high penalty to these variables in objective function.

Step 3: Apply simplex method to the modified LPP. Following cases may arise at the last iteration:

i) At least one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.

ii) At least one artificial variable is present in the basis with positive value. In such case, the given LPP does not possess an optimum solution. The given problem is said to have a pseudo-optimum basic feasible solution.

PROBLEM 1:

Use penalty (or Big M) method to

Maximize $Z= 6X_1+4X_2$

Subject to $2X_1+3X_2 \leq 30$

$3X_1+2X_2 \leq 24$

$X_1+X_2 \geq 3$ And $X_1, X_2 \geq 0$

Solution:

Introducing slack variables S_1 &S_2 , surplus variable S_3 we get

Maximize $Z= 6X_1+4X_2+0S_1+0S_2-MS_3$

Subject to the contraints,

$2X_1+3X_2 +S_1= 30$

$3X_1+2X_2 +S_2 =24$

$X_1+X_2 -S_3+A_1= 3$

And $X_1, X_2 ,S_1, S_2, S_3 \geq 0$

INITIAL ITERATION:

		C_j	6	4	0	0	(-M)	Ratio
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	S_3	
0	S_1	30	2	3	1	0	0	30/2=15
0	S_2	24	3	2	0	1	0	24/3=8
(-M)	S_3	3	1	1	0	0	-1	3/1=3
	Z_j	0	(-M)	(-M)	0	0	M	
	Z_j- C_j		(-M-6)	(-M-4)	0	0	2M	

S_3 leaves the basis and X_1 enters the basis.

New X_1 = Old S_3

New S_1 = Old S_1-2(New X_1)

New S_2 = Old S_2-3(New X_1)

First Iteration:

		C_j	6	4	0	0	(-M)
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	S_3
0	S_1	24	0	1	1	0	2
0	S_2	15	0	-1	0	1	3
6	X_1	3	1	1	0	0	-1
	Z_j		6	6	0	0	-6
	Z_j- C_j	18	0	2	0	0	M-6

Since all Z_j- C_j are positive and X_1=3, X_2=0

Maximize Z = $6X_1+4X_2$

Maximize Z= 6(3)+4(0)

Z= 18

PROBLEM 2:

Maximize Z = $3X_1+2X_2$ Subject to the constraints,

$2X_1+X_2 \leq 2$

$3X_1+4X_2 \geq 12$

And $X_1, X_2 \geq 0$

Use penalty method to solve the LPP .

Solution:

Introducing slack variable S_1 & surplus variable S_2 we get

Maximize $Z= 3X_1+2X_2+0S_1-MS_2$

Subject to the constraints,

$2X_1+X_2 +S_1 = 2$

$3X_1+4X_2 -S_2+A_1 = 12$

And $X_1, X_2, S_1, S_2 \geq 0$

INITIAL ITERATION:

		C_j	**3**	**2**	**0**	**(-M)**	**Ratio**
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**S_1**	**S_2**	
0	**S_1**	2	2	1	1	0	2
(-M)	**S_2**	12	3	4	0	-1	3
	Z_j	0	(-3M)	(-4M)	0	M	
	Z_j- C_j		(-3M-3)	(-4M-2)	0	2M	

S_1 leaves the basis and X_2 enters the basis.

New S_2 = Old S_2 – 4(New X_2)

FIRST ITERATION:

		C_j	**3**	**2**	**0**	**(-M)**
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**S_1**	**S_2**
2	**X_2**	2	2	1	1	0
(-M)	**S_2**	4	-5	0	-4	-1
	Z_j	0	5M+4	0	4M+1	M
	Z_j- C_j		5M+1	0	4M+2	2M

Here the coefficient of M in Each Z_j- C_j is non – negative and an artificial vector appears in the basis and not at the zero level. Thus the given LPP does not possess any feasible solution.

UNIT – V

TRANSPORTATION PROBLEM:

The transportation problem is one of the subclasses of LPP. In which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

Definition of Transportation problem:

Transportation deals with the transportation of a commodity from 'm' sources to 'n' destinations.

1. Level of supply at each sources and the amount of demand at each destinations and
2. The unit transportation cost of commodity from each sources to each destination are known (given). It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

Mathematical formulation of a Transportation problem:

A transportation problem (TP) is a special type of LPP. Its mathematical model is as follows

$$Minimize\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{n} X_{ij} = a_i \quad i = 1, 2, 3, \ldots, m$$

$$\sum_{i=1}^{n} X_{ij} = b_j \quad j = 1, 2, 3, \ldots, n.$$

And non – negativity restrictions.

$X_{ij} \geq 0\ \forall$ i and j.

Note: The two sets of constraints will be consistent is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) (total demand)

Which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called Balanced Transportation problems.

STANDARD TRANSPORTATION TABLE:

Symbols: Let there be m sources or origins $O_1, O_2, ...O_i,...O_m$. Let there be n destination

$D_1, D_2,...,D_j,...D_n$. Let the cost of transportation of one unit of the commodity from the i^{th} source O_i to the j^{th} destination D_j be C_{ij}.

Let the no.of units of the commodity transported from the i^{th} source O_i to the j^{th} destination D_j be X_{ij}.

Let a_i be the no.of units available (availability (or) supply (or) capacity) i^{th} source.

$$\therefore a_i = \sum_{j=1}^{n} X_{ij}.$$

Let b_j be the number of units required (requirement (or) demand) at j^{th} destination.

$$\therefore b_j = \sum_{i=1}^{m} X_{ij}.$$

The following is a model transportation table which consists of a cost matrix and rim requirements.

TRANSPORTATION TABLE:

Destination

		1	2	…	n	Supply
Origin	1	C_{11}	C_{12}	…	C_{1n}	a_1
	2	C_{21}	C_{22}	…	C_{2n}	a_2
	…	…	…	…	…	…
	m	C_{m1}	C_{m2}	…	C_{mn}	a_m
	Demand	b_1	b_2	…	b_n	

Definitions:

Feasible solution:

A set of non – negative values of X_{ij} (Allocations made from m sources to n destinations) which satisfy the rim conditions is called a feasible solution to be the transportation problem.

Basic feasible solution to the LPP

A feasible solution to a transportation problem that contains no more than m+n-1 non negative allocations is called basic feasible solution to a transportation problem.

Non Degenerate Basic Feasible Solution:

A basic feasible solution to a transportation problem is said to be non degenerate if its contains exactly m+n-1 non negative allocations in independent solutions.

Degenerate Basic Feasible Solution:

A basic feasible solution that contains less than m+n-1 degenerate basic feasible solution.

Optimal Solution:

A feasible solution that minimizes the transportation cost is called an optimal solution.

BALANCED AND UNBALANCED TRANSPORTATION PROBLEMS:

A transportation problem is said to be balanced if the total supply from all the sources is equal to the total demand in all the destinations.

i.e., $\sum a_i = \sum b_j$

A transportation problem is said to be unbalanced if the total supply from all the sources is not equal to the total demand in all the destinations.

i.e., $\sum a_i \neq \sum b_j$

By adding a dummy destination with zero unit costs and demand = $\sum a_i - \sum b_j$ if

$\sum a_i > \sum b_j$ or a dummy sources with zero unit costs and supply $= \sum b_j - \sum a_i$ if $\sum a_i < \sum b_j$, a balanced transportation problem can be got from an unbalanced one.

FINDING AN INITIAL BASIC FEASIBLE SOLUTION :

There are three methods available to obtain the IBFS. They are

i) North-West corner Rule(NWCR)
ii) Least cost method(LCM)
iii) Vogel's Approximation method(VAM)

North-West corner Rule(NWCR)

It is a simple and efficient method. Various steps of the method are:

Step 1

First step is to check whether the demand= supply, then it is a balanced transportation problem if not, convert the unbalanced transportation problem in to balanced one by adding dummy row or column with zero cost.

Step 2

Select the upper left hand corner of the transportation problem. Allocate the maximum possible units between the supply and demand requirements. Either the capacity or supply or the requirement or demand is satisfied.

Step 3

Delete that row or column which has no values ie fully exhausted for supply or demand

Step 4

Then the new reduced table again select the north west corner cell and allocate the available values.

Step 5

Repeat steps 2 and 3 untill all the supply and demand values are zero.

PROBLEM 1:

Find the IBFS for the following transportation problem.

	D	E	F	F	Supply
A	4	8	10	16	100
B	7	2	3	1	200
C	5	9	11	12	300
Demand	160	240	105	95	

Solution:

$\sum a_i = \sum b_j = 600$. It is balanced transportation problem.

100 4	8	10	16	~~100~~
7	2	3	1	200
5	9	11	12	300
~~160~~ 60	240	105	95	

60 7	2	3	1	~~200~~ 140
5	9	11	12	300
~~60~~	240	105	95	

140 7	3	1	~~140~~
5	11	12	300
~~240~~ 100	105	95	

100 5	11	12	~~300~~ 200
~~100~~	105	95	

105 11	12	~~200~~ 95
~~105~~	95	

95 12	~~95~~
~~95~~	

100 4	8	10	16
60 7	140 2	3	1
5	100 9	105 11	95 12

Transportation cost = 100x4 + 7x60 + 2x140 + 9x100 + 11x105 + 12x95= 4295

1. Find IBFS to the transportation problem

	D	**E**	**F**	**G**	**Supply**
A	1	2	1	4	30
B	3	3	2	1	50
C	4	2	5	9	20
Demand	20	40	30	10	

Solution:

$\sum a_i = \sum b_j = 100$. It is balanced transportation problem.

20 1	2	1	4	~~30~~ 10
3	3	2	1	50
4	2	5	9	20
~~20~~	40	30	10	

10 2	1	4	10
3	2	1	50
2	5	9	20
40 30	30	10	

30 3	2	1	50 20
2	5	9	20
30	30	10	

20 2	1	20
5	9	20
30 10	10	

10 5	10 9	20
10	10	

20 1	10 2	1	4
3	30 3	20 2	1
4	2	10 5	10 9

Transportation cost = (20x1) + (10x2) + (30x3) + (20x2) + (10x5) + (10x9) = 310

2. Find IBFS for the following transportation problem

	D	E	F	G	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

Solution:

Since $\sum a_i = \sum b_j = 950$. It is balanced transportation problem.

200 / 11	13	17	14	~~250~~ 50
16	18	14	10	300
21	24	13	10	400
~~200~~	225	275	250	

50 / 13	17	14	~~50~~
18	14	10	300
24	13	10	400
~~225~~ 175	275	250	

175 / 18	14	10	~~300~~ 125
24	13	10	400
~~175~~	275	250	

125 / 14	10	~~125~~
13	10	400
~~275~~ 150	250	

150 / 13	250 / 10	~~400~~
~~150~~	250	

200 / 11	50 / 13	17	14
16	175 / 18	125 / 14	10
21	24	150 / 13	250 / 10

Transportation cost = (11x200)+(13x50)+(18x175)+(14x125)+(13x150)+(10x250) = 12,200

LEAST COST METHOD(LCM):

This method takes into account the minimum unit cost and can be summarized as follows:

Step 1:

Select the cell having lowest unit cost in the entire table and allocate the minimum of supply and demand values in that cell.

Step 2:

Then eliminate the row or column in which supply and demand values are same,either of the row or column can be eliminated.

Step 3:

In case, the smallest cost is not unique, then select the cell where maximum allocation can be made.

Step 4:

Repeat the process with next lowest unit cost and continue until the entire available supply at various sources and demand at destinations are satisfied.

PROBLEMS:

1. Obtain IBFS using LCM.

	S	T	U	V	Suply
P	11	13	17	14	250
Q	16	18	14	10	300
R	21	24	13	10	40
Demand	200	225	275	250	

Solution:

Since $\sum a_i = \sum b_j = 950$. It is balanced transportation problem.

11	13	17	14	250
16	18	14	10	300
21	24	13	250 10	~~400~~ 150
200	225	275	~~250~~	

13	17	~~250~~ 50
18	14	300
24	13	150
225	275	

13	17	50
18	14	300
24	150 13	~~150~~
225	~~275~~ 125	

[50] 13	17	~~50~~
18	14	300
~~225~~ 175	125	

[175] 18	[125] 14	300
175	125	

[200] 11	[50] 13	17	14
16	[175] 18	[125] 14	10
21	24	[150] 13	[250] 10

Transportation cost = (11x200)+(13x50)+(18x175)+(14x125)+(13x150)+(10x250) = 12,200

2. Obtain IBFS for the transportation problem.

	G1	**G2**	**G3**	**Supply**
F1	10	9	8	8
F2	10	7	10	7
F3	11	9	7	9
F4	12	14	10	4
Demand	10	10	8	

Solution:

Since $\sum a_i = \sum b_j = 28$. It is balanced transportation problem.

10	9	8	8
10	[7] 7	10	~~7~~
11	9	7	9
12	14	10	4
10	~~10~~ 3	8	

10	9	8	8
11	9	[8] 7	~~9~~ 1
12	14	10	4
10	3	~~8~~	

10	[3] 9	~~8~~ 5
11	9	1
12	14	4
10	~~3~~	

[5] 10	5
[1] 11	1
[4] 12	4
10	

[5] 10	[3] 9	8
10	[7] 7	10
[1] 11	9	[8] 7
[4] 12	14	10

Transportation cost = (10x5)+(9x3)+(7x3)+(11x1)+(12x4)+(7x8)

= 241

3. Find IBFS using LCM.

1	2	3	4	6
4	3	2	0	8
0	2	2	1	10
4	6	8	6	

Solution:

Since $\sum a_i = \sum b_j = 24$. It is balanced transportation problem.

1	2	3	4	6
4	3	2	[6] 0	~~8~~ 2
0	2	2	1	10
4	6	8	~~6~~	

1	2	3	6
4	3	2	2
[4] 0	2	2	~~10~~ 6
~~4~~	6	8	

[6] 2	3	~~6~~
3	2	2
2	2	6
~~6~~	8	

[2] 2	2
[6] 2	6
8	

1	[6] 2	3	4
4	3	[2] 2	[6] 0
[4] 0	2	[6] 2	1

Transportation cost = (2x6)+(2x2)+(0x4)+(2x6)+(0x6) = 28

VOGELS APPROXIMATION METHOD(VAM):

The Vogel's approximation method takes into account not only the least cost c_{ij} but also the costs that just exceed c_{ij}. The steps of the method are given below.

Step 1

Calculate penalties for each row and each column by taking the difference between the smallest cost and next highest cost available in that row or column. If there are two costs then the penalty is zero.

Step 2:

Select the row or column which has largest penalty and make the allocation in the cell having least cost in the selected row or column. If two or more equal penalties exists select one where a row or column contains minimum unit cost.

Step 3:

Delete the row or column which has satisfied the supply and demand

Step 4:

Repeat step 1 and 2 until entire supply and demands are satisfied.

PROBLEMS:

1. Find out IBFS using VAM.

9	12	9	6	9	10	**5**
7	3	7	7	5	5	**6**
6	5	9	11	3	11	**2**
6	8	11	2	2	10	**9**
4	**4**	**6**	**2**	**4**	**2**	

Solution:

Since $\sum a_i = \sum b_j = 22$. It is balanced transportation problem.

9	12	9	6	9	10	**5**	(3)
7	3	7	7	5	[2] 5	~~**6**~~ **4**	(2)
6	5	9	11	3	11	**2**	(2)
6	8	11	2	2	10	**9**	(0)
4	**4**	**6**	**2**	**4**	~~**2**~~		
(0)	(2)	(2)	(4)	(1)	**(5)**		

9	12	9	6	9	**5**	(3)
7	3	7	7	5	**4**	(2) (2)
6	5	9	11	3	**2**	
6	8	11	$\boxed{2}$ 2	2	~~**9**~~ **7**	(0)
4	**4**	**6**	~~**2**~~	**4**		
(0)	(2)	(2)	**(4)**	(1)		

9	12	9	9	**5**	(0)
7	3	7	5	**4**	(2) (2)
6	5	9	3	**2**	
6	8	11	$\boxed{4}$ 2	~~**7**~~ **3**	**(4)**
4	**4**	**6**	~~**4**~~		
(0)	(2)	(2)	(1)		

9	12	9	**5**	(0)
7	$\boxed{4}$ 3	7	~~**4**~~	**(4)** (1)
6	5	9	**2**	
6	8	11	**3**	(2)
4	~~**4**~~	**6**		
(0)	(2)	(2)		

9	9	**5**	(0)
6	9	**2**	(3)
$\boxed{3}$ 6	11	~~**3**~~	**(5)**
~~**4**~~ **1**	**6**		
(0)	(0)		

9	9	**5**	(0)
$\boxed{1}$ 6	9	~~**2**~~ **1**	**(3)**
~~**1**~~	**6**		
(3)	(0)		

$\boxed{5}$ 9	**5**	(9) **(9)**
$\boxed{1}$ 9	**1**	
6		
(0)		

9	12	[5] 9	6	9	10
7	3	7	7	5	[2] 5
[1] 6	5	[1] 9	11	3	11
[3] 6	8	11	[7] 2	[4] 2	10

Transportation cost = (9x5)+(5x2)+(6x1)+(9x1)+(6x3)+(2x7)+(2x4) = 122

2. Find IBFS for the transportation problem.

40	25	22	33	**200**
44	35	30	30	**60**
38	38	28	30	**140**
200	**40**	**120**	**40**	

Solution:

Since $\sum a_i = \sum b_j = 400$. It is balanced transportation problem.

40	[40] 25	22	33	**~~200~~ 160**	(3)
44	35	30	30	**60**	(0)
38	38	28	30	**140**	(2)
200	**~~40~~**	**120**	**40**		
(2)	**(10)**	(6)	(0)		

40	[120] 22	33	**~~160~~ 40**	**(11)**
44	30	30	**60**	(0)
38	28	30	**140**	(2)
200	**~~120~~**	**40**		
(2)	(6)	(0)		

40	[40] 33	**~~40~~**	**(11)**
44	30	**60**	(0)
38	30	**140**	(2)
200	**~~40~~**		
(2)	(0)		

[60] 44	**60**	(44)
[140] 38	**140**	(38)
200		
(6)		

40	[40] 25	[120] 22	33
[60] 44	35	[40] 30	30
[140] 38	38	28	30

Z= (25x40)+(22x120)+(30x40)+(44x60)+(38x140) = 12640

MODIFIED DISTRIBUTION METHOD(MODI):

<u>PROCEDURE FOR MODI METHOD</u>:

Step 1:

Find the initial basic feasible solution using VAM or NWCR or LCM method.

Step 2:

Check the number of occupied cells. if there are less than m+n-1,degenerancyexistsand we introduce a very small quantity € equal to zero in suitable independent situations.so that the unoccupied cells is exactly equal to m+n-l

Step 3:

Find U_i and V_j starting U_i orV_j for which the corresponding row or column has maximum number of allocations.

Step 4:

Find $d_{ij=\, Cij}$-$(u_{i+}v_j)$ from the non allocated cell.

Step 5:

Examine the net evaluations

If all d_{ij}> 0, then the solution under the test is optimal and unique

If all d_{ij}> 0, with at least one dij = 0 then the solution under the test is optimal and the alternative optimal solution exists.

If all d_{ij}<0, then the solution is not optimal. Go to next step.

Step 6:

Form a loop starting from the negative d_{ij} named as $+\theta$ and $-\theta$ alternatively at the corners of the loop.

Step 7:

Choose the minimum of the allocations from the cells having $-\theta$ .add this minimum allocations to the cells with $+\theta$ and subtract this minimum allocations to the cells with $-\theta$

Step 8:

Continue the procedure from step 2 to 5 until all the dij's are positive and optimum solution will be reached .

<u>Degeneracy in transportation problem:</u>

In a transportation problem with m sources of supply and n demand destinations ,the test of optimality of any feasible solution requires allocations in m+n-1.if this is not satisfied then the transportation problem is said to be a degenerate problem. Degeneracy may exists at the initial stage or intermediate stage.

Resolving degeneracy transportation problem:

To resolve degeneracy transportation problem, we allocate a small quantity to one or more empty cells of a transportation table, so that the occupied cells becomes m+n-1 at independent positions. We denote this small amount by $\in$ (epsilon) satisfying the following conditions:

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$

(ii) $x_{ij} \pm \in = x_{ij}$, for all $x_{ij} > 0$

The cells containing $\in$ are then treated like other occupied cells and the problem is solved in the usual way. The $\in 's$ are kept till optimum solution is attained. Then we let each $\in \rightarrow 0$.

Problems:

1. Find the optimal solution to the transportation problem

	W1	**W2**	**W3**	**Supply**
M1	12	11	20	1000
M2	10	12	14	2000
M3	15	12	10	1500
Demand	1200	1900	1400	

Solution:

Since $\sum a_i = \sum b_j = 4500$. It is balanced transportation problem.

12	11	20	1000	(1)
10	12	14	2000	(2)
15	12	1400 10	1500 100	(2)
1200	1900	1400		
(2)	(1)	**(4)**		

12	11	1000	(1)
10	12	2000	(2)
15	100 12	100	**(3)**
1200	1900 1800		
(2)	(1)		

12	11	1000	(1)
1200 10	12	2000 800	(2)
1200	1800		
(2)	(1)		

1000 11	1000	(1)
800 12	800	(2)
1800		
(1)		

12	1000 11	20
1200 10	800 12	14
15	100 12	1400 10

Z= (10x1200)+(11x1000)+(12x800)+(12x100)+(10x1400)

Z = 47800

To find the optimal solution:

For basic cells (or) occupied cells:

	11		u_1=11
10	12		u_2=12
	12	10	u_3=12
v_1= -2	v_2=0	v_3= -2	

$C_{ij} = u_i+v_j$

$12=u_3+v_2$ $\quad$ $11=u_1+v_2$ $\quad$ $12=u_2+v_2$ $\quad$ $10=u_3+v_3$ $\quad$ $10= u_2+v_1$

$12 =u_3+0$ $\quad$ $11=u_1+0$ $\quad$ $12= u_2+0$ $\quad$ $10=12+v_3$ $\quad$ $10=12+v_1$

$\mathbf{12=u_3}$ $\quad$ $\mathbf{11= u_1}$ $\quad$ $\mathbf{12 = u_2}$ $\quad$ $\mathbf{-2 = v_3}$ $\quad$ $\mathbf{-2 = v_1}$

For non-basic cells:

12		20	$u_1=11$
		14	$u_2=12$
15			$u_3=12$
$v_1= -2$	$v_2=0$	$v_3= -2$	

$d_{ij} = c_{ij} – (u_i+v_j)$

$d_{11} = c_{11}-(u_1+v_1) = 12 - (11+ (-2)) = 3$

$d_{13} = c_{13}-(u_1+v_3) = 20 - (11+(-2)) = 11$

$d_{23} = c_{23}-(u_2+v_3) = 14 – (12+(-2)) = 4$

$d_{31} = c_{31}-(u_3+v_1) = 15 – (12+(-2)) = 5$

Since all the d_{ij} values are positive, the IBFS is optimal.

2. Find the optimal solution to the transportation problem.

19	30	50	12	7
70	30	40	60	10
40	10	60	20	18
5	8	7	15	

Solution:

Since $\sum a_i = \sum b_j = 35$. It is balanced transportation problem

By VAM we get the following solution

5 19	30	50	2 12
70	30	7 40	3 60
40	8 10	60	10 20

Z= (19x5)+(10x8)+(40x7)+(12x2)+(60x3)+(20x10) = 859

To find the optimal solution:

For basic cells (or) occupied cells:

19			12	u_1=12
		40	60	u_2=60
	10		20	u_3=20
v_1= 7	v_2= -10	v_3= -20	v_4=0	

$C_{ij} = u_i+v_j$

$12=u_1+v_4$	$19=u_1+v_1$	$60=u_2+v_4$	$40=u_2+v_3$	$10= u_3+v_2$
$12 =u_1+0$	$19=v_1+12$	$60= u_2+0$	$40=60+v_3$	$10=20+v_2$
$\mathbf{12=u_1}$	$\mathbf{7= v_1}$	$\mathbf{60 = u_2}$	$\mathbf{-20 = v_3}$	$\mathbf{-10 = v_1}$

For non-basic cells:

	30	50		u_1=12
70	30			u_2=60
40		60		u_3=20
v_1= 7	v_2= -10	v_3= -20	v_4=0	

$d_{ij} = c_{ij} – (u_i+v_j)$

$d_{12} = c_{12}-(u_1+v_2) = 30 - (12+ (-10)) = 28$

$d_{13} = c_{13}-(u_1+v_3) = 50 - (12+(-20)) = 58$

$d_{21} = c_{21}-(u_2+v_1) = 70 – (60+7) = 3$

$d_{22} = c_{22}-(u_2+v_2) = 30 – (60+(-10)) = -20$

$d_{31} = c_{31}-(u_3+v_1) = 40 – (20+7) = 13$

$d_{33} = c_{33}-(u_3+v_3) = 60 – (20+(-20)) = 60$

Hence there is negative in d_{ij}.

5 | 19			2 | 12
	+ϴ30	7 | 40	3 | 60 -ϴ
	8 | 10 -ϴ		10 | 20 +ϴ

5 | 19			2 | 12
	3 | 30	7 | 40	60
	8 | 10		10 | 20

The optimal solution is = (19x5)+(12x2)+(30x3)+(40x7)+(10x8)+(20x10) = 799.

Maximization problem:

Find the maximum profit for the following T.P

15	51	42	33	23
80	42	26	81	44
90	40	66	60	33
23	31	16	30	

Solution:

15	51	42	33	23
80	42	26	81	44
90	40	66	60	33
23	31	16	30	

Subtract each value from the maximum value 90.

75	39	48	57	23
10	48	64	9	44
[23] 0	50	24	30	~~33~~ 10
~~23~~	31	16	30	

39	48	57	23
48	64	[30] 9	~~44~~ 14
50	24	30	10
31	16	~~30~~	

39	48	23
48	64	14
50	[10] 24	~~10~~
31	~~16~~ 6	

(23) 39	48	~~23~~
48	64	14
~~31~~ 8	6	

(8) 48	(6) 64	14
8	6	

75	(23) 39	48	57
10	(8) 48	(6) 64	(30) 9
(23) 0	50	(10) 24	30

Transportation cost = (23x39)+(48x8)+(0x23)+(24x10)+(9x30)

= 2175

ASSIGNMENT PROBLEM:

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have ‘n’ jobs to be performed on ‘m’ machines (one job to one machine) and our objective to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be started in the form of $n \times n$ matrix $\left(c_{ij}\right)$ called a cost matrix (or) Effectiveness matrix where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

	A_1	A_2	…	A_n	Available
R_1	C_{11}	C_{12}	…	C_{1n}	1
R_2	C_{21}	C_{22}	…	C_{2n}	1
…	..	…	…	…	…
R_n	C_{n1}	C_{n2}	…	C_{nn}	1
Required	1	1	…	1	

MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM

Consider an assignment problem to assigns n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

$$\text{Let } x_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, \text{ if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is given by the following LPP

$$\text{Minimize Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to constraints

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1,2, \dots , n$$

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1,2, \dots, n$$

and x_{ij} = 0 (or) 1.

PROCEDURE FOR ASSIGNMENT PROBLEM:-

Step :1

First check whether the assignment problem is balanced or unbalanced by finding out whether the no. of rows is equal to no. of columns.

Step :2

(Reduce the element of each row)

Identify the smallest cost in each row and subtract it from all the elements of the row and get the reduced matrix.

Step :3

(Reduce the element of each column)

From the reduced matrix find the smallest cost element in each column and subtract it from all the elements of the column and get the reduced matrix.

Step :4

(make the assignments)

a) Examine the rows one by one .if there is only one zero in a row , make the assignment by drawing a square around the zero [0] and cross out all the zero ~~0~~ in its column.

b) Examine the columns one by one .if there is only one zero in a column, make the assignment by drawing a square around the zero [0] and cross out all the zero ~~0~~ in its row.

Problems:

1.Solve the following assignment problem.

Machines

		1	2	3	4
Jobs	A	9	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution:

No.of rows =No.of.columns.

It is balanced assignment problem.

Consider row:

0	17	8	2
9	24	0	22
23	4	3	0
9	16	14	0

Consider column:

	1	2	3	4
A	[0]	13	8	2
B	9	20	[0]	22
C	23	[0]	3	~~0~~
D	9	12	4	[0]

The assignment is A→ 1, B→3, C→2, D→4Assignment cost = 9+4+19+10 = 42

2.Solve the following assignment problem

		Machines			
		1	2	3	4
Jobs	A	7	6	8	4
	B	8	9	2	5
	C	11	1	6	7
	D	5	4	9	6

Solution:

No.of rows = No.of. columns.

It is balanced assignment problem.

Consider row:

3	2	4	0
6	7	0	3
10	0	5	6
1	0	5	2

Consider column:

	1	2	3	4
A	2	2	4	[0]
B	5	7	[0]	3
C	9	[0]	5	6
D	[0]	~~0~~	5	2

The assignment is A→4, B→3, C→2, D→1

The assignment cost = 4+2+1+5 = 12

3.Solve the following assignment problem

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Solution:

No.ofrows =No.of.columns. It is balanced assignment problem.

Consider row:

8	2	0	12	0
0	16	12	19	4
0	14	3	11	4
19	3	0	5	5
11	7	0	5	1

Consider column:

8	0	0	7	[0]
[0]	14	12	14	4
0	12	8	6	4
17	1	[0]	0	5
11	5	0	[0]	1

The assignment is

12	[0]	0	7	0
[0]	10	8	10	0
0	8	4	2	[0]
23	1	[0]	0	5
15	5	0	[0]	1

The assignment cost is 3+1+4+0+6 = 14

4.Solve the following assignment problem

41	72	39	52
22	29	49	65
27	39	60	51
45	50	48	52

Solution:

No.ofrows =No.of.columns.

It is balanced assignment problem.

Consider row:

2	33	0	13
0	7	27	43
0	12	33	24
0	5	3	7

Consider column:

~~2~~	~~28~~	[0]	~~6~~
[0]	2	27	36
~~0~~	7	33	17
~~0~~	~~0~~	~~3~~	[0]

The assignment is

2	28	[0]	6
~~0~~	[0]	25	34
[0]	5	31	15
2	~~0~~	3	[0]

The assignment cost is = 39+29+27+52 = 147

Unbalanced assignment problem:

5.Solve the following assignment problem

20	21	14	12	18
17	21	20	24	24
15	16	19	22	24
23	25	21	20	17

Solution:

No.of rows $\neq$ No.of. columns.

It is not a balanced assignment problem. So we add a dummy row and we get

20	21	14	12	18
17	21	20	24	24
15	16	19	22	24
23	25	21	20	17
0	0	0	0	0

Consider row:

8	9	2	0	6
0	4	3	7	5
0	1	4	7	9
6	8	4	3	0
0	0	0	0	0

Consider column:

8	9	2	[0]	6
[0]	4	3	7	5
0	1	4	7	9
6	8	4	3	[0]
0	0	[0]	0	0

The assignment is

9	9	2	[0]	6
[0]	3	2	6	4
0	[0]	3	6	8
7	8	4	3	[0]
0	0	[0]	0	0

The assignment cost is 12+17+16+0+17 = 62