Mathematical Statistics

Paper code: 21USTA01

Unit-I Probability, Random variable and mathematical Expectation:-

Definition - Addition and Multiplication theorem of probability - condition probability - Random variable (discrete & continuous) - Distribution functions - marginal & conditional Distributions - mathematical Expection - moment generating function - characteristic function (concept only) - Tchebychev's inequality - simple problem.

Unit-II Discrete & continuous Distributions.

Binomial Poisson Distribution - Derivations. Proporties and Application simple problems. Normal distribution - Derivation - Properties & Application - Simple problem.

Unit-III Measures of centeral Tendency, measures of Dispersion and skewness:-

Definition - mean, median, mode, Geometric mean, Hormonic mean - merits and demerits Range, Quartile deviation mean, deviation, & their coefficient - standard deviation - co-efficient of variation merits and demerits - measure of skewness;

Unit - IV Curve Fitting.

Method of least square - Fitting of a straight line & second degree parabola, Fitting of Power curve and Exponential curves - Simple Problems.

Unit - V (Correlation and Regression).

Definition - Types and methods of measuring correlation - Scatter diagram, Karl Person's correlation coefficient and Spearman's rank correlation coefficient - Regression lines - Regression coefficient - Properties. Regression equation.

Unit - 2

Random Variable

Let a real number 'x' which is connected with the outcome of experiment is called random variable.

A random variable (r.v) is a function x(w) with domain S and range (-∞, ∞) such that for every real number 'a', the event

$$[w : x(w) \leq a] \in B$$

Example:

Suppose two coins are tossed then we get

outcome : HH, HT, TH, TT

Value of X : 2, 1, 1, 0

Discrete Random Variable.

If a random Variable takes almost a countable number of values, it is called a Discrete random Variable.

In other words, a real valued function defined on a discrete a sample place is called a discrete random variable.

Example:

i). Marks obtained in a test.

ii) Number telephones calls per unit time.

Probaility Mass Function (PMF)

Suppose 'X' is a one dimensional discrete random variable taking almost a countably infinite a number of values.

$x_1, x_2, \ldots, x_n$

It each possible outcome x_i, we associate a number $p_i = P(x = x_i) = P(x_i)$, called the probability of x; the number $P(x_i)$; $i = 1, 2 \ldots$ must satisfy the following conditions.

i) $P_\infty (x_i) \geq 0 \forall i$

ii) $\sum_{i=1}^{'} P(x_i) = 1$

The function 'p' is called the ~~most~~ mass function of the random variable 'X' and the set $\{x_i, P(x_i)\}$ is called the Probability distribution of the random variable 'X'

Discrete Distribution function:

In this case, there are a countable number of points.

$x_1, x_2, x_3 \ldots$ and number $P_i \geq 0$,

$\sum_{i}^{\infty}{}' p_i = 1$, such that

$$F(x) = \sum{}'_{i} P_i$$
$$i : x_i \leq x$$

continuous random variable:

A random variable 'x' is said to be, continuous if it can take all possible values between certain limites.

In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondents with a set of positive integers.

Eg: Age; Hight, Weight.

Probability density function:

A function $f(x)$ is said to be the Probability density function (pdf) of a continuous random variable x if the following continuous are satisfied.

i) $f(x) \geq 0 \;\forall x$

ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$ $\quad \sum{}'_{i} P(x) = 1$.

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Distribution function (or) Cumulative distribution function:

Let X be a random variable discrete or continuous them, the function $f(x)$ defined by,

$F(x) = P(X \leq x)$ is called the distribution function of X.

If X is a discrete random variable when, $F(x) = \sum_{x_i \leq x} p(x_i)$

If X is a continuous random variable then, $F(x) = \int_{-\infty}^{x} f(x)\,dx$

Properties of Distribution function:

i) $0 \leq f(x) \leq 1$

ii) $F(x) \leq F(y)$ if $x < y$

iii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

iv) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

v) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$

$= F(b) - F(a)$

problems:-

1. Find the mean and variance of the random variable x having the following probability distribute.

x	1	2	3	4	5
P(x)	0.1	0.2	0.3	0.2	0.2

Mean, $E(x) = \Sigma' , x\, P(x)$

$= (1\times 0.1) + (2\times 0.2) + (3\times 0.3) + (4\times 0.2) + (5\times 0.2)$

$= 0.1 + 0.4 + 0.9 + 0.8 + 1.0$

$E(x) = 3.2$

Varriance, $V(x) = E(x^2) - [E(x)]^2$

Then, $E(x^2) = \Sigma' , x^2\, P(x)$

$= (1\times 0.1) + (4\times 0.2) + (9\times 0.3) + (16\times 0.2) + (25\times 0.2)$

$= 0.1 + 0.8 + 2.7 + 3.2 + 5.0$

$E(x^2) = 11.8$

$\therefore V(x) = 11.8 - (3.2)^2 = 11.8 - 10.24$

$V(x) = 1.56$

Result:

Mean $E(x) = 3.2$, Varriance $V(x) = 1.56$

2. Find the mean ... distribution:

x	:	0	1	2	3
P(x)	:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean, $E(x) = \Sigma' \, x P(x)$

$= (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8})$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$= \frac{12}{8}$

$\therefore E(x) = \frac{3}{2}$

Varriance, $V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \Sigma' \, x^2 P(x)$

$= (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (4 \times \frac{3}{8}) + (9 \times \frac{1}{8})$

$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8}$

$= \frac{24}{8}$

$E(x^2) = 3$

$V(x) = E(x^2) - [E(x)]^2$

$= 3 - (\frac{3}{2})^2$

$= 3 - \frac{9}{4}$

$V(x) = \frac{3}{4}$

Result:

Mean $E(x) = \frac{3}{2}$, varriance $V(x) = \frac{3}{4}$.

1. A Continuous random variable X has P.d.f
$f(x) = k ; 0 < x < 1$ find, constant K and also
$P(X \leq 1/4)$

we know that the probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(K) \cdot dx = 1$$

$$K \int_0^1 k \cdot dx = 1$$

$$K \int_0^1 dx = 1$$

$$K [x]_0^1 = 1$$

$$K [1 - 0] = 1$$

$$K [1] = 1$$

$$\therefore K = 1$$

$$P(X \leq 1/4) = \int_0^{1/4} f(x)\,dx$$

$$= \int_0^{1/4} K \cdot dx$$

$$= \int_0^{1/4} \cdot dx$$

$$= (x)_0^{1/4}$$

$$= \frac{1}{4} - 0$$

$$\therefore P(x \leq 1/4) = 1/4$$

2. Given that the pdf of a random variable x, $f(x) = kx$; $0 < x < 1$; Find k and $P(x \geq 0.5)$

W.K.T the probability dereity function.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 k\,x\,dx = 1$$

$$k\int_0^1 x\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$k\left[\frac{1}{2} - \frac{0}{2}\right] = 1$$

$$\frac{k}{2} = 1$$

$$\therefore k = 2$$

$P(x > 0.5) \Rightarrow P(x > \frac{1}{2})$

$P(x > 1/2) \Rightarrow \int_{1/2}^{1} f(x)\,dx$

$$= \int_{1/2}^{2} kx\,dx$$

$$= \int_{1/2}^{1} 2x \cdot dx$$

$$= 2\int_{1/2}^{1} x \cdot dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= 2\left[\frac{1}{2} - \frac{1/4}{2}\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{8}\right]$$

$$= 2\left[\frac{6}{16}\right]$$

$$= 2\left[\frac{3}{8}\right]$$

$\therefore P(X > 0.5) = \frac{3}{4}$

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3. If x is the random variable, probability density function $f(x) = \lambda x(2-x)$; $0 < x < 2$. Find the value of λ

W.K.T

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^2 \lambda x(2-x)\,dx = 1$$

$$\lambda \int_0^2 2x - x^2 . dx = 1$$

$$\lambda \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$\lambda \left[4 - \frac{8}{3} - 0\right] = 1$$

$$\lambda \left[\frac{4}{3}\right] = 0$$

$$\therefore \lambda = \frac{3}{4}.$$

4. If x is a discrete random variable having the probability distribution.

x :	1	2	3
$P(x)$:	K	K	K^2

Find the value of K

Solu:

We know that

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

$$K + K + K^2 = 1$$

$$2K + K^2 = 1$$

$$K^2 + 2K - 1 = 0$$

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{(2)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm \sqrt{4\times 2}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= \frac{2(-1 \pm \sqrt{2})}{2}$$

$$\therefore k = -1 \pm \sqrt{2}$$

Marginal distribution function:-

We joined distribution function $F_{xy}(x,y)$, it is possible to obtain be individual distribution functions $F_X(x)$ and $F_Y(y)$ which are termed as marginal distribution function of X and Y, respectively, with respect to we joined distribution function $F(x,y)$

$$F(x) = P(X \le x) = P(X \le x, y < \infty) \Rightarrow \lim_{y \to \infty} F_{x,y}(x,y)$$

$$= F_{xy}(x, \infty)$$

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$$F(y) = P(Y \leq y) = P(X < \infty, Y \leq y)$$

$$= \lim_{x \to \infty} F_{xy}(x, y)$$

$$= F_{xy}(\infty, y)$$

Discrete case:

Marginal distribution function of x

$$F(x) = \Sigma_y f(x, y) = P(x = x)$$

Marginal distribution function of y

$$F(x) = \Sigma_x f(x, y) = P(Y = y).$$

Continuous Case:

Marginal density function of x is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

Marginal density function of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

And, joint density function of x, y is

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{xy} f(x, y)\, dx\, dy.$$

Conditional distribution function:

The conditional distribution $F_{XY}(x,y)$ denotes the distribution of x, when Y already assumed the particular value y.

Conditional distribution of x given Y is

$$F_{x|y}(x|y) = F_{X|Y}(X=x|Y=y) = \frac{F(x,y)}{F(y)}$$

Conditional distribution of Y given X is

$$F_{Y|X}(y|x) = F_{Y|X}(Y=y|X=x) = \frac{F(x,y)}{F(x)}.$$

Joint Probability distribution function:

Let X, Y be a two dimensional random variable and the joint probability distribution function is denoted by $F_{XY}(x,y)$

$$F_{XY}(x,y) = P(-\infty \leq X \leq x, -\infty \leq Y \leq y)$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

where,

$f(x,y) \geq 0$ and

$$\int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dx\,dy = 1$$

(or)

$$\Sigma_x \Sigma_y f(x,y) = 1$$

Problems :

1. From the following joint distribution function x, y. Find the marginal distri…

Y/x	0	1	2
0	$3/28$	$\frac{9}{28}$	$\frac{3}{28}$
1	$\frac{3}{14}$	$\frac{5}{14}$	0
2	$\frac{1}{28}$	0	0

Y/X	0	1	2	P(Y)
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{6}{14}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
P(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$P(X=0) = \frac{10}{28}$, $P(Y=0) = \frac{15}{28}$

$P(X=1) = \frac{15}{28}$ $\quad$ $P(Y=1) = \frac{6}{14}$

$P(X=2) = \frac{3}{28}$ $\quad$ $P(Y=2) = \frac{1}{28}$

Joint Probability distribution function

Let (x, y) be a two dimensional descrete random variable. Let $P(x_i, y_i)$, $i, j = 1, 2, \ldots$ be a real number associated with each them P is called the joint probability function of (x, y), if the following condition are satisified.

i) $P(x_i, y_i) \geq 0$ for all $i, j = 1, 2, \ldots$

ii) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(x_i, y_i) = 1$

Let (x, y) be a two dimensional continuous random variable them $f(x, y)$ is called the probability density function if the following conditions are satisfied.

i) $f(x, y) \geq 0$ for all x, y

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

Conditional Probability distribution:

Let (x, y) be a two dimensional descrete random variable with joint Probability function $P(x, y)$

Let $P(x)$ and $P(y)$ be the marginal probability function of x and y respectively.

Then the conditional distribution of Y ...
x is defined by $P(x = x \mid Y = y) = \frac{P(x,y)}{P(y)}$

$|||^{ly}$

$$P(y = y \mid X = x) = \frac{P(x, y)}{P(x)}$$

Let (x, y) be a two dimensional continuous random variable with join probability function $f(x,y)$

Then, the conditional probability function of X given that Y = y is defined by

$$f(x \mid y) = \frac{f(x, y)}{f(y)}$$

and, the conditional probability density function of Y given that X = x is defined by

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

1. Given the following bivariate probability distribution obtain.

i) Marginal distribution of X and Y

ii) The conditional distribution of X given Y = 2

Y/X	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

i)

Y/X	-1	0	1	Total
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{4}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{6}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{5}{15}$
Total P(x)	$\frac{6}{15}$	$\frac{5}{15}$	$\frac{4}{15}$	1

$$P(X=x) = P(X=-1) = P(x=-1, Y=0) + P(X=-1, Y=1) + P(X=-1, Y=2)$$

$$= \frac{1}{15} + \frac{3}{15} + \frac{2}{15}$$

$$P(X=-1) = \frac{6}{15}$$

|||ly

$$P(X=0) = \frac{5}{15}$$

$$P(X=1) = \frac{4}{15}$$

Then,

$P(Y=y) = P(Y=0) = P(Y=0, X=-1) + P(Y=0, X=0) + P(Y=0, 1)$

$P(Y=0) = \frac{4}{15}$

III^{ly}

$P(Y=1) = \frac{6}{15}$ and $P(Y=2) = \frac{5}{15}$

obtain marginal distribution function of X

values of X, x :	-1	0	1
$P(x)$:	$\frac{6}{15}$	$\frac{5}{15}$	$\frac{4}{15}$

and Marginal distribution function of Y

values of Y, y :	0	1	2
$P(y)$:	$\frac{4}{15}$	$\frac{6}{15}$	$\frac{5}{15}$

ii) Conditional distribution function of X given $y=2$

$P(X=-1/Y=2) = \frac{P(X=-1, Y=2)}{P(Y=2)}$

$= \frac{2/15}{5/15}$

$\therefore P(X=-1/Y=2) = \frac{2}{5}$

$P(X=0/y=2) = \frac{P(X=0, Y=2)}{P(Y=2)}$

$= \frac{1/15}{5/15}$ $\therefore P(X=0/Y=2) = \frac{1}{5}$

2.

$$P(X=1 \mid Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)}$$

$$= \frac{2/15}{5/15}$$

$$\therefore P(X=1 \mid Y=2) = \frac{2}{5}.$$

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2. For the join probability distribution of two random variable x and y given below

x\y	1	2	3	4
1	4/36	3/36	2/36	1/36
2	1/36	3/36	3/36	2/36
3	5/36	1/36	1/36	1/36
4	1/36	2/36	1/36	5/36

Find

i) The marginal distribution of x and y

ii) Conditional distribution of x given the value of y = 1 that of the given value of x = 2.

X\Y	1	2	3	4	Total
1	4/36	3/36	2/36	1/36	10/36
2	1/36	3/36	3/36	2/36	9/36
3	5/36	1/36	1/36	1/36	8/36
4	1/36	2/36	1/36	5/36	9/36
Total P(y)	11/36	9/36	7/36	9/36	1

The Marginal distribution function of 'x' defined as

$$P(X=x) = \Sigma_y P(X=x, Y=y)$$

$\therefore P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$P(X=1) = \frac{10}{36}$$

||ly

$P(X=2) = \frac{9}{36}$, $P(X=4) = \frac{9}{36}$ $P(X=3) = \frac{8}{36}$

Similarly obtain the marginal distribution of 'y' defined as

$$P(Y=y) = \Sigma_x P(Y=y, X=x)$$

$$P(Y=1) = \frac{11}{36}$$

$P(Y=2) = \frac{9}{36}$

$P(Y=3) = \frac{7}{36}$

$P(Y=4) = \frac{9}{36}$

Marginal distribution of 'x'

Values of X, x :	1	2	3	4
$P(x)$:	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

Marginal distribution of 'y'

values of Y, y :	1	2	3	4
$P(y)$:	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

ii) The conditional distribution probability function of X given y is defined as follow

$\therefore P(X=x \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)}$

$= \frac{4/36}{11/36}$

$\therefore P(X=x \mid Y=1) = \frac{4}{11}$

$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)}$

$= \frac{1/36}{11/36}$

$\therefore P(X=2 \mid Y=1) = \frac{1}{11}$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)}$$

$$= \frac{5/36}{11/36}$$

$$\therefore P(X=3 \mid Y=1) = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)}$$

$$= \frac{1/36}{11/36}$$

$$\therefore P(X=4 \mid Y=1) = \frac{1}{11}$$

Similarly ~~data~~ obtain the conditional distribution function of Y given $X=2$ defined as given below

$$P(Y=1 \mid X=2) = \frac{P(Y=1 \mid X=2)}{P(X=2)}$$

$$= \frac{1/36}{9/36}$$

$$= \frac{1}{9}$$

$$P(Y=2 \mid X=2) = \frac{P(Y=2, X=2)}{P(X=2)}$$

$$= \frac{3/36}{9/36}$$

$$= \frac{3}{9}$$

$$P(Y=3 \mid X=2) = \frac{P(Y=3, X=2)}{P(X=2)}$$

$$= \frac{3/36}{9/36} = \frac{3}{9}$$

$$P(Y=4 \mid X=2) = \frac{P(Y=4, X=2)}{P(X=2)}$$

$$= \frac{\frac{2}{36}}{\frac{9}{36}} = \frac{2}{9}$$

Hence the conditional distribution function of X given Y = 1

x :	1	2	3	4
$P(X=x \mid Y=1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

And the conditional distribution function of Y given X = 2

y :	1	2	3	4
$P(Y=1 \mid X=2)$:	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{3}{9}$	$\frac{2}{9}$

3. For the following bivariate probability distribution of X and Y

X\Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Find i) $P(X \le 1)$ iv) $P(X \le 1, Y \le 3)$

ii) $P(X \le 1, Y = 2)$ v) $P(X + Y \le 4)$

iii) $P(Y \le 3)$ vi) $P(X \le 1 \mid Y \le 3)$

X\Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	2/8
2	1/32	1/32	1/64	1/64	0	2/64
P(y)	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$

i) $P(X \le 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$P(X \le 1) = \frac{28}{32}$$

ii) $P(X \le 1, Y=2) = P(X=0,1, Y=2)$

$$= P(X=0, Y=2) + P(X=1, Y=2$$

$$= 0 + 1/16$$

$$P(X \le 1, Y=2) = \frac{1}{16}$$

iii) $P(Y \le 3) = P(Y=1,2,3)$

$$= P(Y=1) + P(Y=2) + P(Y=3)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$P(Y \le 3) = \frac{23}{64}$$

iv) $P(X \leq 1, Y \leq 3) = P(X=0,1,\ Y=1,2,3)$

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) +$
$P(X=1, Y=1) + P(X=1, Y=2) +$
$P(X=1, Y=3)$

$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$P(X \leq 1, Y \leq 3) = \frac{9}{32}$

v) $P(X+Y \leq 4) = P(X=0,1,2,\ Y=1,2,3,4)$

$= P(X=0, Y=1) + P(X=0, Y=2) +$
$P(X=0, Y=3) + P(X=0, Y=4) +$
$P(X=1, Y=1) + P(X=1, Y=2) +$
$P(X=1, Y=3) + P(X=1, Y=4) +$
$P(X=2, Y=1) + P(X=2, Y=2) +$
$P(X=2, Y=3) + P(X=2, Y=4)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{32} +$
$\frac{1}{32} + \frac{1}{64} + \frac{1}{64}$

$= \frac{2+4+4+4+8+8+2+2+1+1}{64}$

$= \frac{36}{64}$

$\therefore P(X+Y \leq 4) = \frac{18}{32}$

vi) $P(X\leq 1 \mid Y\leq 3) = \dfrac{P(X\leq 1, Y\leq 3)}{P(Y\leq 3)}$

$= \dfrac{9/32}{23/64}$

$\frac{9}{32}\times\frac{64}{23}$

$\therefore P(X\leq 1 \mid Y\leq 3) = \frac{18}{23}$

4. Find the marginal density function X and Y is, If $f(x,y) = \frac{2}{5}(2x+3y)$, $0\leq x\leq 1, 0\leq y\leq 1$.

Marginal density function of 'x' is

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$f(x) = \int_0^1 \frac{2}{5}(2x+3y)\,dy$

$= \frac{2}{5}\int_0^1 (2x+3y)\,dy$

$= \frac{2}{5}\left\{(2xy)_0^1 + \left(\frac{3y^2}{2}\right)_0^1\right\}$

$= \frac{2}{5}\left\{(2x-0) + \left(\frac{3}{2}-0\right)\right\}$

$= \frac{2}{5}\left(2x + \frac{3}{2}\right)$

$= \frac{2}{5}\left(\frac{4x+3}{2}\right)$

$\therefore f(x) = \frac{4x+3}{5}$

Marginal density function of 'y' is

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_0^1 \frac{2}{5}(2x+3y)\,dx$

$= \frac{2}{5}\int_0^1 (2x+3y)\,dx$

$= \frac{2}{5}\left\{\left[\frac{2x^2}{2}\right]_0^1 + \left[3xy\right]_0^1\right\}$

$= \frac{2}{5}\left([1-0]+[3y-0]\right)$

$= \frac{2}{5}[1+3y]$

$\therefore f(y) = \frac{2+6y}{5}$

---

5. Let X and Y have joint distribution function $f(x,y)=2,\ 0<x<y<1$. Find

i) Marginal density function and

ii) Conditional density function of Y given $X=x$

i) $f(x) = \int_x^1 2\,dy$

$= (2y)_x^1$

$= 2-2x$

$\therefore f(x) = 2(1-x)$

$f(y) = \int_0^y 2\,dx$

$= (2x)_y^0$

$= ~~2y-0~~\ 0-2y = -2y$

$\therefore f(y) = -2y$

ii) Conditional density function

$X = x$

$$= \frac{f(x,y)}{f(x)}$$

$$= \frac{2}{2(1-x)}$$

$$= \frac{1}{1-x}$$

$\therefore$ Conditional density function

x is $\frac{1}{1-x}$

6. The joint probability density of the two dimensional rand[om] variable is $f(x,y) = \begin{cases} \frac{8}{9} xy ; \\ 0 \quad ; ot[herwise] \end{cases}$

i) find the marginal density func[tion of] X and

ii) find the conditional density f[unction] of X given Y & Y given X

Solution:

i) Marginal density function of

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{2} \frac{8}{9} xy \; dy$$

$$= \int_{x}^{2} \frac{8}{9} xy \; dy$$

$$= \frac{8}{9} \int_x^2 xy\, dy$$

$$= \frac{8}{9} \left[\frac{xy^2}{2} \right]_x^2$$

$$= \frac{8}{9} \left[\frac{4x}{2} - \frac{x^3}{2} \right]$$

$$= \frac{8}{9} \left[\frac{4x - x^3}{2} \right]$$

$$= \frac{4}{9} \left[4x - x^3 \right]$$

$$= \frac{16x - 4x^3}{9}$$

$$\therefore f(x) = \frac{4x(4-x^2)}{9}$$

Marginal density function of Y is

$$F(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_1^y \frac{8}{9} xy\, dy$$

$$= \frac{8}{9} \int_1^y xy\, dx$$

$$= \frac{8}{9} \left[\frac{x^2 y}{2} \right]_1^y$$

$$= \frac{8}{9} \left[\frac{y^2}{2} - \frac{y}{2} \right]$$

$$= \frac{8}{9} \left[\frac{y^3 - y}{2} \right]$$

$$= \frac{8(y^3 - y)}{18}$$

$$\therefore f(y) = \frac{4y^3 - 4y}{9}$$

ii) Conditional density function of x given y

$$= \frac{f(x,y)}{f(y)}$$

$$= \frac{\frac{8}{9}xy}{4y(y^2-1)/9}$$

$$= \frac{8xy}{9} \times \frac{x}{4y(y^2-1)}$$

$$x/Y = \frac{2x}{(y^2-1)}$$

conditional density function of Y given x

$$= \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{8}{9}xy}{\frac{4x(4-x^2)}{9}}$$

$$= \frac{8xy}{9} \times \frac{9}{4x(9-x^2)}$$

$$Y/X = \frac{2y}{4-x^2}$$

---

1. Two random variable have been following joint probability density function

$$f(x,y) = \begin{cases} 2-x-y & ; 0 \le x \le 1, 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Find i) Co-variance (x,y) ii) Correlation.

1) Marginal density function of 'x'

$$f(x) = \int_{-\infty}^{\infty} f(x-y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \int_0^1 2\,dy - \int_0^1 x\,dy - \int_0^1 y\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= \left[2 - x - \frac{1}{2} - 0\right]$$

$$\therefore f(x) = \frac{3}{2} - x$$

Marginal density function of 'y'

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \int_0^1 2\,dx - \int_0^1 x\,dx - \int_0^1 y\,dx$$

$$= \left[2x - \frac{x^2}{2} - xy\right]_0^1$$

$$= \left[2 - \frac{1}{2} - y\right]$$

$$\therefore f(y) = \frac{3}{2} - y$$

$$E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^1 x\left(\frac{3}{2} - x\right)dx \Rightarrow \int_0^1 \left(\frac{3x}{2} - x^2\right)dx$$

$$= \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^1 = \frac{3x^2}{4} - \frac{x^3}{3}$$

$$= \frac{3}{4} - \frac{1}{3} \quad \frac{9-4}{12} = \frac{5}{12}$$

$$= \left[\frac{3}{4} - \frac{1}{3} - 0 \right]$$

$$= \left[\frac{9-4}{12} \right]$$

lll^ly

$$E(Y) = \int_{-\infty}^{\infty} y F(y)\, dy$$

$$\therefore E(Y) = \frac{5}{12}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 \left(\frac{3}{2} - x \right) dx$$

$$= \int_0^1 \left(\frac{3x^2}{2} - x^3 \right) dx$$

$$= \left[\frac{3x^3}{2x^3} - \frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{3}{6} - \frac{1}{4} - 0 \right]$$

$$= \frac{6}{24}$$

$$\therefore E(x^2) = \frac{1}{4}$$

lll^ly

$$E(y^2) = \int_{-\infty}^{\infty} y^2 F(y)\, dy$$

$$\therefore E(y^2) = \frac{1}{4}$$

$$\sigma x^2 = Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{4} - \left[\frac{5}{12} \right]^2$$

$$= \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144}$$

$\sigma x^2 = \frac{11}{144}$

$\sigma x = \sqrt{\frac{11}{144}}$

$\therefore \sigma_x = \frac{\sqrt{11}}{12}$

III ly

$\sigma y^2 = E(Y^2) - [E(Y)]^2$

$= \frac{1}{4} - \left[\frac{5}{12}\right]^2$

$= \frac{1}{4} - \frac{25}{144}$

$= \frac{11}{144}$

$\therefore \sigma_y = \frac{\sqrt{11}}{12}$

$E(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^1 xy(2 - x - y)\, dx\, dy$

$= \int_0^1 \int_0^1 (2xy - x^2 y - xy^2)\, dx\, dy$

$= \int_0^1 \left[\frac{2x^2 y}{2} - \frac{x^3 y}{3} - \frac{x^2 y^2}{2}\right]_0^1 dy$

$= \int_0^1 \left[y - \frac{y}{3} - \frac{y^2}{2}\right] dy$

$= \left[\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6}\right]_0^1$

$= \left[\frac{1}{2} - \frac{1}{6} - \frac{1}{6} - 0\right]$

$\therefore E(x, y) = \frac{1}{6}$

ii) Correlation $r = \frac{cov(X,Y)}{\sigma_x \sigma_y}$

where

$cov(x,y) = E(X,Y) - E(X)E(Y)$

$= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12}$

$= \frac{1}{6} - \frac{25}{144}$

$cov(X,Y) = \frac{-1}{144}$

$\therefore r = \frac{\frac{-1}{144}}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}} = \frac{-1}{144} \times \frac{144}{11}$

$\therefore r = \frac{-1}{11}$

Result:

Co-Varriance $(X,Y) = \frac{-1}{144}$

Correlation $r = \frac{-1}{11}$

Unit-2

Mathematical Expectation:

Let x be a discrate random variable taken posible outcome values

$x_1, x_2, \ldots, x_n$

Let $P(x_1) = P(X = x_1)$, $i = 1, 2, \ldots$

then the expecterd value of x, denote by $E(x)$ is defined as

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i) \text{ if the series}$$

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$ ~~if the series~~

convergent absolutely

If x be the continuous random variable with p.d.f of $f(x)$ then

$$E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

If this integral is absolutely convergent

Function of a random variable.

Let X be the random variable and $y = H(X)$. If Y is a discrate random variable with posible values $y_1, y_2 \ldots y_n$ and $q(y_1), q(y_2) \ldots$ are the Probability Expectation of Y

E(y) is defined as

$E(Y) = \sum_{i=1}^{\infty} y\, q(y)$. for the series of

R.H.S is absolutely convergent

If 'Y' is a random variable with p

q f(y) then, we defined $E(y) = \int_{-\infty}^{\infty} y\, q$

if it exists.

The following theorem are the ma

on finding E(y) without the know

of the p.d. of y.

Theorem:

Let X be the random variable and $Y = H(x)$

i) If 'X' be is discrete with probability function $p(x_i)$, then

$$E(Y) = \sum_{i=1}^{\infty} H(x_i)\, p(x_i)$$

ii) If X' be the continuous random variable with p.d.f $f(x)$ then

$$E(Y) = \int_{-\infty}^{\infty} H(x)\, f(x)\, d(x).$$

1. A random variable x has a P.f. has follows

values of x : -1 0 1

P : 0.2 0.3 0.5

Evaluate :

i) $E(3x+1)$, ii) $E(X^2)$

i) $E(3X+1) = \Sigma(3X+1)\,P(x)$

$= (-3+1)(0.2) + (0+1)(0.3) + (3+1)(0.5)$

$= -0.4 + 0.3 + 2$

$E(3X+1) = 1.9$

ii) $E(X^2) = \Sigma X^2 P(X)$

$= (1 \times 0.2) + (0 \times 0.3) + (1 \times 0.5)$

$= 0.2 + 0 + 0.5$

$\therefore E(X^2) = 0.7$

---

2. A random Variable 'x' has probability function given.

X :	-3	6	9
$P(X=x_i)$:	1/6	1/2	1/3

Find $E(2X+1)$

$E(X) = \Sigma x\,P(x)$

$= (-6+1)\frac{1}{6} + (12+1)\frac{1}{2} + (18+1)\frac{1}{3}$

$= -(5)\frac{1}{6} + (13)\frac{1}{2} + (19)\frac{1}{3}$

$= \frac{-5}{6} + \frac{13}{2} + \frac{19}{3}$

$= \frac{72}{6}$

$\therefore E(K) = 12.$

Expectation of two dimensional random variable:

Let (x, y) be a two dimensional random variable and $z = H(x, y)$ be a real valued function of (x, y). Then 'z' is a one dimensional random variable and $E(z)$ is defined as follows.

i) If 'z' is a discrete random variable with possible values $z_1, z_2, \ldots$ and $P(z_i) = P(z = z_i)$

Then, $E(z) = \sum_{i=1}^{\infty} z_i \, P(z_i)$. If this series are R.H.S is absolutely convergent.

ii) If 'z' is a continuous random variable with p.d.f $f(z)$ then

$$E(z) = \int_{-\infty}^{\infty} z f(z) dz$$

If the integral is absolutely convergent we shall now state the theorem (without proof) with the help of which we can determine.

E(z), without knowing the probability distribution of z.

Theorem:

Let (X, Y) be a two dimensional random variable and $Z = H(X, Y)$

If (x, y) is a discrete random variable with probability function $P(x_i, y_i)$ then,

$$E(z) = \sum_j \sum_i H(x_i, y_i)$$

$P(x_i, y_i)$ if it exists.

If (x, y) is a continuous random variable with probability density function $f(x, y)$. Then,

$$E(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) \text{ of } (x, y) dx\, dy.$$

Properties of expected values:

Theorem 1:

If c is a constant $E(c) = c$.

Proof:

$$E(c) = \int_{-\infty}^{\infty} c f(x) dx$$

$$= c \int_{-\infty}^{\infty} f(x) dx$$

$$= c(1)$$

$\therefore E(c) = c$

$\therefore$ Hence, proved.

Theorem 2 :

If c is a constant and $Y = H(x)$ is a [cut off] variable then.

$$E\{c \cdot H(x)\} = c E\{H(x)\}$$

Proof :

$$E\{c \cdot H(x)\} = \int_{-\infty}^{\infty} c \cdot H(x) f(x) dx$$

$$= c \int_{-\infty}^{\infty} H(x) f(x) dx$$

$$= c E[H(x)]$$

∴ Hence, Proved.

Thearem 3 :

If X is a random variable and g_1 [cut off] and $g_2(x)$ are two functions of x then

$$E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x$$ [cut off]

assuming that the exists.

Proof :

$$E[g_1(x) + g_2(x)] = \int_{-\infty}^{\infty} [g_1(x) + g_2(x)] f(x$$ [cut off]

$$= \int_{-\infty}^{\infty} g_1(x) f(x) dx +$$

$$\int_{-\infty}^{\infty} g_2(x) f(x) dx$$

$$= E[g_1(x)] + E[g_2(x)]$$

∴ Hence, Proved.

Theorem 4:

$$E[g_1(x)] \le E[g_2(x)]$$

if $g_1(x) \le g_2(x)$ for all x.

Proof:

$$E[g_2(x)] - E[g_1(x)] = E[g_2(x) - g_1(x)]$$

$$= \int_{-\infty}^{\infty} [g_2(x) - g_1(x)] f(x) dx$$

$$\ge 0$$

$$\therefore E[g_1(x)] \le E[g_2(x)]$$

Theorem 5:

$$|E[g(x)]| \le E[|g(x)|]$$

Proof:

$$|E[g(x)]| = \left| \int_{-\infty}^{\infty} g(x) f(x) dx \right|$$

$$\le \int_{-\infty}^{\infty} |g(x)| f(x) dx$$

$$\therefore |E[g(x)]| \le E|g(x)|$$

$\therefore$ Hence, proved.

3. Theoretical Standard Distribution

Discrete Distribution	Continuous Distribution
Binomial Distribution	Uniform Distribution
Poisson Distribution	Normal Distribution.

Binomial Distribution:

Let a random experiment be performed repeatedly. each repetation being called a trial and be the occurrence of an event in an trial be called a success and its non-occurance a failure.

The experession for the addition theorem of probability is given by

$$^nC_x \cdot p^x \cdot q^{n-x}$$

The Probability distribution of the number of success obtained is called the Binomial probability distribution for the obvious reson that the probabilities of 0, 1, 2, ... n success viz.

$$q^n . {}^nC_1 . p^1 q^{n-1}, {}^nC_2 . p^2 . q^{n-2} ... p^n$$

are the successive terms of the binomial expansion $(q+p)^n$

<u>Probability mass function:</u>

A random variable is "x" said to follow binomial distribution. If it assumes only non-negative values and it is given by :

$$P(X=x) = P(x) = \begin{cases} {}^nC_x \; p^x q^{n-x}; & x = 0, 1, 2, 3, \dots n \\ 0; \text{ otherwise} & \end{cases}$$

$q = 1-p$

The Probability

$P(x) = {}^nC_x \; p^x q^{n-x}, x = 0, 1 .... n$ is also sometimes denoted by the $b(x, n, p)$.

Total Probability of Binomial distribution

$P(X=x) = P(x) = {}^nC_x\, p^x q^{n-x}, \quad x = 0,1,2,\ldots$

$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x}$

$= {}^nC_0\, p^0 q^{n-0} + {}^nC_1\, p^1 q^{n-1} + \ldots + {}^nC_n\, p^n q^{n-n}$

$= q^n + {}^nC_1\, p^1 q^{n-1} + \ldots + {}^nC_n\, p^n \quad \text{--- (1)}$

$= (q+p)^n$

$= (1)^n$

$\therefore P(x) = 1$

Mode of Binomial Distribution:

we have:

$$\frac{P(x)}{P(x-1)} = \frac{{}^nC_x\, p^x q^{n-x}}{{}^nC_{x-1}\, p^{x-1} q^{n-(x-1)}}$$

$$= \frac{\frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}}{\frac{n!}{(x-1)!\,(n-x+1)!}\, p^{x-1} q^{n-x+1}}$$

$$= \frac{\frac{1}{1\cdots(x-1)x \cdot 1\cdots(n-x)}}{\frac{1}{1\cdots(x-1)\cdot 1\cdots(n-x)(n-x+1)\, p^{-1} q^{-1}}}$$

$$= \frac{\frac{1}{x}}{\frac{q}{(n-x+1)p}} = \frac{1}{x} \cdot \frac{(n-x+1)p}{q}$$

Mode is the value of x for which $P(x)$ is maximum.

$$\frac{P(x)}{P(x-1)} = \frac{1}{x} \cdot \frac{(n-x+1)p}{q}$$

~~characteristics function of Binomial distribution~~

1. The mean of a binomial distribution is 5 and standard deviation is 2. Determine the distribution.

Solution:

Given: mean $np = 5 \rightarrow (1)$

Standard deviation $\sqrt{npq} = 2$

varriance $npq = 4$

$$npq = 4$$

$$5q = 4$$

$$q = \frac{4}{5}$$

$$p = 1 - q$$

$$= 1 - \frac{4}{5} = \frac{5-4}{5} = \frac{1}{5}$$

$$\therefore p = \frac{1}{5}$$

$(1) \Rightarrow n\left(\frac{1}{5}\right) = 5 \qquad n = 25.$

Hence Binomial distribution is

$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}; \; x = 0, 1, 2 \dots n$

$P(x) = {}^{25}C_{x} \cdot \left(\frac{1}{5}\right)^{x} \left(\frac{4}{5}\right)^{25-x}; \; x = 0, 1, \dots 25$

The Binomial Distribution for the mean is 4 and Varriance is 3 and also

$P(x) = 15$

Solution:

mean $np = 4 \rightarrow (1)$

varriance, $npq = 3$

$npq = 3$

$(4)q = 3$

$q = 3/4$

$p = 1 - q = 1 - \frac{3}{4} = \frac{4-3}{4} \therefore p = \frac{1}{4}$

$(1) \rightarrow n\left(\frac{1}{4}\right) = 4$

$n = 16$

Hence the Binomial Distribution is

$P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$; $x = 0, 1, 2, \ldots n$

$P(x = 15) = 16C_{15}\, p^{15}\, q^{16-15}, x$

$= {}^{16}C_{15}\left(\frac{1}{4}\right)^{15}\left(\frac{3}{4}\right)$

$= 16 \times 3.75 \times 9.31 \times 0.0$

Find the Binomial Distribution if $n = 6$ and if

$9(P)(x=4) = P(x=2)$

Solution:

$$n = 6$$

$$P(x) = {}^6C_x \, p^x q^{6-x} \; ; \; x = 0, 1, \dots 6$$

$$9P(x=4) = P(x=2)$$

$$9\,{}^6C_4 \, p^4 q^{6-4} = {}^6C_2 \, p^2 q^{6-2}$$

$$9 \cdot 15 p^4 q^2 = 15 p^2 q^4$$

$$9 p^2 = q^2$$

$$3p = q$$

$$3p = 1 - p$$

$$3p + p = 1$$

$$4p = 1$$

$$\therefore p = 1/4$$

$$q = 1 - p = 1 - \frac{1}{4} = \frac{4-1}{4} \Rightarrow q = \frac{3}{4}$$

Binomial Distribution is

$$P(x) = {}^6C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^2 \text{ for } (x=4)$$

$$P(x) = {}^6C_2 \left(1/4\right)^2 \left(3/4\right)^4 \text{ for } (x=2).$$

4. 10 coins are tossed simultaneously, find Probability of getting.

i) At least 7 heads

ii) Exactly 7 heads

iii) At Least the most 7 heads.

solution:

Number of trials = 10

p = Probability of head in the tossed of coin = $\frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{2-1}{2} \therefore q = \frac{1}{2}$

i) Probability of getting at least 7 heads

$P(X \geq 7) = P(7) + P(8) + P(9) + P(10)$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} +$

$\quad {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10-10}$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)$

$\quad {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^{10} + {}^{10}C_8 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10} + {}^{10}C_{10} \left(\frac{1}{2}\right)$

$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$

$= \frac{1}{1024} [120 + 45 + 10 + 1]$

$= \frac{176}{1024} \quad \therefore P(X \geq 7) = \frac{11}{64}$

ii) $P(x=7) = {}^{10}C_7 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-7}$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^{10}$$

$$= 120 \times \frac{1}{1024}$$

$$\therefore P(X=7) = \frac{15}{128}$$

iii) $P(X \leq 7) = P(0) + P(1) + \dots + P(7)$

$$= 1 - \frac{1}{1024} \{P(x > 7)\}$$

$$= 1 - \frac{1}{1024} \{P(8) + P(9) + P(10)\}$$

$$= 1 - \frac{1}{1024} \{45 + 10 + 1\}$$

$$= 1 - \frac{1}{1024} \times 56$$

$$= 1 - \frac{7}{128}$$

$$= \frac{128-7}{128}$$

$$\therefore P(X \leq 7) = \frac{121}{128}$$

Normal Distribution.

A random variable 'x' is said to have a normal distribution with Parameter μ & σ^2, if its Probability density function (p.d.f) is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$ $-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

M.G.F of Normal Distributions

we have,

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let

$$z = \frac{x-\mu}{\sigma} \Rightarrow \sigma z = x-\mu \Rightarrow x = \mu + \sigma z$$

$$dx = 0 + \sigma dz$$

$$dx = \sigma dz$$

$$E(e^{tx}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-1/2 z^2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu + t\sigma z} e^{-1/2 z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu} e^{t\sigma z} e^{-1/2 z^2} dz$$

=

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[(z - t\sigma)^2 - t^2\sigma^2]} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-t\sigma)^2} e^{1/2 t^2 \sigma^2} dz$$

$$= \frac{e^{t\mu + \frac{1}{2} t^2 \sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2} dz$$

Let $u = z - t\sigma$, $du = dz$

$$= \frac{e^{t\mu + 1/2 t^2 \sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 u^2} du$$

$$= \frac{e^{t\mu + 1/2 t^2 \sigma^2}}{\sqrt{2\pi}} \sqrt{2\pi} = e^{t\mu + \frac{1}{2} t^2 \sigma^2}$$

Mean and variance of Normal distribution we have

$$M_x(t) = e^{t\mu + 1/2 \sigma^2 t^2}$$

Mean $\mu_1' = \left[\frac{d}{dt} M_x(t)\right] = \frac{d}{dt}\left[e^{t\mu + 1/2 t^2 \sigma^2}\right]_{t=0}$

$$= \left[(\mu + t\sigma^2) e^{t\mu + \frac{1}{2} t^2 \sigma^2}\right]_{t=0}$$

$\mu_1' = \mu$

Variance $\mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]$

$= \frac{d}{dt}\left[\underbrace{(\mu + t\sigma^2)}_{u}\underbrace{(e^{t\mu + \frac{1}{2}t^2\sigma^2})}_{v}\right]_{t=0}$

$= [(0+\sigma^2)e^{t\mu + \frac{1}{2}t^2\sigma^2} + (\mu + t\sigma^2)(\mu + t\sigma^2)$

$(e^{t\mu + \frac{1}{2}t^2\sigma^2})$

$= [\sigma^2(1) + \mu(\mu)(1)]$

$= \sigma^2 + \mu^2$

$\mu_2 = \sigma^2 + \mu^2 - \mu^2$

$= \sigma^2$

$\mu_2 = \sigma^2.$

Students "t" distribution:

Let x_i : ($i = 1, 2, \ldots, n$) be a random sample of size "n" from a normal Population with mean μ and varriance σ.

When student's "t" distribution, is defined by statistics

$$t = \frac{\bar{x} - \mu}{S\sqrt{n}}$$

where.

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, is the sample mean and

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

is an unbiased estimate of the population variance σ^2, and it follows student's 't' distribution with $v = (n-1)$ d.f (degrees of freedom) with p.d.f.

$$f(t) = \frac{1}{\sqrt{v} \cdot B\left(\frac{1}{2}, \frac{v}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{v}\right)^{\frac{v+1}{2}}} \quad -\infty < t < \infty$$

"F" Distribution

If x and y are two independent chi square variates with v_1 and v_2 d.f respectively, then F-statistics is defined by

$$F = \frac{x/v_1}{y/v_2}$$

In otherwords, F is defined as the ratio of two idependent chi-square variates divided by the respective degrees of freedom and it it follows Snedecor's F-distribution with (v_1, v_2) d.f with probability function given by

$$f(F) = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \cdot \frac{F^{\frac{v_1}{2}-1}}{\left(1+\frac{v_1}{v_2}F\right)^{\frac{v_1+v_2}{2}}}, \quad 0 \le F < \infty$$

chi-Square Distribution

The Square of a standard normal variate is known as chi-square variate with one degree of freedom (d.f)

Thus if $X \sim N(\mu, \sigma^2)$. Then,

$z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and

$z^2 = \left(\frac{x-\mu}{\sigma}\right)^2$ is a chi-square variate with 1 (d.f)

In general if x_i ($i = 1, 2, 3, \ldots n$) are "n" independent normal variates with mean "μ_i" and "σ_i"; $i = 1, 2, 3, \ldots n$

Then, $\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma}\right)^2$ is a

chi-square (χ^2 variate with "n" d.f)

and $dP(\chi^2) = \frac{\mu}{2^{n/2}\sqrt{\frac{n}{2}}} e^{-\chi^2/2} (\chi^2)^{n/2 - 1} d\chi^2$

$, 0 \le \chi^2 < \infty$

which is the required p.d.f of chi square (χ^2) distribution with "n" d.f.

1. Let x is a possion distribution random variable such that $P(X=1)=0.3$ & $P(x=2)=0.2$ find $P(X=0)$

solution:

If x is a possion random variable with parameters λ

$$P(x=1) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-\lambda}\lambda^{1}}{1!} = \frac{e^{-\lambda}\lambda}{1!} = 0.3 \rightarrow (1)$$

$$P(x=2) = \frac{e^{-\lambda}\lambda^{2}}{2!} = 0.2 \rightarrow (2)$$

equation $2/1 \Rightarrow$

$$\frac{\frac{e^{-\lambda}\lambda}{1!}}{\frac{e^{-\lambda}\lambda^{2}}{2!}} = \frac{0.3}{0.2} \quad \frac{\frac{e^{-\lambda}\lambda^{2}}{2!}}{\frac{e^{-\lambda}\lambda}{1!}} = \frac{0.2}{0.3}$$

$$\frac{\lambda}{2!} = \frac{0.2}{0.3} = \lambda = 4/3 = 1.333$$

Then

$$P(X=0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-\lambda} = e^{-4/3} = e^{-1.333}$$

$$P(X=0) = 0.264.$$

2. If a random variable x follows a poisson distribution such that P(x=2) = P(x=1)
find $P(X=0)$

Solu:

Given that.

$P(X=2) = P(X=1)$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{e^{-\lambda}\lambda^{1}}{1!} \Rightarrow \lambda = 2$$

Then

$$P(X=0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-\lambda} = e^{-2} = 0.1353$$

---

3. The no of accidents in a year attributed to taxi drivers in a city follow poission distribution with mean 3 and of 1000 taxi derivers find approximation the no. of. driver with

i) No accidents in a year.

ii) more that 3 accidents in a year

Solution.

Given that

$\lambda = 3 \quad N = 1000$

$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \frac{e^{-3}3^{x}}{0!} = e^{-3} = P(x=0)$

$P(x=0) = 0.05$

i) Number of drivers with no accidents.

$= NP(x) = 1000 \times 0.05 = 50$

ii) $P(x>3) = 1 - P(x \leq 3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$= 1 - \left[\frac{e^{-3}3^{0}}{0!} + \frac{e^{-3}3^{1}}{1!} + \frac{e^{-3}3^{2}}{2!} + \frac{e^{-3}3^{3}}{3!}\right]$

$= 1 - e^{-3}\left[1 + 3 + 9/2 + 27/6\right]$

$= 1 - e^{-3}\left[156/12\right]$

$= 1 - (0.05)(13) = 1 - 0.65 = 0.35$

Number of derivers with more than 3 accidents in a year.

$= NP(X)$

$= 1000 \times 0.35$

$= 350.$

Tchebychev's inequality:-

Statement:

If x is a random Variable with mean μ and Variance is σ^2 then for any positive number k, we have,

$$p\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2} \quad (\text{or})$$

$$p\{|x-\mu| < k\sigma\} \geq 1-\frac{1}{k^2}$$

Proof:

Case (i) x is a continuous random variable we know that

$$Var(x) = \sigma^2$$

$$= E[x - E(x)]^2$$

$$= E(x-\mu)^2$$

By the definition of probability density function.

$$Var(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$\int_{-\infty}^{\mu-k\sigma} + \int_{\mu-k\sigma}^{\mu+k\sigma} + \int_{\mu+k\sigma}^{\infty}$$

$$Var(x) = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x) dx$$

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\, dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx \quad \rightarrow (1)$$

Consider I st integral

$$x \leq \mu - k\sigma$$

$x-\mu \leq -k\sigma$

$(x-\mu)^2 \leq (-k\sigma)^2$

$(x-\mu)^2 \leq k^2\sigma^2$

$(x-\mu)^2 \leq k^2\sigma^2$

Consider 2nd integral

$x \geq \mu+k\sigma$

$x-\mu \geq k\sigma$

$(x-\mu)^2 \geq k^2\sigma^2$

① $\Rightarrow \sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)$

$\frac{\sigma^2}{k^2\sigma^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx$

$\frac{1}{k^2} \geq P(x \leq \mu-k\sigma) + P(x \geq \mu+k\sigma)$

$\frac{1}{k^2} \geq P(|x-\mu|) \geq k\sigma$

$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

Also,

$P\{|x-\mu| \geq k\sigma\} + P\{|x-\mu| < k\sigma\} = 1$

$P\{|x-\mu| < k\sigma\} = 1-P\{|x-\mu| \geq k\sigma$

$P\{|x-\mu| < k\sigma\} \geq 1-\frac{1}{k^2}$

Case (ii)

X is a discrete random Variable the proof follows exactly similarly on replacing integration by summation.

Note (or) Remark:-

If we take $k\sigma = c > 0$

$$k = \frac{c}{\sigma}$$

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2} \text{ and}$$

$$P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$P\{|X-E(X)| \geq c\} \leq \frac{Var(X)}{c^2}$$

and $P\{|X-E(X)| \quad \} \geq \frac{1-Var(X)}{c^2}$

① A geometric dice is thrown 600 times find the lower bound for the probability of getting 80 to 120 lines.

Soln:-

$$\left.\begin{aligned} P\{|X-\mu| \geq k\sigma) &\leq \frac{1}{k^2} \\ P[|X-\mu| < k\sigma) &\geq 1 - \frac{1}{k^2} \end{aligned}\right\} \rightarrow ①$$

$n = 600$, $\mu \Rightarrow E(X) = np$, $\sigma^2 = Var(X) = npq$

$P = 1/6$

$p + q = 1$

$\frac{1}{6} + q = 1 \qquad q = 5/6$

$\mu = E(X) = np$

$= 600 \times \frac{1}{6}$

$\boxed{\mu = 100}$

$\sigma^2 = Var(X) = npq = 600 \times \frac{1}{6} \times \frac{5}{6}$

$\sigma^2 = \frac{500}{6}$

$\sigma^2 = \frac{250}{3}$

$\sigma = \frac{\sqrt{250}}{\sqrt{3}}$

① ⇒ $P[\,|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$P\left[100 - k\frac{\sqrt{250}}{\sqrt{3}} < x < 100 + k\frac{\sqrt{250}}{\sqrt{3}}\right] \geq 1 - \frac{1}{k^2}$

$P\left(100 - \left(\frac{\sqrt{3}}{\sqrt{250}} \times 20\right)\frac{\sqrt{250}}{\sqrt{3}} < x < 100 + \left(\frac{\sqrt{3}}{\sqrt{250}} \times 20\right)\frac{\sqrt{250}}{\sqrt{3}}\right) \geq \left(1 - \frac{1}{\left(\frac{\sqrt{3}}{\sqrt{250}} \times 20\right)^2}\right)$

$P\{80 < x < 120\} \geq 1 - \frac{250}{3 \times 400}$

$\geq 1 - \frac{250}{1200}$

$\geq \frac{1200 - 250}{1200}$

$\geq \frac{950}{1200} = 0.7916$

$k\sigma = c$

$k = \frac{20}{\sqrt{250}/\sqrt{3}}$

$k = \frac{\sqrt{3}}{\sqrt{250}} \times 20$

2) A symmetric dice is thrown 360 times find the lower bound for the probability of getting 55 to 65 sixes

Soln:

$$\left.\begin{aligned} P(|x-\mu| \geq k\sigma) &\leq \frac{1}{k^2} \\ P(|x-\mu| < k\sigma) &\geq 1-\frac{1}{k^2} \end{aligned}\right\} \rightarrow (1)$$

$n = 360 \quad \mu = E(x) = np$

$\sigma^2 = Var(x) = npq$

$p = \frac{1}{6} \quad p+q = 1$

$\frac{1}{6}+q = 1 \quad q = 5/6$

$\mu = E(x) = np = 360 \times \frac{1}{6} = 60$

$\sigma^2 = Var(x) = npq$

$= 360 \times \frac{1}{6} \times \frac{5}{6}$

$= 50$

$= \sqrt{25 \times 2}$

$\sigma = 5\sqrt{2}$

$k\sigma = c$

$k = c/\sigma$

$= \frac{5}{5\sqrt{2}}$

$(1) \Rightarrow P(|x-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$

$P(60 - k5\sqrt{2} < x < 60 + k5\sqrt{2}) \geq 1-\frac{1}{k^2}$

$P(60 - \frac{5}{5\sqrt{2}} 5\sqrt{2} < x < 60 + \frac{5}{5\sqrt{2}} \times 5\sqrt{2}) \geq 1-\frac{1}{(5/5\sqrt{2})^2}$

$P(60-5 < x < 60+5) \geq 1-\frac{1}{1/2}$

$P[55 < x < 65] \geq 1-2$

$\therefore P(55 < x < 65) \geq -1$

There is no possible to get si in b/w 55 to 65 because we get negative value.

3) If x is the number of source in an thrown of die show that $P\{|X-\mu| > 2.5\} \leq 0.47$

Soln:

On rolling a dice

x	1	2	3	4	5	6
$P(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \sum_x x f(x)$$

$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$

$$= \frac{21}{6} = 3.5$$

$$E(X^2) = \sum x^2 f(x)$$

$$= \frac{1 + 4 + 9 + 16 + 25 + 36}{6}$$

$$= 91/6 = 15.16$$

$$Var[X] = E(X^2) - (E(X))^2$$

$$= 15.16 - (3.5)^2$$

$$= 15.16 - 12.25$$

$$= 2.91$$

1.7058

$$P\{|x-\mu| > c\} \leq \frac{\sigma^2}{c^2}$$

$$P\{|x-\mu| > 2.5\} \leq \frac{2.91}{6.25}$$

$$\leq 0.4656$$

$$P(|x-\mu| > 2.5) \leq 0.47$$

4) Two unbiased dice are thrown. if x is the sum of the number of source in a n thrown of dice show that $P\{|x-7| \geq 3\} = \frac{35}{54}$

Soln:

x	1	2	3	4	5	6	7
P(X=x)	0	1/36	2/36	3/36	4/36	5/36	6/36

8	9	10	11	12
5/36	4/36	3/36	2/36	1/36

$$\text{Mean}(\mu) = E(x) = \sum_x x f(x)$$

$$= 0 + \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36}$$

$$+ \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36}$$

$$= \frac{252}{30}$$

$$\boxed{\mu = 7}$$

$$E(x^2) = \frac{4+18+48+100+180+294+330+324+300+242+144}{36}$$

$$= 1974/36 = 54.83$$

$Var(x) = E(x^2) = \sigma^2 = E(x^2) - (E(x))^2$

$= 54.83 - (7)^2$

$= 54.83 - 49$

$\sigma^2 = 5.83 \qquad \sigma = 2.4145$

$P\{|x-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$

$P\{|x-7| \geq 3\} \leq \frac{5.83}{9} \times \frac{6}{6}$

$P\{(|x-7|) \geq 3\} \leq \frac{34.98}{54}$

$\leq \frac{35}{54}$

Characteristic function:

The characteristic function is defined as

$\phi_x(t) = E[e^{itx}]$

where $E[e^{itx}] = \begin{cases} \int_{-\infty}^{\infty} e^{itx} f(x)dx \text{ (for con. pro function)} \\ \sum_x e^{itx} p(x) \text{ (for dis. pro fun.)} \end{cases}$

If $F_x(x)$ is the distribution function of a continuous random variable X then

$\phi_x(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$

Properties of Characteristic function

i) For any real, we have

$$\phi(t) = \int_{-\alpha}^{\alpha} dF(x) = 1$$

$$|\phi(t)| \leq 1 = \phi(0)$$

ii) $\phi(t)$ is continuous everywhere,

i.e) $\phi(t)$ is continous function of t in $(-\alpha, \alpha)$.

Rather $\phi(t)$ is uniformly continuous in "t"

iii) $\phi_x(-t)$ and $\phi_x(t)$ are conjugate function

i.e) $\phi_x(-t) = \overline{\phi_x(t)}$ where $\bar{a}$ is the conjugate of a.

iv) If the distribution function of a random variable x is symmetrical about zero.

If $1 - F(x) = F(-x)$.

$\Rightarrow f(-x) = f(x)$ then $\phi_x(t)$ is real valued and even function of t.

v) If x is same random variable and continuous function $\phi_x(t)$ and if $\mu_r' = E(x^r)$ Exists then

$$\mu_r' = (-i)^r \left[\frac{\partial^r}{\partial t^r} \phi(t)\right]_{t=0}$$

vi) $\phi_{cx}(t) = \phi_x(ct)$. c being a constant

viii) If x_1 & x_2 are independent random variables then

$\phi_{x_1}(t) + \phi_{x_2}(t) = \phi_{x_1}(t) \cdot \phi_{x_2}(t)$.

viii) If $u = \frac{x-a}{h}$ where a and h are constant then $\phi_u(t) = e^{-iat/h} \phi_x(t/h)$

ix) If $|\phi_x(s)| = 1$ for some $s \neq 0$. Then for some real a, x-a is a lattice variable with mesh $h = \frac{2\pi}{|s|}$.

Definition : Lattice and Mesh

A random variable x is said to be a lattice variable (or) lattice distribution if for sum $h > 0$.

$P(x/h \text{ is an integer}) = 1$

h is called the mesh.

Problems:-

① A discrete random variable has the probability function given by

x	-1	1
p(x)	1/2	1/2

find its characteristic function

Soln

$\phi_x(t) = E[e^{itx}]$

$E[e^{itx}] = \sum_x e^{itx} p(x)$

$= \sum_{x=-1} e^{itx} p(x)$

x - real valued function

M.G.F.

$M_X(t) = \int e^{tx} f(x) dx$

Mean $= E(X)$

Var. $= E(x^2) - E(x)^2$

$= e^{-it x_1} p(x_1) + e^{itx_2} p(x_2)$

$= e^{it(-1)} p(-1) + e^{it(1)} p(1)$

$= e^{-it}(1/2) + e^{it}(1/2)$

$= \frac{1}{2}(e^{-it} + e^{it}) = \cos ht$

2) Find the moment generating function of $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-e^{-x} & \text{for } x > 0 \end{cases}$

Soln:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-e^{-x} & \text{for } x > 0 \end{cases}$$

$$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}(1-e^{-x})$$

$$= 0 - (-e^{-x})$$

$$f(x) = e^{-x}$$

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_{-\alpha}^{\alpha} e^{tx} f(x)\,dx$$

$e^{-it(\frac{1}{2})} + e^{it(\frac{1}{2})}$

$$= \int_{-\alpha}^{0} e^{tx}(0) + \int_{0}^{\alpha} e^{tx} e^{-x}\,dx$$

$$= \int_{0}^{\alpha} e^{(t-1)x}\,dx$$

$$= \left(\frac{e^{x(t-1)}}{t-1}\right)_0^{\alpha} = 0 - \frac{1}{t-1}$$

$$= \frac{-1}{t-1} = \frac{-1}{-(1-t)} = \frac{1}{1-t}$$

3) Find the moment generating function of random Variable x with Probability density function

$$f(x) = \begin{cases} \frac{1}{2} e^{|x|} & -\infty < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Soln:

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{tx} \frac{1}{2} e^{-x}\, dx + \int_{0}^{\infty} e^{tx} \frac{1}{2} e^{x}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{tx} e^{-x}\, dx + \frac{1}{2} \int_{0}^{\infty} e^{tx} e^{x}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{(t-1)x}\, dx + \frac{1}{2} \int_{0}^{\infty} e^{(t+1)x}\, dx$$

$$= \frac{1}{2} \left(\frac{e^{x(t-1)}}{t-1}\right)_{-\infty}^{0} + \left(\frac{e^{x(t+1)}}{(t+1)}\right)_{0}^{\infty}$$

$$= \frac{1}{2} \left(\frac{1}{t-1} - 0 + 0 - \frac{1}{t+1}\right)$$

$$= \frac{1}{2} \left(\frac{1}{t-1} - \frac{1}{t+1}\right)$$

$$= \frac{1}{2} \left(\frac{t+1-t+1}{t^2-1}\right)$$

$$= \frac{1}{2} \times \frac{2}{t^2-1}$$

$$= \frac{1}{t^2-1}$$

Moment of binomial distribution

(or) find the mean & variable of binomial distribution:-

$$E[x] = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x \left(nC_x\, p^x q^{n-x} \right)$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x} \qquad nC_r = \frac{n!}{r!\,(n-r)!}$$

$$= \sum_{x=1}^{n} \frac{x\, n\, (n-1)!}{x(x-1)!\,(n-x+1-1)!} p^x q^{n-x}$$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} p^{x+1-1} q^{n-1-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} p^{x-1} q^{n-1-(x-1)} \quad (p+q)^{n-1}$$

①

$$= np\,(p+q)^{n-1}$$

$$= np\,(1)^{n-1}$$

$$E(x) = np$$

$$E(x^2) = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} (x^2 + x - x)\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x)\, n_{C_x} p^x q^{n-x} + \sum_{x=0}^{n} x\, n_{C_x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\, n_{C_x} p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=1}^{n} \frac{x(x-1)\, n(n-1)!}{x(x-1)!\,(n-x+2-2)} p^x q^{n-x} + np$$

$$= n \sum_{x=1}^{n} \frac{(x-1)(n-1)!}{(x-1)(x-2)!(n-2-(x-2)} p^{x+2-2} q^{n-x} + np$$

$$= n \sum_{x=2}^{n} \frac{(n-1)(n-2)}{(x-2)!((n-2)-(x-2))!} p^{x-2} q^{n-2(x)} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$E(x^2) = n(n-1)p^2 + np$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$Var(x) = npq$$

Moment generating function using binomial distribution:-

$$M_x(t) = E[e^{tx}]$$

$$E(x) = \Sigma x\, p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, \frac{nc_x\, p^2\, (n-2)\, pq}{(pe^t)^x\, q^{n-x}}$$

$$= np - np^2$$

$$= np(1-p)$$

$$var(x) = npq$$

Moment generating function using binomial distribution:-

$$M_x(t) = E[e^{tx}]$$

$$E(x) = \Sigma x\, p(x)$$

$$M_x(t) = \sum_{x=0}^{n} e^{tx}\, p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, nC_x\, p^x\, q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x\, (pe^t)^x\, q^{n-x}$$

$$M_x(t) = (pe^t + q)^n$$

Characteristic function using binomial distribution:-

$$\phi_x(t) = E[e^{itx}]$$

$$= \sum_{x=0}^{n} e^{itx} p(x)$$

$$= \sum_{x=0}^{n} e^{itx} \, nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^{it})^x q^{n-x}$$

$$\phi_x(t) = (pe^{it} + q)^n$$

(7). In a precision bombing attack there is a 50% chance that anyone bomb will strike the t[arget]. Two direct hits are required t[o] destroy the target completely. How many bomb must be dropp[ed] to given 99% chance or better completely destroying the targe[t].

Soln:

p = Probability that bom[b] strikes the targ[et]

$p = 50\%$

$p = \frac{50}{100}$ $\quad p = 1/2$.

Let n be the number of bom[bs] which should be dropped to ensure 99% chance or better of

Completely destroying the target-

⇒ The Probability that out of n bombs atleast two strikes the target is greater than 0.99%.

Let X be a random Variable representing the number of bombs striking the target. Then,

$$X \sim B(n, p = 1/2)$$

$$p(x) = p(X=x) = nC_x p^x q^{n-x}.$$

$$= nC_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}$$

$$= nC_x \left(\frac{1}{2}\right)^n \quad (x = 0, 1, 2 \ldots n)$$

$$p(X \geq 2) \geq 0.99$$

$$1 - p(X \leq 1) \geq 0.99$$

$$\{1 - p(0) + p(1)\} \geq 0.99$$

$$1 - \left\{nC_0\left(\frac{1}{2}\right)^n + nC_1\left(\frac{1}{2}\right)^n\right\} \geq 0.99$$

$$1 - \left(nC_0\left(\frac{1}{2}\right)^n + nC_1\left(\frac{1}{2}\right)^n\right) \geq 0.99$$

$$1 - 0.99 \geq [nC_0 + nC_1]\left(\frac{1}{2}\right)^n$$

$nC_0 = 1$
$nC_1 = n$

$$0.01 \geq (1+n)\frac{1}{2^n}$$

$$0.01 \geq \frac{1+n}{2^n}$$

$$2^n \geq \frac{1+n}{0.01}$$

$$2^n \geq (1+n)(100)$$

$$2^n \geq 100 + 100n$$

By trial method we find the inequality is satisfied by $\boxed{n = 11.}$

Hence the minimum number of bombs needed to destroy the target completely is 11.

$2^n = 10100$

Central moments of binomial distribution.

$$\mu_1 = np$$

$$\mu_2 = npq$$

$$\mu_3 = npq(q-p)$$

$$\mu_4 = npq(1+3(n-2)pq)$$

Skewness:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(npq(q-p))^2}{(npq)^3}$$

$$= \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$$

$$= \frac{(q-p)^2}{npq}.$$

$$= \frac{(1-p-p)^2}{npq}$$

$$\beta_1 = \frac{(1-2p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\,(1+3(n-2)pq)}{(npq)^2}$$

$$= \frac{1+3(n-2)pq}{npq}$$

$$= \frac{1+3npq-6pq}{npq}$$

$$= \frac{3npq}{npq} + \frac{1-6pq}{npq}$$

$$= 3 + \frac{1-6pq}{npq}$$

$$= \frac{1+3npq-6pq}{npq} = \frac{1+3npq-6pq}{npq}$$

$$= \frac{3npq}{npq} + \frac{1-6pq}{npq}$$

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

Kurtosis:-

$$\delta_1 = \sqrt{\beta_1} = \sqrt{\frac{(1-2p)^2}{npq}}$$

$$\delta_1 = \frac{1-2p}{\sqrt{npq}}$$

$$\delta_2 = \beta_2 - 3$$

$$= 3 + \frac{1-6pq}{npq} - 3$$

$$\delta_2 = \frac{1-6pq}{npq}$$

8) Recurrence relation for the moments of binomial distribution [Renovsky formula]

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right)$$

$$\mu_{r+1} = pq\left(\frac{nr}{\mu_{r-1}} + \frac{d\mu_r}{dp}\right)$$

Proof:

By a definition

$$\mu_r = E(X - E(X))^r$$

$$= E(X - np)^r$$

$p+q=1$

$q = 1-p$

$$= \Sigma((x-np)^r) \cdot p(x)$$

$$= \Sigma (x-np)^r \, nC_x \, p^x q^{n-x}$$

$$= \Sigma (x-np)^r \, nC_x \, p^x (1-p)^{n-x}$$

$$= \Sigma nC_x \left((x-np)^r \cdot p^x (1-p)^{n-x}\right)$$

D.w.r to 'p' we get

$$\frac{d\mu_r}{dp} = \Sigma nC_x \left[(x-np)^r (p^x (n-x)\right.$$

$$(1-p)^{n-x-1}(-1) + (1-p)^{n-x} \, x p^{x-1}) +$$

$$p^x \, (1-p)^{n-x} \, r \, (x-np)^{r-1} (-n)$$

$$= \sum {}^nC_x \left(-(x-np)^r \, p^x \, (n-x) \, (1-p)^{n-x-1} \right.$$

$$+ \sum {}^nC_x \left[x(x-np)^r . (1-p)^{n-r} \, p^{x-1} \right) -$$

$$\sum {}^nC_x \, n r p^x \left((1-p)^{n-x} (x-np)^{r-1} \right)$$

$$= (x-np)^r \left(- \sum {}^nC_x \, p^x \, (1-p)^{n-x-1} (n-x) \right.$$

$$+ \sum {}^nC_x \, (1-p)^{n-x} \, x \, p^x p^{-1} - n r \sum {}^nC_x$$

$$(1-p)^{n-x} \, (x-np)^{r-1} \Big\}.$$

$$= -nr \sum {}^nC_x p^x \, (1-p)^{n-x} \, (x-1) \, p^{r-1}$$

$$+ (x-np)^r \sum {}^nC_x \, p^x (1-p)^{n-x} \left[-(1-p)^{-1} \right.$$

$$\left. (n-x) + x p^{-1} \right)$$

$$= -nr \sum p(x) \, (x-np)^{r-1} + (x-np)^r \sum p(x)$$

$$\left(\frac{x}{p} - \frac{(n-x)}{1-p} \right)$$

$$= -nr \sum p(x) \, (x-np)^{r-1} + (x-np)^r \sum p(x)$$

$$\left(\frac{x(1-p) - p(n-x)}{p(1-p)} \right)$$

$$= -nr \sum p(x) \, (x-np)^{r-1} + \sum p(x) \, (x-np)^r$$

$$\left(\frac{x - xp - pn + px}{pq} \right)$$

$$= -nr\sum (x-np)^{r-1} p(x) + \frac{1}{pq} \sum (x-np)^{r+1} p(x)$$

$$\frac{d\mu_r}{dp} = -nr\mu_{r-1} + \frac{1}{pq} \mu_{r+1}$$

$$\frac{d\mu_r}{dp} + nr\mu_{r-1} = \frac{1}{pq} \mu_{r+1}$$

$$\mu_{r+1} = pq \left(\frac{d\mu_r}{dp} + nr\mu_{r-1} \right)$$

Theorem:

Additive property of binomia distribution.

Statement:

If $X_1 \sim B(n_1, p_1)$ & $X_2 \sim B(n_2, p_2)$ are two independent variable then

$$X_1 + X_2 \sim B(n_1 + n_2 p)$$

Proof:

Given that X_1 & X_2 are two independent random variable with parameters n_1, n_2 and p respectively

Let us consider moment generating function of X_1 and X_2 about origin.

$M_{X}(t) = E(e^{tx})$

$M_{X_1}(t) = (q + \ldots)^{n_1}$

$M(x_2)(t) = (pe^t + q)^{n_2}$

But the property of moment functio

$Mx_1(t) + Mx_2(t)$

$= Mx_1(t) M(x_2) t$

$= (pe^t + q)^{n_1} \cdot (pe^t + q)^{n_2}$

$= (pe^t + q)^{n_1 + n_2}$

This is the moment generating function of $x_1 + x_2$ with function $n_1 + n_2 p$

Hence by Unique theorem $x_1 + x_2$ is a binomial variable with parameter $n_1 + n_2$ & p.

Problems:

① The mean and variance of binomial distribution 4 and $\frac{4}{3}$ find $P(x \geq 1)$.

Soln:

$\text{Mean} = np = 4 \rightarrow ①$

$\text{Variance} = npq = \frac{4}{3} \rightarrow ②$

$x \sim B(n, p)$

$② \Rightarrow npq = \frac{4}{3}$

$4q = \frac{4}{3}$

$q = \frac{1}{3}$

$p + q = 1$

$p + 1/3 = 1 \qquad p = 2/3$

$p = 1 - 1/3$

$= \frac{3-1}{3}$

$= 2/3$

① ⇒ $n\left(\frac{2}{3}\right) = 4$

$$\boxed{n = 6}$$

$p(x \geq 1) = p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5) + p(x=6)$.

$= 6C_1 p^1 q^{6-1} + 6C_2 p^2 q^{6-2} + 6C_3 p^3 q^{6-3} + 6C_4 p^4 q^{6-4} + 6C_5 p^5 q^{6-5} + 6C_6 p^6 q^{6-6}$

$= 6C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + 6C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 + 6C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 + 6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + 6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + 6C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0$

$= 6\left(\frac{2}{3}\right)\left(\frac{1}{243}\right) + 15\left(\frac{4}{9}\right)\left(\frac{1}{81}\right) + 20\left(\frac{8}{27}\right)\left(\frac{1}{27}\right) + 15\left(\frac{16}{81}\right)\left(\frac{1}{9}\right) + 6\left(\frac{32}{243}\right)\left(\frac{1}{3}\right) + \left(\frac{64}{729}\right)(1)$

$= \frac{12 + 60 + 160 + 240 + 192 + 64}{729}$

$= \frac{728}{729}$

$p(x \geq 1) = 0.9986$

Unit-5-

Poisson Distribution:

A random Variable x is said to follow a poisson distribution if it assumes only non-negative value and its probability mass function is given by

$$p(x,\lambda) = p(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^{x}}{x!} & x=0,1,2\ldots \lambda>0 \\ 0 & \text{otherwise.} \end{cases}$$

Here λ is known as the parameter of the distribution.

$X \sim P(\lambda)$ denote that X is a Poisson variate with parameter λ.

$\frac{1}{2}\times\left(\frac{1}{2}\right)^{31}$

$\frac{1}{2\times 31}$

EX:

i) Number of suicides reported in particular city.

ii) The number of defective material is a packing manufactured by a good.

4294967296

iii) Number faulty blades is a placed of 100.

iv) Number of air accidents is some cities.

v) Number of printing mistakes at each page of the book.

Poisson distribution - Derivation (as a limiting case of the binomial distribution).

Poisson distribution is a limiting case of the binomial distribution under the following condition.

i) n, the number of trials is indefinitely large i.e $n \to \infty$.

ii) p, the constant probability of success for each trial is indefinitely small i.e $p \to 0$.

iii) $np = \lambda$ (say) is finite

$$p = \lambda / n$$

$$q = 1 - \lambda/n$$ where λ is a +ve real number.

The probability of x success in a series of n independent trials is

$$B(x; n, p) = nC_x p^x q^{n-x} \quad x = 0, 1, 2 \ldots$$

Applying limiting values on both sides.

$$\lim_{n \to \infty} B(x; n, p) = \lim_{n \to \infty} nC_x p^x q^{n-x}$$

$$\lim_{n \to \infty} B(x; n, p) = \lim_{n \to \infty} \frac{n!}{x!(n-x)!}$$

$$\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

using stirling's approximately on for $n!$ as $n \to \infty$

$$\lim_{n \to \infty} n! = \sqrt{2\pi}\, e^{-n} n^{n+1/2}$$ we get

$$\lim_{n \to \infty} B(x; n, p) = \lim_{n \to \infty} \left\{ \frac{\sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}}{x! \sqrt{2\pi}\, e^{-(n-x)} (n-x)^{n-x+\frac{1}{2}}} \right.$$

$$= \lim_{n \to \infty} \left\{ \frac{e^{-n} n^{n+\frac{1}{2}}}{x!\, e^{-n} e^{x} (n-x)^{n-x+\frac{1}{2}}} \right\} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^x x!} \lim_{n \to \infty} \left\{ \frac{n^{n+1/2}}{(n-x)^{n-x+\frac{1}{2}}} \right\} \frac{1}{n^x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^{\lambda} x!} \lim_{n \to \infty} \left\{ \frac{n^{-x} n^{n+1/2}}{(n-x)^{n-x+\frac{1}{2}}} \right\} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^{\lambda} x!} \lim_{n \to \infty} \left\{ \frac{n^{n-x+1/2}}{n^{n-x+1/2} \left(1 - \frac{x}{n}\right)^{n-x+\frac{1}{2}}} \right\} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^{\lambda} x!} \lim_{n \to \infty} \left(\frac{\left(1-\frac{\lambda}{n}\right)^{n-x}}{\left(1-\frac{x}{n}\right)^{n-x+\frac{1}{2}}} \right)$$

$$= \frac{\lambda^x}{e^{\lambda} x!} \lim_{n \to \infty} \left\{ \frac{\left(1-\frac{\lambda}{n}\right)^{n} / \left(1-\frac{\lambda}{n}\right)^{x}}{\left(1-\frac{x}{n}\right)^{n} \left(1-\frac{x}{n}\right)^{\frac{1}{2}}} \right.$$

Moment of poisson distribution

$\mu_1' = \lambda, \quad \mu_2' = \lambda^2 + \lambda$

$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$

$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$

$\mu_1 = \lambda, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda$

$\mu_4 = 3\lambda^2 + \lambda$

Skewness & kurtosis :

Skewness $\quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3}$

$\beta_1 = \frac{1}{\lambda}$

$\beta_2 = \frac{\mu_4}{\mu_2^2}$

$= \frac{3\lambda^2 + \lambda}{\lambda^2}$

$\beta_2 = 3 + 1/\lambda$

Kurtosis $\quad \delta_1 = \sqrt{\beta_1} = \sqrt{1/\lambda}$

$$\delta_2 = \beta_2 - 3$$

$$= 3 + \frac{1}{\lambda} - 3$$

$$\delta_2 = \frac{1}{\lambda}$$

Mean & variance of poisson distribution:-

$$\text{Mean} = E(x)$$

$$= \sum_{x=0}^{\alpha} x\, p(x, \lambda)$$

$$= \sum_{x=0}^{\alpha} x \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\alpha} \frac{x \cdot e^{-\lambda} \lambda^{x+1-1}}{x(x-1)!}$$

$$= \sum_{x=0}^{\alpha} \frac{e^{-\lambda} \lambda^{x-1} \lambda}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots\right)$$

$$= \lambda e^{-\lambda} (e^{\lambda})$$

$$\text{mean} = \lambda$$

$$E(x^2) = \sum_{x=0}^{\alpha} p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda} \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} (x^2 + x - x) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} (x^2 - x) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x(x-1)!} + \lambda$$

$$= \sum_{x=2}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^x}{(x-1)(x-2)!} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2+2}}{(x-2)!} + \lambda$$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} + \lambda$$

$E(X^2) = \lambda^2 + \lambda$

Variance $= E(X^2) - (E(X))^2$

$= \lambda^2 + \lambda - \lambda^2$

Variance $= \lambda$

Recurrence relation for moments of the poisson Distribution.

$$\mu_r = E(X - E(X))^r$$
$$= E(X-\lambda)^r$$
$$= \sum_{x=0}^{\infty} (x-\lambda)^r \, p(x,\lambda)$$
$$\mu_r = \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

D. w.r to λ

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \Big(r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!} +$$
$$\frac{(x-\lambda)^r}{x!}\left(e^{-\lambda}x\lambda^{x-1}\right) + \lambda^x e^{-\lambda}(-1)\Big)$$
$$= -r\sum_{x=0}^{\infty}\left((x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!}\right) + \sum_{x=0}^{\infty}$$
$$\frac{(x-\lambda)^r}{x!}e^{-\lambda}x\lambda^{x-1} - \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\lambda^x e^{-\lambda}$$
$$= -r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}\frac{\lambda^x e^{-\lambda}}{x!} + \frac{e^{-\lambda}\lambda^x}{\lambda}\sum_{x=0}^{\infty}$$
$$\frac{(x-\lambda)^r}{x!}\left[\left(\frac{x}{\lambda}-1\right)\right] \quad \left(\frac{x-\lambda}{\lambda}\right)$$
$$= -r\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r-1} + \frac{1}{\lambda}\sum_{x=0}^{\infty}$$
$$\frac{e^{-\lambda}\lambda^x}{x!}\cdot(x-\lambda)^r(x-\lambda)^1$$
$$= -r\mu_{r-1} + 1/\lambda\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r+1}$$

$$\frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$\frac{d\mu_r}{d\lambda} + r\mu_{r-1} = \frac{1}{\lambda}\mu_{r+1}$$

$$\mu_{r+1} = \lambda\left(\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right)$$

Moment generating function of poisson distribution

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx}\left(\frac{e^{-\lambda}\lambda^x}{x!}\right).$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\left(1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} \dots\right.$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(-1+e^t)} \qquad = e^{\lambda(e^{t}-1)}$$

$$M_x(t) = e^{\lambda(e^t-1)}$$

Characteristic function of poisson distribution.

$$P_x(t) = E[e^{itx}]$$

$$= \sum_{x=0}^{\infty} e^{itx} \left(\frac{e^{-\lambda} \lambda^x}{x!} \right)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} (\lambda e^{it})^x}{x!}$$

$$= e^{-\lambda} \left(1 + \frac{\lambda e^{it}}{1!} + \frac{(\lambda e^{it})^2}{2!} + \dots \right)$$

$$P_x(t) = e^{\lambda(e^{itx}-1)}$$

(*) imp Additive property of poisson distribution (or) Reproductive property of independent poisson Variate.

Statement:-

Sum of independent poisson variate is also a poisson variate more elebrately, if $x_i = (i = 1, 2, 3 \dots n)$ are independent poisson variate with parameter $\lambda_i = (i = 1, 2 \dots n)$ respectively then $\sum_{i=1}^{n} x_i$ is also a poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$

Proof:

$$M_{x_i}(t) = e^{\lambda_i(e^t - 1)}$$

WKT

$$M_{x_1+x_2+x_3+\dots x_n}(t) = M_{(x_1)}t \dots M_{x_n}(t)$$

Since x_i ($i=1,2,3\dots$ are indepen

$$M(\lambda_i(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \dots e^{\lambda_n(e^t-1)}$$

$$M(\lambda_i(t)) = e^{(e^t-1)(\lambda_1+\lambda_2+\lambda_3+\dots}$$

which is the m.g.f of a poisson variable with paramet $\lambda_1 + \lambda_2 + \dots \lambda_n$

Hence by uniqueness theorem $\sum_{i=1}^{n} x_i$ is also a poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

1. A manufacturer of 10 pins knows that 5% of his product is defective if he sells cotton pins is bones of 100 and graduates that not more than 10 pins will be defective. What is the approximately probabili

that a box will fail to meet the grauanted quality.

Soln: Given, $n = 100$

p = Probability of defective pins

$= 5\%$

$= 5/100 = 0.05$

$\lambda = np$

$= 100 \times 0.05$

$\boxed{\lambda = 5}$

$\therefore P(X \geq 10) = 1 - P(X \leq 10)$

$= 1 - \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)$

$= 1 - \left(\frac{e^{-5}5^x}{x!}\right)$

$= 1 - e^{-5}\left(\frac{5^x}{x!}\right)$

$= 1 - e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!} + \frac{5^8}{8!} + \frac{5^9}{9!} + \frac{5^{10}}{10!}\right.$

$= 1 - e^{-5}\left(1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24} + \frac{3125}{120} + \frac{15625}{720} + \frac{78125}{5040}\right.$

$+ \frac{390625}{40320} + \frac{1953125}{362880} + \frac{97656...}{362...}$

$= 1 - 0.006737\,94(1 + 5 + 12.5 + 20.83 + 26.04 + 26.09 + 21.70 + 15.50 + 9.68 + 5.38 + 2.69)$

$= 1 - 0.006737\,94(146.36)$

$= 1 - 0.9861648984$

$p(x \geq 10) = 0.014$

2) A manufactured who produce medicine bottles find that 0.1% of the bottles are defective the bottles are placed in boxes contains 500 bottles. A manufacture buys 100 boxes from the producer of bottles using P.D. find how many bottle will contains i) no defective ii) atleast 2 defective with 100

Soln:-

$n = 500$

$p = 0.1\% = \frac{0.1}{100}$

$\boxed{p = 0.001}$

$\lambda = np$

$= 500 \times 0.001$

$\lambda = 0.5$

i) $p(x=0) = \frac{e^{-\lambda}\lambda^{x}}{x!} \times 100$

$= \frac{e^{-0.5}(0.5)^{0}}{0!} \times 100$

$= e^{-0.5} \times 100 = 0.6065 \times 100$

$= 60.6$

$p(x=0) = 61$

ii) $p(x > 2) = 1 - P(x \leq 2) \times 100$

$= 1 - [P(x=0) + p(x=1) + p(x=2)] \times 100$

$= 1 - e^{-0.5}(1 + 0.5 + 0.125) \times 100$

$= 1 - e^{-0.5}(1.625) \times 100$

$= 1 - 0.6065(1.625) \times 100$

$= 1 - 0.9097 \times 100$

$= 1 - 0.9097 \times 100$

$= 9.03$

3) An insurance company insures 4000 people against of less

less of both eyes in a co
accidents. Based on precious
data the rate were Competed
on the assumption that on the
average 10 persons in 1,00,000
will have can accident each
year that result in this
type of injury. what is the
probability that more than 3
of the insured will collect
on their policy in a given
year.

Soln: $n = 4000$ p = probability of injured in car accident

$$p = \frac{10}{10,0000}$$

$$p = 0.0001$$

$$\lambda = np$$

$$= 4000 \times 0.0001$$

$$\lambda = 0.4$$

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - (P(X=0) + P(X=1) + P(X=2) + P(X=3))$$

$$= 1 - e^{-0.4}\left(\frac{(0.4)^0}{0!} + \frac{(0.4)^1}{1!} + \frac{(0.4)^2}{2!} + \frac{(0.4)^3}{2!}\right)$$

$$= 1 - e^{-0.4}(1 + 0.4 + 0.08 + 0.0106)$$

$$= 1 - e^{-0.4}(1.4906)$$

$$= 1 - (0.6703)(1.4906)$$

$$= 1 - 0.9991$$

$$p(x > 3) = 0.0009$$