Measure theory & Integration.

5/6/19

Paper code :- 19PMA10

Unit - 1 :-

Lebesgue Measure.

Introduction - outer Measure - Measurable sets and Lebesgue Measure - Measurable function - Little Wood's three principles (Chap 3 : Sec 1, 2, 3, 5, 6

Unit -2 :-

Lebesgue Integral.

The Riemann Integral - Lebesgue Integral - Lebesgue integral of bounded functions over a set of finite measures - The integral of a non-negative function - The General Lebesgue integral. (Chap 4 : Sec 1, 2, 3, 4)

Unit - 3 :- Differentiation & Integration.

Differentiation & Integration - Differentiation of Monotone functions - Functions of bounded variation - Differentiation of an integral - absolute Continuity (Chap 5 : Sec 1, 2, 3, 4)

Unit-4 :-

General Measure & Integration.

General Measure & Integration - Measure Spaces - Measurable functions - Integration - Signed Measure - The Radon - Nikodym theorem (chap 11 : Sec 1, 2, 3, 5, 6)

Unit -5 :-

Measure & outer Measure

Measure and outer Measure - outer Measure and Measurability - The extension theorem - Product Measure

(Chap 12 : Sec : 1, 2, 4)

Text book :-

H. L. Roider

"Real Analysis"

Ref book :-

1. G. D. Barold

   Me [illegible] Theory & Inte

2. P. K. Jain [illegible] Gupta

   [illegible]gue Measure & Inte.

3. Walter Rudin

   Real & Complex Analysis.

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Unit - 1

Lebesgue Measure.

Definition :- The length $l(I)$ of an Interval I is defined to be the difference of the (two) end points of the interval.

Length is an example of set function.

i.e. The function that associates an extended Real number to each set in a collection of sets.

In this case, the domain in the collection of all intervals. we would like to construct 2m [a set function 'm' that assigns to each set E. in some collection μ. of set of Real numbers, a non-negative extended Real number mE is called Measure of E.] 2m

U.Q 2m (E.T)

The set function m possess the following properties :-

1.) mE is defined for each set E of Real numbers.

Power series

i.e. $\boxed{\mu = P(R)}$

2.) The measure of an interval is its length.

i.e. $\boxed{mI = l(I)}$

(3.) Measure is translation invariant.

if E is a set for which m is defined and if $E+y$ is the set $\{x+y \,/\, x \in E\}$ obtained by replacing each point x in E by the point $x+y$, then the Measure of $(E+y)$ is defined as, $\boxed{m(E+y) = mE}$

(4.) Measure is Countably Additive. over countable disjoint Union of sets

i.e) if $\{E_n\}$ is a sequence of disjoint set then $\boxed{m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} mE_n}$

Definition :-

The collection of Measurable sets μ (+ve measure) is called Sigma Algebra (σ-Algebra) 2m

if every union of countable collection of sets in μ is again in μ.

Result :-

Let n be Countably Additive Measure defined for all sets in a σ-Algebra μ, then

① If A and B are two sets in μ with $\underline{A \subset B}$, then $\boxed{mA \leq mB}$ (monotonicity Property)

② Let $\{E_n\}$ be any sequence of sets in μ, then $\boxed{m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} mE_n}$

(Countably subadditive property)

③ If there is a set A in μ. Such that $\underline{mA < \infty}$, then $\boxed{m\phi = 0}$

<u>Outer Measure</u> :-

Let A be set of Real numbers.

2m

The outer measure m^*A is defined as

$$\boxed{m^*A = \operatorname{Inf} \sum_{A \subset \cup I_n} l(I_n)}, \quad A \subset R$$

where $l(I_n) \Rightarrow$ length of I_n

<u>Results</u> :-

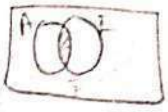
i) $m^*A \geq 0$

ii) $m^*A \leq m^*B$ if $A \subset B$

iii) $m^*[x] = 0 \quad \forall x \in R$

iv) $m^*\phi = 0$

* $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$ (Lebesgue measure m)



V.Q 10 m

Theorem 1 :-

The outer Measure of an Interval is its length.

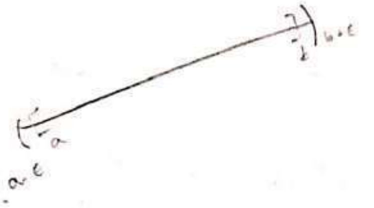
Proof :-

Case ① :-

Consider the closed finite interval $[a,b]$, the open interval $(a-\epsilon, b+\epsilon)$ contains $[a,b]$ for each +ve ϵ. We have $m^*[a,b] = m^*(a-\epsilon, b+\epsilon)$

$m^*A \leq m^*B$ if $A \subset B$

$$\leq l(a-\epsilon, b+\epsilon)$$



$$\leq b+\epsilon-a+\epsilon$$

$$\leq b-a+2\epsilon$$

$$\boxed{m^*[a,b] \leq b-a} \rightarrow (1)$$

$\because \epsilon$ is arbitrary

Next, we will prove $m^*[a,b] \geq b-a$

But it is equivalent to showing that

If $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open bounded interval covering $[a,b]$, then

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a \rightarrow (2)$$

By Heine - Borel theorem,

Any collection of open interval covering $[a,b]$ has a finite subcollection that also covers $[a,b]$

Since, Sum of the length of the finite Sub-collection is no greater than the sum of the length of the original collection.

It is enough to prove that

$$\sum_{k=1}^{n} l(I_k) > b-a \rightarrow (3)$$

and eqn (2) will hold.

Since, $a \in \bigcup_{k=1}^{n} I_k$

there must be one of the I_k's that contain 'a', Select such one interval and denote it by (a_1, b_1)

we have, $a_1 < a < b_1$

if $b_1 \geq b$, the inequality (3) is established

Since, $\sum_{k=1}^{n} l(I_k) \geq b_1 - a_1 > b - a$

$b_1 \geq b$

if $b_1 \leq b$, $b_1 \in [a, b]$

and since $b_1 \notin (a_1, b_1)$

there must be an interval (a_2, b_2)

$b_1 \leq b$

Such that $a_2 < b_1 < b_2$

in the collection of sequence $\{I_k\}_{k=1}^{n}$

if $b_2 > b$, then inequality (3) is established

$$\sum_{k=1}^{n} l(I_k) \geq (b_2 - a_2 + b_1 - a_1)$$
$$> b_2 - (a_2 - b_1) - a_1$$
$$> b_2 - a_1$$
$$> b - a$$

if $b_2 < b$, we continue the same process and we get the sequence $(a_1, b_1)(a_2, b_2)$ $(a_3, b_3) \ldots (a_K, b_K)$ form the collection of $\{I_K\}_{K=1}^{n}$ such that $a_i < b_{i-1} < b_i$

Since $\{I_K\}_{K=1}^{n}$ is a finite collection, our process must terminate with some interval (a_N, b_N).

But it terminates only if $b \in (a_N, b_N)$

therefore

$$\sum_{K=1}^{n} l(I_K) \geq \sum_{K=1}^{N} l(a_K, b_K)$$

$$\geq b_N - a_N + b_{N-1} - a_{N-1} + \ldots + b_1 - a_1$$

$$\geq b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \ldots - (a_2 - b_1)$$

$$\sum_{K=1}^{n} l(I_K) \geq b_N - a_1 \rightarrow (4) \qquad (\because a_i < b_{i-1})$$

Since $b_N > b$ and $a_1 < a$

we have $b_N - a_1 > b - a$

$\therefore$ equ (4) $\Rightarrow \sum_{K=1}^{n} l(I_K) > b - a$

$\therefore$ equ (3) holds.

$$\boxed{\therefore \sum_{K=1}^{\infty} l(I_K) \geq b - a}$$

$\therefore \quad m^*[a, b] \geq b - a \rightarrow (5)$

Case ② :-

If I be any finite Interval

i.e. $I = [a, b)$ or

$I = (a, b)$ or

$I = (a, b]$

for given $\epsilon > 0$, $\exists$ $J = [a + \epsilon/4 \, , \, b - \epsilon/4]$

$$l(J) = b - \frac{\epsilon}{4} - a - \frac{\epsilon}{4}$$

$$= b - a + (-\epsilon/2)$$

$$l(J) = l(I) + (-\epsilon/2)$$

$$l(J) + \epsilon/2 > l(I) - \epsilon/2$$

$$l(J) > l(I) - \epsilon$$

$$l(I) - \epsilon < l(J)$$

$$\Rightarrow l(I) - \epsilon < l(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = l(\bar{I}) = l(I)$$

where $\bar{I} = [a, b]$

$$l(I) - \epsilon < m^* I = l(I)$$

$$\Rightarrow l(I) - \epsilon < m^* I = l(I) + \epsilon$$

$\boxed{m^*(I) = l(I)}$, I is any finit interval

Case ③ :-

Let I be an infinite Interval

then given any real number Δ, there is a closed interval $J \subset I$ with $l(J) = \Delta$

$\Rightarrow \quad m^*(J) \le m^*(I)$

hence $m^*(I) \ge m^* J = l(J) = \Delta$

Since, $m^*(I) \ge \Delta \;\forall\; \Delta$

we have $\boxed{m^*(I) = \infty = l(I).}$

Hence the theorem.

Theorem 2 :-

outer measure is Countably Sub-Additively. ie) if $\{A_n\}$ is any Countable collection of sets, then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^* A_n$

Proof:-

For each 'n' and for any $\epsilon > 0$

$\exists$ a sequence of interval $\{I_{n,i}\}_{i=1}^{\infty}$

of open intervals.

such that $A_n \subset \bigcup_{i=1}^{n} I_{n,i}$

and $\sum_{i=1}^{\infty} l(I_{n,i}) < m^*(A_n) + \frac{\epsilon}{2^n} \rightarrow (*)$
(i, n)

Now $\{I_{n,i}\}$ is a Countable collection of open intervals that covers $\bigcup_{n=1}^{\infty} A_n$ is Countable.

Since, Union of Countable number of Countable collection is Countable.

$\therefore m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n,i} l(I_{n,i})$

$$\leq \underbrace{\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} l(I_{n,i})}_{\text{by } (*)}$$

$$< \sum_{n=1}^{\infty} \left[m^*(A_n) + \frac{\epsilon}{2^n} \right]$$

$$< \sum_{n=1}^{\infty} m^* A_n + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}$$

$$< \sum_{n=1}^{\infty} m^* A_n + \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \cdots \right)$$

$$\boxed{\therefore m^*\left(\bigcup_{n=1}^{\infty} A_n\right) < \sum_{n=1}^{\infty} m^* A_n}$$

/ $\because \epsilon$ is abrbitrary

Collary :- ①

If A is Countable, $m^* A = 0$

(or) Show that outer measure of a Countable Set is zero.

2m

Proof :-

Let $A = \{x_1, x_2, \dots x_n \dots\}$

$$m^* A = m^*\left(\bigcup_{i=1}^{\infty} x_i\right)$$

$$= \sum_{i=1}^{\infty} m^* x_i$$

/ $\therefore m^*[x] = 0 \quad \forall x \in R$

$$\boxed{m^* A = 0}$$

Collary :- ②

2m

Prove that $[0,1]$ is not Countable.

Proof:- Suppose $[0,1]$ is Countable

By Previous collary, $m^*[0,1] = 0$.

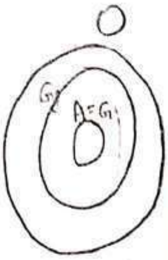
By theorem ① $\Rightarrow$ $m^*[0,1] = 1 - 0 = 1$ (which is Contradicti to our assumption

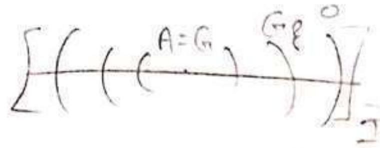
$\therefore [0,1]$ is not countable

Theorem 3 :-

Given any $\epsilon > 0$ and any set A there is an open set O. Such that $A \subset O$ and $m^* O \leq m^* A + \epsilon$, there is a $G \in G_\delta$ such that $A \subset G$ and $m^* A = m^* G$.

$A \subset G \subset O$





Proof:-

Let A be a set of real numbers then ∃ an open interval $A \subset \bigcup I_n$

and $\sum_{n=1}^{\infty} l(I_n) \leq m^* A + \epsilon$

Let $O = \bigcup I_n$, $A \subset O$

$\Rightarrow m^* O = \sum l(I_n)$

$\underline{m^* O = m^* A + \epsilon}$

Let for each +ve number ∃ O_n such that

$A \subset O_n$

and $\underline{m^* O_n \leq m^* A + \epsilon}$

Let $G = \bigcap O_n$ then $G \in G_\delta$ and $A \subset G$

$(\because G \subset O_n)$

$\hookrightarrow \quad m^* G \leq m^* O_n$

$\leq m^* A + \epsilon$

Since ϵ is arbitrary

$m^* G \leq m^* A \longrightarrow (1)$

Since $G \supset A$ (or) $A \subset G$, we have

$m^* A \leq m^* G \longrightarrow (2)$

From (1) & (2) $\Rightarrow$ $\boxed{m^* A = m^* G}$

Result :-

Prove that if $m^* A = 0$ then $m^*(A \cup B) = m^*(B)$.

Proof:-

$$m^*(A \cup B) \leq m^*(A) + m^*(B) \qquad \text{/ Additive condition}$$

$$\leq 0 + m^*(B) \qquad / m^* A = 0$$

2M $$m^*(A \cup B) \leq m^*(B) \rightarrow (1)$$

W.K.T $B \subset A \cup B$.

$$m^*(B) \leq m^*(A \cup B) \rightarrow (2)$$

From (1) & (2), we get

$$\boxed{m^*(A \cup B) = m^*(B)}$$

Measurable sets and ~~[illegible]~~ Lebesgue Measure :-

Measurable sets :-

I.Q. A set 'E' is said to be Measurable

2M if for each set A we have $\boxed{m^* A = m^*(A \cap E) + m^*(A \cap \bar{E})}$

Note 1 :-

$A = A \cap E \cup A \cap \bar{E}$

We always have $m^* A \leq m^*(A \cap E) + m^*(A \cap \bar{E})$

hence E is Measurable iff for each A,

$$m^* A \geq m^*(A \cap E) + m^*(A \cap \bar{E})$$

Note 2 :- The definition of Measurable is symmetric in E and $\bar{E}$. ie) $\bar{E}$ is Measurable whenever E is Measurable.

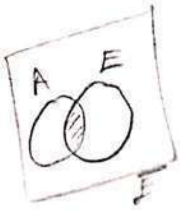
Note 3 :-

ϕ and the set R of all real numbers are Measurable.

Lemma 1 :-

If $m^* E = 0$ then E is Measurable

Proof:-

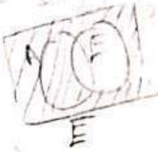


Let A be any set, then $A \cap E$

$$A \cap E \subset E$$

$$m^*(A \cap E) \leq m^* E$$

$\Rightarrow$ $m^*(A \cap E) \leq 0$ /gn.



Also, $A \cap \bar{E} \subset A$

$$m^*(A \cap \bar{E}) \leq m^* A$$

$$0 + m^*(A \cap \bar{E}) \leq m^* A$$

$$m^*(A \cap E) + m^*(A \cap \bar{E}) \leq m^* A \quad \rightarrow (1)$$

$|||^{ly}$ $m^* A \leq m^*(A \cap E) + m^*(A \cap \bar{E}) \rightarrow (2)$

(1) & (2) $\Rightarrow$ $\boxed{m^* A = m^*(A \cap E) + m^*(A \cap \bar{E})}$

$\therefore E$ is Measurable

Hence the Proof.

Lemma 2 :-

If E_1 and E_2 are Measurable. So is $E_1 \cup E_2$.

Proof :-

Let A be any set.

Given, E_1 and E_2 are Measurable.

we have to prove $(E_1 \cup E_2)$ is Measurable

ie) we have to show that

$$m^* A = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (\overline{E_1 \cup E_2}))$$

Since E_2 is Measurable

$$\therefore m^* A = m^*(A \cap E_2) + m^*(A \cap \bar{E_2}) \rightarrow (1)$$

Replace A by $A \cap \bar{E_1}$ (*)

$$(1) \Rightarrow m^*(A \cap \bar{E_1}) = m^*((A \cap \bar{E_1}) \cap E_2) + m^*((A \cap \bar{E_1}) \cap \bar{E_2})$$

$$\therefore m^*(A \cap \bar{E_1}) = m^*[A \cap (\bar{E_1} \cap E_2)] + m^*[A \cap (\bar{E_1} \cap \bar{E_2})] \rightarrow (2)$$

Since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$

$= (A \cap E_1) \cup (A \cap E_2 \cap R)$ Real line

$= (A \cap E_1) \cup [A \cap E_2 \cap (E_1 \cup \bar{E_1})]$

$= (A \cap E_1) \cup [(A \cap E_2 \cap E_1) \cup (A \cap E_2 \cap \bar{E_1})]$

$\because A \cap (E_2 \cap E_1) \subset A \cap E_1$

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup [A \cap E_2 \cap \bar{E_1}]$$

De morgan's law.

$R = E_1 \cup \bar{E_1}$

$(A \cap E_1) \cup (A \cap E_1) \cup (A \cap E_2 \cap \bar{E_1})$

$(A \cap E_1) \cup (A \cap E_2 \cap \bar{E_1})$

Now,

$$m^*[A\cap(E_1\cup E_2)] \le m^*(A\cap E_1)+m^*(A\cap \ldots$$

Adding $m^*[A\cap(\overline{E_1\cup E_2})]$ on both sides

$$m^*[A\cap(E_1\cup E_2)]+m^*[A\cap(\overline{E_1\cup E_2})] \le$$

$$m^*(A\cap E_1)+m^*(A\cap E_2\cap \overline{E_1})+m^*(A\cap(\overline{E_1\cup \ldots$$

$\overline{E_1\cup E_2} = \ldots$

$\because \overline{E_1}\cap\overline{E_1}=\overline{E_1}$, $E_2\cap\overline{E_2}=R$

$$\le m^*(A\cap E_1)+m^*(A\cap E_2\cap\overline{E_1})+m^*((\overline{E_1}\cap\overline{E_2} \ldots$$

$$\le m^*(A\cap E_1)+m^*(A\cap\overline{E_1})$$

$$\therefore m^*[A\cap(E_1\cup E_2)]+m^*[A\cap(\overline{E_1\cup E_2})] \le m^*A \rightarrow (3)$$

$\because$ Since E_1 is measurable

w.k.T IIIly,

$$m^*A \le m^*[A\cap(E_1\cup E_2)]+m^*[A\cap(\overline{E_1\cup E_2})] \rightarrow (4)$$

From (3) & (4)

$$m^*A = m^*[A\cap(E_1\cup E_2)]+m^*[A\cap(\overline{E_1\cup E_2})]$$

$\therefore E_1\cup E_2$ is Measurable.

Hence the Proof

Lemma 3 :-

Let A be any set $E_1, E_2, \ldots, E_n$ be a finite sequence of disjoint measurable sets then $m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i)$

Proof:-

we prove the Lemma

By induction on n :-

It is clearly true for $n=1$

and we assume

It is true if we have $(n-1)$ sets of E_i

Since, E_i are disjoint sets.

we have, $A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n = A \cap E_n \rightarrow (1)$

and $A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap \tilde{E}_n = A \cap \left(\bigcup_{i=1}^{n-1} E_i\right) \rightarrow (2)$

By definition,

E_n is Measurable.

$$m^* A = m^*(A \cap E_n) + m^*(A \cap \tilde{E}_n)$$

Replace A by $A \cap \bigcup_{i=1}^{n} E_i$

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = m^*\left(A \cap \bigcup_{i=1}^{n} E_i \cap E_n\right) + m^*\left(A \cap \bigcup_{i=1}^{n} E_i \cap \tilde{E}_n\right)$$

$$= m^*(A \cap E_n) + m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i)$$

$$\boxed{\therefore m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)}$$

Theorem 4:-

The Collection μ of Measurable sets is σ-Algebra.

ie) The Complement of Measurable set is Measurable and union (& intersection) of countable collection of Measurable set is Measurable.

Moreover, Every set with outer Measure "zero" is Measurable

Proof:-

To Prove:- Complement of a Measurable set is Measurable.

Let E be a Measurable set in μ

Then $m^*A = m^*(A \cap E) + m^*(A \cap \bar{E})$

$$m^*A = m^*(A \cap \bar{E}) + m^*(A \cap \bar{\bar{E}})$$

$\therefore \Rightarrow \bar{E}$ is Measurable.

$\therefore$ The Complement of Measurable set is Measurable

To Prove:- Union of countable collection of Measurable set is Measurable

Let $E = \bigcup_{i=1}^{\infty} E_i$, where E_i's are disjoint Measurable sets

By Previous theorem,

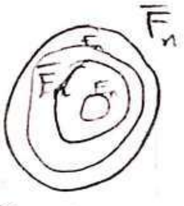
$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i) \rightarrow (1)$$

Put $F_n = \bigcup_{i=1}^{n} E_i$

① $\Rightarrow$ $m^*(A \cap F_n) = \sum_{i=1}^{n} m^*(A \cap E_i) \rightarrow$ ②

Since F_n is Measurable and

$\overline{F_n} \supset \overline{E_n}$



$$m^* A = m^*(A \cap F_n) + m^*(A \cap \overline{F_n})$$

$$\geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*(A \cap \overline{F_n})$$

$$\geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \overline{E_n})$$

$$m^* A \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \overline{E_n})$$

$\therefore$ Left Hand Side of the inequality is <u>independent of n</u>.

$$\therefore \quad m^* A \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \overline{E})$$

$$\geq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*(A \cap \overline{E})$$

$$m^* A \geq m^*(A \cap E) + m^*(A \cap \overline{E}) \rightarrow \text{Ⓐ}$$

<u>WKT</u>

$$m^* A \leq m^*(A \cap E) + m^*(A \cap \overline{E}) \rightarrow \text{Ⓑ}$$

From Ⓐ & Ⓑ

$$\Rightarrow \boxed{m^* A = m^*(A \cap E) + m^*(A \cap \overline{E})}$$

ie) $E = \bigcup_{i=1}^{\infty} E_i$ is Measurable.

To Prove :-

Every set with outer Measure zero is Measurable.

Write Lemma 1 :-

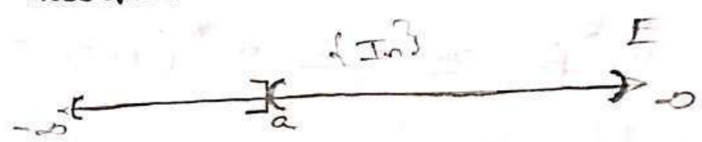
If $m^* E = 0$, then E is Measurable

$\therefore E$ is Measurable.

Hence The Proof.

Theorem 5 :- Every interval is measurable.

The interval (a, ∞) is measurable.



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U.Q

Proof :-

Let 'A' be any set.

To Prove :- (a, ∞) is measurable

It is enough to show that

$$m^* A \geq m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a])$$

$\because \overline{(a,\infty)} = (-\infty, a]$

ie) $m^* A \geq m^* A_1 + m^* A_2$

where $A_1 = A \cap (a, \infty)$

$A_2 = A \cap (-\infty, a]$

If $m^* A = \infty$, the Result is trivial

If $m^* A < \infty$, Given $\epsilon > 0$, $\exists$ a countable Collection of $\{I_n\}$ of open which covers A

$A \subset \bigcup I_n$

and $\sum l(I_n) \leq m^* A + \epsilon \to (*)$

Now, $l(I_n) = l(I_n \cap \mathbb{R})$

$$= l\left(I_n \cap \left((a,\infty) \cup (-\infty,a]\right)\right)$$

$$= l\left(I_n \cap (a,\infty)\right) \cup l\left(I_n \cap (-\infty,a]\right)$$

$$l(I_n) = l(I_n') \cup l(I_n'')$$

$$\therefore \; m^* I_n = m^* I_n' + m^* I_n'' \to (1).$$

$= m^*(I_n' + I_n'')$

Since $A \subset \bigcup_{n=1}^{\infty} I_n$ $\quad$ " $= \Sigma$ "

$$\Rightarrow A \cap (a,\infty) \subset \bigcup_{n=1}^{\infty} I_n \cap (a,\infty)$$

$$\Rightarrow m^*(A \cap (a,\infty)) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\Rightarrow m^*(A \cap (a,\infty)) \leq \sum_{n=1}^{\infty} m^* I_n' \to (2).$$

Again, $A \subset \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow A \cap (-\infty,a] \subset \bigcup_{n=1}^{\infty} I_n \cap (-\infty,a]$$

$$\Rightarrow m^*(A \cap (-\infty,a]) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n''\right).$$

$$\Rightarrow m^*(A \cap (-\infty,a]) \leq \sum_{n=1}^{\infty} m^* I_n'' \to (3)$$

equ (2) + (3) $\Rightarrow$

$$m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty,a]) \leq \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'')$$

$$m^* A_1 + m^* A_2 \leq \sum_{n=1}^{\infty} m^*(I_n' + I_n'')$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n)$$

$$\leq \sum_{n=1}^{\infty} l(I_n)$$

$$m^* A_1 + m^* A_2 \leq m^* A + \epsilon \quad \text{/}\because \text{by } (*)$$

So, we have.

$$m^*A_1 + m^*A_2 \le m^*A$$

$$\Rightarrow \quad m^*A \ge m^*A_1 + m^*A_2$$

$$\Rightarrow \quad m^*A \ge m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a$$

hence, (a, ∞) is Measurable.

Hence the Proof.

Repetition - 10

Borel Set :-

2m U.Q.

A Collection $\mathcal{B}$ of Borel set is the Smallest σ- Algebra which contains all the open sets

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σ- Algebra :- The σ- Algebra of subsets of X generated by a set A is the Smallest σ- Algebra including A

σ- Algebra :-

1. $\phi \in \mathcal{F}$
2. If $E \in \mathcal{F}$, Then its Complement E^c is al
3. If $E_1, E_2, \ldots E_n$ is a Countable Collection of sets in $\mathcal{F}$ then their $\bigcup_{n=1}^{\infty} E_n$ is in

---

Theorem 6 :-

10 M U.Q

Every Borel Set is Measurable.

Proof :-

By theorem 4,

The collection $\mathcal{M}$ of a Measurable sets is σ - Algebra.

By theorem 5,

(a, ∞) is measurable.

$\therefore (a, \infty) \in \mu$.

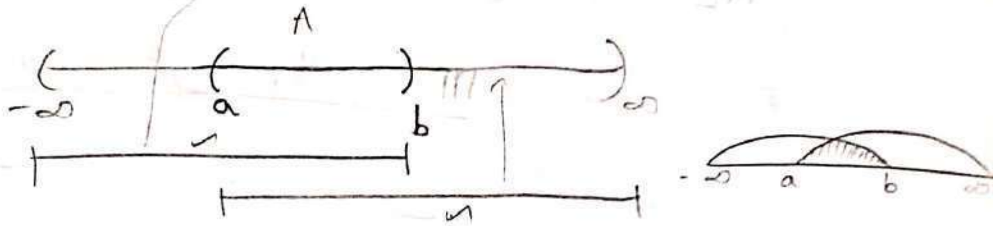
$\therefore \overline{(a, \infty)} = [-\infty, a]$ is also measurable.

for any $b \in R$.

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \tfrac{1}{n}]$$

Since each of $(-\infty, b - \frac{1}{n}]$ is measurable and countable union of Measurable set is Measurable

$\therefore (-\infty, b)$ is Measurable.



Consider an open interval (a, b)

which can be written as

$$(a, b) = (-\infty, b) \cap (a, \infty)$$

Since both $(-\infty, b)$ and (a, ∞) are Measurable

$\therefore$ We get (a, b) is Measurable.

But, each open set is the Union of countable number of open interval.

and so, Must be Measurable.

Thus, μ is a σ-Algebra containing open sets and Must therefore contain the family $\mathcal{B}$

of borel set, $\beta \subseteq \mu$

Since β is a smallest σ-Algebra containing the open sets.

$\therefore$ Every Borel set is Measurable

Hence The Proof.

Lebesgue Measure :-

If "E" is a Measurable set we define the Lebesgue Measure mE to be the outer Measure of E. Thus, m is the set function obtained by restricting the set function m^* to the family μ of Measurable set

Theorem 7 :-

Let $\{E_i\}$ be the sequence of Measurable sets, then $m(\cup E_i) \leq \sum mE_i$. If the set E_i are pair wise disjoint then

$$m(\cup E_i) = \sum m E_i$$

Proof :-

i) write the theorem 2.

Hence the Proof.

ii) $E_1, E_2, \ldots E_n$ be a finite sequence of disjoint measurable sets.

By Lemma 3,

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Take $A = R$

$$m^*\left(R \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(R \cap E_i)$$

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \overset{\text{outer Measure}}{m^*(E_i)}$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \overset{\text{Lebesgue Measure}}{m E_i} \rightarrow (1)$$

$\therefore$ m is finitely additive.

Let $\{E_i\}$ be an infinite sequence of pair wise disjoint Measurable sets, then

$$\bigcup_{i=1}^{\infty} E_i \supseteq \bigcup_{i=1}^{n} E_i$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m \bigcup_{i=1}^{n} E_i$$

By equ (1)

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{n} m E_i$$

$\therefore$ L.H.S of inequality is independent on n.

So, we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m E_i \rightarrow (2)$$

From Countable sub additive

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m E_i \rightarrow (3)$$

from (2) & (3), we have

$$\boxed{m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} mE_i}$$

Hence the Proof.

Unit-4, 2nd theorem also in this way.

U.Q

(X)

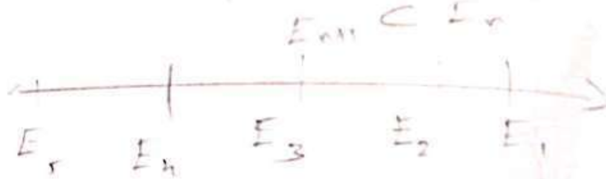
10 m

Theorem 8 :- Continuity of Measure.

Let $\{E_n\}$ be an infinite decreasing sequence of Measurable sets i.e., a sequence with $E_{n+1} \subset E_n$ for each n. Let mE_i be finite then $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$

$\wedge$

Proof:-



Gn:

E_n is a decreasing sequence of Measurable sets.

$$E_{n+1} \subset E_n$$

Claim :- (1) $E_1 - E = \cup F_i$ and F_i are pairwise disjoint

Let, $x \in E_1 - E$

$\Rightarrow x \in E_1$ and $x \notin E = \cap E_i$

$\Rightarrow x \in E_1$ and $x \notin E_i$ (for some i)

Now,

$$E_1 \supset E_2 \supset E_3 \supset \dots \supset E_i$$

Let 'i' be the smallest suffix

$\ni: x \notin E_i$

ie) $x \notin E_j$ then $x \in E_{j-1}$

ie) $\underline{x \in E_{j-1}}$ and ($x \notin E_j \Rightarrow \underline{x \in \overline{E_j}}$).

$\Rightarrow x \in E_{j-1} \cap \overline{E_j}$

$\Rightarrow x \in E_{j-1} - \overline{E}_j$

$\Rightarrow x \in F_i$

$\Rightarrow x \in \bigcup_{i=1}^{\infty} F_i$

$\Rightarrow E_1 - E \subset \bigcup_{i=1}^{\infty} F_i \to (1)$.

Conversely,

Let $x \in \bigcup_{i=1}^{\infty} F_i$

$x \in F_i$ (for some i)

$\Rightarrow x \in E_i - E_{i+1}$

$\Rightarrow x \in E_i$ and $x \notin E_{i+1}$

$\Rightarrow x \notin \cap E_i = E$ So $x \in \overline{E}$

also, since

$E_1 \supset E_i$ we have $x \in E_1$

$\therefore x \in E \cap \overline{E_1}$

ie) $x \in E_1 - E$

$\bigcup_{i=1}^{\infty} F_i \subset E_1 - E \to (2)$.

From (1) & (2)

$$\boxed{E_1 - E = \bigcup_{i=1}^{\infty} F_i} \to (A)$$

— x — (Regularity of the Lebesgue measure m)

If E is measurable and $\epsilon > 0$, then $\exists$ an open set E_1 and a closed set E_2 in R such that $E_2 \subset E \subset E_1$, $m(E_1 \cap E^c) < \epsilon$ and $m(E \cap E_2^c) < \epsilon$.

Claim (2) :-

To Prove F_i's are Pairwise disjoint.

Let $i \neq j$ without loss of generality

$\therefore E_i \supset E_j$

if $x \in F_i$ then $x \in E_i - E_{i+1}$

i.e) $x \in E_i$ and $x \notin E_{i+1}$

Since , $x \notin E_{j+1} \subset E_j$ $\left(\because F_j = E_j \right.$

Let , $y \in F_j$ then $y \in E_j - E_{j+1}$

i.e) $y \in E_j$ and $y \notin E_{j+1}$

Since $E_i \supset E_{i+1} \supset \dots E_j \supset E_{j+1}$

we have $y \notin F_i$

$F_i \cap F_j = \phi$ $(\because x \cap y = \phi)$

$\therefore F_i$'s are pairwise disjoint. $\rightarrow$ (3)

Now,

(A) $\Rightarrow$ $E_1 - E = \bigcup_{i=1}^{\infty} F_i$

$m(E_1 - E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$

$= \sum_{i=1}^{\infty} m F_i$

$$\boxed{m E_1 - m E = \sum_{i=1}^{\infty} m\left(E_i - E_{i+1}\right)} \rightarrow (4)$$

Since $E \subset E_1$ we have $E_1 = E \cup (E_1 - E)$ disjoint

Since $E = \bigcap E_i$ is Measurable set in we have,

$$E_1 - E \text{ is Measurable set.} \rightarrow (5)$$

$E_{i+1} \subset E_i$

$$\Rightarrow mE_i = mE_{i+1} + m(E_i - E_{i+1}) \rightarrow (6)$$

$\therefore$ equ (4) $\Rightarrow$ $mE_1 - mE = \sum_{i=1}^{\infty} m(E_i - E_{i+1})$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} \left[mE_1 - \cancel{mE_2} + \cancel{mE_2} - \cancel{mE_3} + \cancel{mE_3} - \dots \cancel{mE_{n-1}} - mE_n \right]$$

$$= \lim_{n\to\infty} [mE_1 - mE_n]$$

$$\cancel{mE_1} + mE = \cancel{mE_1} - \lim_{n\to\infty} mE_n$$

$$mE = \lim_{n\to\infty} mE_n$$

$$\boxed{m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n}$$

Hence the Proof.

Measurable function :-

Theorem 9 :-

Let f be extended real valued function whose Domain is Measurable. Then the following statement are equivalent.

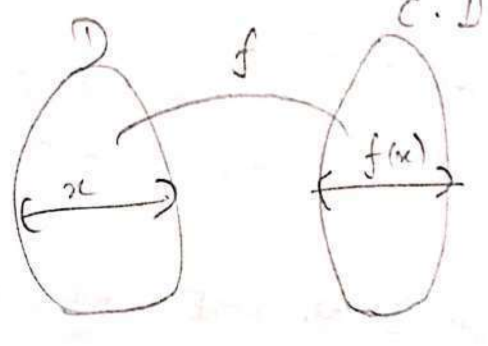
i) For every real number α, $\{x : f(x) > \alpha\}$ is Measurable

ii) For every real number α, $\{x : f(x) \geq \alpha\}$ is Measurable

iii) For every real number α, $\{x : f(x) < \alpha\}$ is Measurable

iv) For every real number α, $\{x : f(x) \leq \alpha\}$ is Measurable

v) For each extended real number α, the $\{x : f(x) = \alpha\}$ is Measurable.



Proof:-

Let D be the Domain of f.

To Prove :-

The 4 statements are equivalent

we will prove that i) $\Rightarrow$ iv)

Assume That: $\{x: f(x) > \alpha\}$ is Measurable

$\therefore \{x : f(x) \leq \alpha\} = D - \{x : f(x) > \alpha\}$

here D and $\{x: f(x) > \alpha\}$ are Measurable.

$\therefore$ The difference of two Measurable sets Measurable.

$\therefore \{x : f(x) \leq \alpha\}$ is Measurable.

Hence the Proof.

$|||^{ly}$, we can Prove iv) $\Rightarrow$ i)

$\therefore$ i) $\Leftrightarrow$ iv)

In the same manner $\boxed{\text{ii)} \Leftrightarrow \text{iii)}}$

To Prove :- i) $\Rightarrow$ ii)

Assume that $\{x : f(x) > \alpha\}$ is Measurable.

$$\therefore \{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \tfrac{1}{n}\}$$

Since i) is Measurable

It is true for all (α), It is true for $(\alpha - \frac{1}{n})$

and also Intersection of sequence of Measurable set is Measurable.

$\therefore \{x : f(x) \geq \alpha\}$ is Measurable.

Hence the Proof

Assume that $\{x : f(x) \geq \alpha\}$ is Measurable.

$$\text{ie)} \ \{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + \tfrac{1}{n}\}$$

Since ii) is Measurable

It is true for all (α), It is true for $(\alpha + \frac{1}{n})$

and also Union of sequence of Measurable set is Measurable.

$\therefore \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + \frac{1}{n}\}$ is Measurable.

$\therefore \{x : f(x) > \alpha\}$ is Measurable.

ii) $\Rightarrow$ i)

$\therefore$ i) $\Leftrightarrow$ ii)

$|||^{ly}$, we can Prove iii) $\Leftrightarrow$ iv)

$\therefore$ we get i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) $\Leftrightarrow$ i)

$\therefore$ All The 4 Statements are equivalent.

Finally, v)

Case (i) :-

If α is any finite number

then, $\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$

Since

from (ii) & (iv) statements.

R.H.S is Measurable.

$\therefore$ L.H.S is Measurable.

ie) $\{x : f(x) = \alpha\}$ is Measurable.

Case (ii) :-

If α is any infinite number.
ie) $\alpha = \infty$

then, $\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}$

$\therefore$ By statement (ii)

R.H.S is Measurable.

$\therefore$ L.H.S is Measurable.

ie) $\{x : f(x) = \infty\}$ is Measurable.

Case (iii) :-

If $\alpha = -\infty$

then, $\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq -n\}$

$\therefore$ By statement (iv)

R.H.S is Measurable.

$\therefore$ L.H.S is Measurable.

ie) $\{x : f(x) = -\infty\}$ is Measurable.

Hence the Proof.

Definition :-

An extended real valued function f is said to be Lebsgue Measure if its Domain (Y) is Measurable and it satisfies one of 1st Four Statement of above theorem.

Theorem 10:-

1 time Let c be a constant and f & g be two Measurable real valued function defined on the same line then the function f+c, cf, f+g, g-f, fg, f^2 are also Measurable.

(5) U.Q

Proof :-

To Prove :- f+c is Measurable.

$\forall \alpha \in R$, $\{x : (f+c)x < \alpha\}$

$\Rightarrow \{x : f(x) + c < \alpha\}$

$\Rightarrow \{x : f(x) < \alpha - c\}$

By Previous theorem. $\{x : f(x) < \alpha - \frac{1}{n}\}$ is Measurable

$\therefore$ $\{x : f(x) < \alpha - c\}$ is Measurable.

$\therefore \{x : (f+c)x < \alpha\}$ is Measurable

$\boxed{\therefore\ f+c \text{ is Measurable}}$

<u>To Prove</u> :- cf is Measurable.

$\forall \alpha \in R$, $\{x : (cf)x < \alpha\}$

$\Rightarrow \{x : f(x)\, c < \alpha\}$

$\Rightarrow \{x : f(x) < \alpha/c\}$

By Previous theorem,

$\therefore \{x : f(x) < \frac{\alpha}{c}\}$ is Measurable.

$\therefore \{x : (cf)x < \alpha\}$ is Measurable.

$\boxed{\therefore\ cf \text{ is Measurable.}}$

<u>To Prove</u> :- $f+g$ is Measurable

$\forall \alpha \in R$, $\{x : (f+g)x < \alpha\}$

$\Rightarrow \{x : [f(x)+g(x)] < \alpha\}$

$\Rightarrow \{x : f(x) < \alpha - g(x)\}$

By the dense set of Rational number Q in R

There is a rational number r for which

$f(x) < r < \alpha - g(x)$ ①

Hence $\{x : f(x)+g(x) < \alpha\} = \bigcup_{r \in Q} \{x : f(x)$

Since, Rational numbers are countable.

$\therefore \{x : f(x) < r\}$ and $\{x : g(x) < \alpha - r\}$

are Measurable for every r

$\therefore \{x : [f(x) + g(x)] < \alpha\}$ is Measurable.

$\boxed{\therefore\ f+g \text{ is Measurable.}}$

To Prove :- $g-f$ is Measurable.

~~$\forall\, \alpha \in R,\ \{x : (g-f)x < \alpha\}$~~

$$g - f = g + (-f)$$
$$= g + (-1) f$$

where $(-1) f = c f$ / $c = -1$

$\therefore -f$ is Measurable.

Gn: g is Measurable.

$\therefore g - f$ is Measurable.

To Prove :- fg is Measurable.

First we will prove f^2 is Measurable.

W.K.T

$$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) > -\sqrt{\alpha}$$

Both the function of 'f' on R.H.S is Measurable

$\therefore$ L.H.S is Measurable.

$\therefore \{x : f^2(x) > \alpha\}$ is Measurable.

$\therefore f^2$ is Measurable.

Finally, To Prove :- fg is Measurable

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

W.K.T

$(f+g)$ & $(f-g)$ are Measurable

$(f+g)^2$ & $(f-g)^2$ are Measurable.

$\therefore \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ is Measurable.

$\therefore$ fg is Measurable.

Hence the Proof.

Theorem 11 :-

Let $\{f_n\}$ be a sequence of Measurable function. Then the function $\sup\{f_1, f_2, \dots f_n\}$, $\inf\{f_1, f_2, \dots f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ are Measurable.

Proof :-

Let $h = \sup\{f_1, f_2, \dots f_n\}$

Let $g = \inf\{f_1, f_2, \dots f_n\}$

we will show that

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

Let $x \in h(x) > \alpha$

$\Rightarrow$ $\sup\{f_i(x)\} > \alpha$

$\exists\, i$ such that $f_i(x) > \alpha$

$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \rightarrow ①$

Conversely,

Let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

then $\exists i$ such that $f_i(x) > \alpha$

Since $h(x) \geq f_i(x) > \alpha$

$\Rightarrow h(x) > \alpha$

$\therefore x \in \{x : h(x) > \alpha\} \rightarrow ②$

From ① & ②

we have, $\boxed{\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}}$

$\therefore$ R.H.S of the above equation is Union of Measurable function which is Measurable

$\therefore \{x : h(x) > \alpha\}$ is Measurable

$\therefore h(x)$ is Measurable

i.e) $\sup\{f_1, f_2, \dots f_n\}$ is Measurable.

~~Let $g = \inf\{f_1, f_2, \dots, f_n\}$~~

Next ~~First~~ we will prove

$\{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

Let $x \in g(x) < \alpha$

$\Rightarrow \inf\{f(x) < \alpha\}$ also $\{g(x) < f_i(x)\} \forall i$

$\exists i$ such that $f_i(x) < \alpha$

$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

Conversely,

let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

$\Rightarrow \exists\, i$ such that $f_i(x) < \alpha$.

since $g(x) \leq f_i(x) < \alpha$ for every i

$\Rightarrow g(x) < \alpha$

$\therefore x \in \{x : g(x) < \alpha\}$

$$\boxed{\therefore \{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}}$$

$\therefore$ R.H.S of above inequality is Measurable

we know that

Union of Measurable set is Measurable

$\therefore \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$ is Measurable

$\therefore \{x : g(x) < \alpha\}$ is Measurable.

$\therefore g(x)$ is Measurable.

ie) $\inf \{f_1, f_2, \ldots f_n\}$ is Measurable.

— x —

<u>To Prove</u> :- $\sup_n f_n$ is Measurable.

w.k.T

ie) $\{x : \sup f_n(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

Measurable.

$\Rightarrow \sup_n f_n(x)$ is Measurable.

$|||^{ly}$ $\{x : \inf f_n(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) < \alpha\}$

is Measurable.

$\Rightarrow \inf_n f_n(x)$ is Measurable.

— × —

To Prove:- $\overline{\lim} f_n$ is Measurable.

Let $\overline{\lim} f_n = \inf_n \sup_{K \geq n} f_K$

$= \inf_n \sup \{f_n, f_{n+1}, \ldots\}$

$\Rightarrow \overline{\lim} f_n$ is Measurable.

$|||^{ly}$ we can Prove $\underline{\lim} f_n$ is Measurable.

— × —

Almost Everywhere :-

If the set of points where it fails to whole set of M_0 in particular we say that $f = g$ Almost everywhere

If f, g have same domain and $m\{x : f(x) \neq g(x)\} = 0$

$mE = 0$

2M

Theorem 12:-

If f is Measurable function and $f = g$ Almost everywhere. Then g is also Measurable.

V.Q

Proof:

Let E be the set

i.e) $E = \{x : f(x) \neq g(x)\}$

By Hypothesis

$$m\{x : f(x) \neq g(x)\} = 0$$

i.e) $mE = 0$

Now

$$\{x : g(x) > \alpha\} = \left[\{x : f(x) > \alpha\} \cup \{x \in E, \ldots\right.$$
$$- \{x \in E : g\ldots$$

Since,

$\therefore$ f is Measurable.

$\{x : f(x) > \alpha\}$ is Measurable and

$\{x \in E : g(x) > \alpha\}$ & $\{x \in E : g(x) \leq \alpha\}$ are subse…

of $\{x : f(x) \neq g(x)\}$

So, $\{x \in E : g(x) > \alpha\}$ & $\{x \in E : g(x) \leq \alpha\}$ …

Measurable

$\therefore$ L.H.S of Eqn ① is Measurable.

$\therefore$ g is Measurable.

Hence the Proof.

Theorem 13:-

Little Wood's 3rd three Principles.

Statement :-

Let E be a Measurable set of finite Measure and $\{f_n\}$ be a sequence of Measurable function defined on E.

Let f be a real valued function such that for each x in E, we have

$$f_n(x) \xrightarrow{\text{converges to}} f(x), \text{ then}$$

Given $\epsilon > 0$ & $\delta > 0$

there is a Measurable set $A \subset E$ with $\boxed{mA < \delta}$ and an integer N such that $\forall x \notin A$ and all $n \geq N$,

$$\boxed{|f_n(x) - f(x)| < \epsilon} \qquad \begin{matrix} x \in E \\ x \notin A \end{matrix}$$

Proof :-

Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$

and set $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon$ for some $n \geq N\}$

$$\therefore E_{N+1} = \bigcup_{n=N+1}^{\infty} G_n$$

we have

$$E_{N+1} \subset E_N$$

Since $f_n(x)$ converges to $f(x)$ for each $x \in E$

$\exists$ be E_N such that $x \notin E_N$

Hence $\bigcap E_n = \phi \rightarrow \circledast$

$x \in E$
$x \notin E_N$
$E \cap E_N = \phi$

each G_n is Measurable set

$\therefore E_N$ is Measurable.

$\because$ f_n & f are Measurable f... $f_n \rightarrow f$ is ... Meas...

Therefore, E is a finite Measure.

$E_N \subset E$

$\therefore$ We have, E_N also finite Measure

Then by theorem 5 $\Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m E_n$

$\therefore \lim_{n\to\infty} m E_n = m\left(\bigcap_{n=1}^{\infty} E_n\right)$ by $\circledast$

$= m\phi$

$= 0$

$$\boxed{\therefore \lim_{n\to\infty} m E_n = 0}$$

Hence, Given $\delta > 0$ $\exists$ a N

such that $m E_N < \delta$

ie) $m\{x \in E : |f_n(x) - f(x)| \geq \epsilon$ for some

Let this E_N be denoted by A

then $mA < \delta$

and $\bar{A} = \{x \in E : |f_n(x) - f(x)| < \epsilon$ $\forall n \geq$

Hence the Proof.

Definitions :-

* A set which is countable union of closed sets is called $\underline{F_\sigma}$

* A set which is countable intersection of open sets is called $\underline{G_\delta}$

Theorem 14 :-

Let E be a given set

Then the following 5 statements are equivalent

i) E is Measurable.

ii) Given $\epsilon > 0$ $\exists$ an open set $O \supset E$ with

$$m^*(O \sim E) < \epsilon$$

iii) Given $\epsilon > 0$ $\exists$ an closed set $F \subset E \subset O$ with

$$m^*(E \sim F) < \epsilon .$$

iv) There is a G in G_δ with $E \subset G$

$$m^*(G \sim E) = 0 .$$

v) There is an F in F_σ with $F \subset E$

$$m^*(E \sim F) = 0$$

If m^*E is finite, the above statements are equivalent to

vi) Given $\epsilon > 0$, $\exists$ a finite Union U of open intervals such that $m^*(U \Delta E) < \epsilon$

Proof :-

First we will prove

i) → ii)   ii) → iv)   iv) → i)

i) → ii) :-

Let E is Measurable

Case 1 :-

Suppose $m^* E = mE = \infty$

Given, $\epsilon > 0$ $\exists$ an open interval $E \subset ($

and $\sum l(I_n) \leq m^* E + \epsilon \rightarrow (1)$

Let $O = \bigcup_{n=1}^{\infty} I_n$ (O is open

$$m^* O = m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$= \sum_{n=1}^{\infty} m^* I_n$$

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$m^* O \leq m^* E + \epsilon$$

$$m^* O - m^* E \leq \epsilon \rightarrow (2)$$

Since $E \subset O$, we have

$$O = E \cup (O - E)$$

$$m^* O = m^* E + m^*(O - E)$$

$$m^* O - m^* E = m^*(O - E)$$

$\therefore m^*(O - E) = m^* O - m^* E \leq \epsilon$ $/ \because$ by (

$$m^*(O - E) < \epsilon$$

Case 2 :-

Suppose $m^* E = mE = \infty$

Let $R = \bigcup_{n=1}^{\infty} I_n$

A Countable Union of disjoint finite set

Let $E_n = E \cap I_n$

$\Rightarrow m^* E_n < \infty$

$\left[\because E = \bigcup_{n=1}^{\infty} E_n \quad E_n : E \cap I_n \to \text{finite} \right]$

$\exists$ an open interval $O_n \supset E_n$

$\ni : m^*(O_n - E_n) < \epsilon/2^n$

Take $O = \bigcup_{n=1}^{\infty} O_n$ is an open set

Then $O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n$

$$\leq \bigcup_{n=1}^{\infty} (O_n - E_n)$$

$$m^*(O - E) = m^* \left[\bigcup_{n=1}^{\infty} (O_n - E_n) \right]$$

$$\leq \sum_{n=1}^{\infty} m^*(O_n - E_n)$$

$$\leq \sum_{n=1}^{\infty} \epsilon/2^n$$

$$m^*(O - E) < \epsilon .$$

— x —

ii) → iv) :-

For each n $\exists$ an open set O_n

Such that $O_n \supset E$

$m^*(O_n - E) < \frac{1}{n}$

Let $G = \bigcap_{n=1}^{\infty} O_n$, E is a set in G

$E \subset G$

$$m^*(G-E) = m^*\left(\bigcap_{n=1}^{\infty} O_n - E\right)$$

$$= m^*\left(\bigcap_{n=1}^{\infty} (O_n - E)\right)$$

$$\leq m^*(O_n - E)$$

$$m^*(G-E) < \frac{1}{n} \qquad \forall n \geq 1$$

$\frac{1}{n} = \frac{1}{\infty}$

$\therefore$ we get $\boxed{m^*(G-E) = 0}$.

hence ii) $\Rightarrow$ iv)

iv) $\rightarrow$ i) :-

Let $m^*(G-E) = 0$ | $m^* A = 0 \Rightarrow A$ is Measurable

$\therefore (G-E)$ is Measurable.

$\therefore G$ being a G_δ set is Measurable

$(G_\delta = G - E)$

$\Rightarrow$ $G - (G-E)$ is Measurable.

$\Rightarrow \therefore E$ is Measurable

we get

i) $\Rightarrow$ ii)

ii) $\Rightarrow$ iv)

iv) $\Rightarrow$ i)

i) $\Rightarrow$ ii) $\Rightarrow$ iv) $\Rightarrow$ i)

— x —

Next, we will prove

$$ii) \Rightarrow iii) \Rightarrow v) \Rightarrow i)$$

<u>ii) $\Rightarrow$ iii) :-</u>

we already proved that

$$ii) \Rightarrow iv) \Rightarrow i)$$

i.e.) ii) $\Rightarrow$ i)

E is Measurable.

$\therefore$ $\bar{E}$ is Measurable

Since $\epsilon > 0$, $\exists$ an open set of $E \subset \bar{E}$

Such that

$$m^*(O - \bar{E}) < \epsilon \rightarrow ①$$

but $(E - \bar{O}) = (O - \bar{E})$

$$\therefore m^*(E - \bar{O}) = m^*(O - \bar{E}) < \epsilon.$$
by ①

Since O is open

$\bar{O}$ is closed.

Thus, we have a closed set $\bar{O}$

Such that

$$m^*(E - \bar{O}) < \epsilon \quad \text{and} \quad \bar{O} \subset E$$

$m^*(E - F) < \epsilon$ ... for $\bar{O} = F \subset E$

$$\therefore ii) \Rightarrow iii)$$

<u>iii) $\Rightarrow$ v) :-</u>

Assume (iii) for each n

$\exists$ a closed set $F_n \subset E$

$$\ni: m^*(E - F_n) < \frac{1}{n}$$

Let $F = \bigcup_{n=1}^{\infty} F_n$

F is a set in F_σ and $F_n \subset E$

$$m^*(E-F) = m^*\left(E - \bigcup_{n=1}^{\infty} F_n\right)$$
$$= m^*\left(\bigcap_{n=1}^{\infty}(E-F_n)\right)$$
$$= m^*(E-F_n)$$
$$< \frac{1}{n}$$

$$\boxed{\therefore\ m^*(E-F) = 0}$$

v) is true

v) $\Rightarrow$ i) :-

F is a set in F_σ

$\Rightarrow$ F is Measurable.

from v) $m^*(E-F) = 0$

$\Rightarrow$ $E-F$ is Measurable.

$\therefore$ $F + (E-F) \cdot F$ is Measurable.

$\therefore$ E is Measurable.

v) $\Rightarrow$ i)

ii) $\Rightarrow$ iii) $\Rightarrow$ v) $\Rightarrow$ i)

— X —

From the above proof

we prove

All the Five statements are equivalent

Now, we will prove that

$$ii) \Rightarrow vi) \Rightarrow ii)$$

ii) $\Rightarrow$ vi) :-

Assume ii) for every $\epsilon > 0$

$\exists$ an open set $O \supset E$

$\ni$: $m^* O \leq m^* E + \epsilon$

Let $J = \bigcup_{i=1}^{n} I_i$ and $U = O \cap J$

$(O \Delta E) \subseteq (O \Delta U) \cup (U \Delta E)$

$m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E)$

Let $U \subseteq J$

$\Rightarrow U - E \subseteq J - E \rightarrow \circledast$

Now, $E - U = E - (O \cap J)$

$= (E - O) \cup (E - J)$

$= \phi \cup (E - J)$

$$\boxed{E - U = E - J}$$

$\therefore U \Delta E = (U - E) \cup (E - U)$

$\overset{\circledast}{\subseteq} (J - E) \cup (E - J)$

$\subset (J \Delta E)$

we have $\boxed{m^*(U \Delta E) < \epsilon}$

here, $E \subseteq U \cup (U \Delta E)$

$m^* E \leq m^* U + m^*(U \Delta E)$

$\boxed{m^* E \leq m^* U + \epsilon}$

ii) is Proved

ii) $\Rightarrow$ vi) $\Rightarrow$ ii)

3/7/17. Unit - 2.

Riemann Integral.

Let f be a bounded real valued function defined on the $[a,b]$.

Let $a = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_n = b$

be a subdivision of $[a,b]$. Then for the subdivision we can define the sum

capital $S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i$ and

small $s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$

where $M_i = \sup_{\xi_{i-1} < x \le \xi_i} f(x)$

$m_i = \inf_{\xi_{i-1} < x \le \xi_i} f(x)$

{We can define the upper Riemann Integral of f by

$$R \overline{\int_a^b} f(x)\,dx = \inf S \text{ (capital)}$$

2m U.Q. with the inf taken overall possible subdivision of $[a,b]$.

$|||^{ly}$, we can define the lower Riemann Integral

$$R \underline{\int_a^b} f(x)\,dx = \sup s \text{ (small)}$$

$\sup \sum (\xi_i - \xi_{i-1}) \inf f$

If the upper integral & lower integral are equal we say f is Riemann Integrable

we denote it by $R\int_a^b f$]

V.Q 2m

By the $\boxed{\text{Step function}}$, we need that a function which has the form $\psi(x) = C_i$, $x_{i-1} < x < x_i$ for some subdivision of $[a,b]$.

and some set of constant c_i

Also $\int_a^b \psi(x)\,dx = \sum_{i=1}^{n} c_i (x_i - x_{i-1})$

$\phi \leq f \leq \psi$

From this, we have Riemann upper Integral

$$R\overline{\int_a^b} f(x)\,dx = \text{Inf} \int_a^b \psi(x)\,dx$$ for every step function $\psi(x) \geq f(x)$

and R.L.I :-

$$R\underline{\int_a^b} f(x)\,dx = \sup \int_a^b \phi(x)\,dx$$ for every step function $\phi(x) \leq f(x)$

Lebesgue Integral :-

The Lebesgue integral of a bounded function over a set of finite Measure.

Simple function :-

2m

The function χ_E defined by

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called a characteristic function of E.

A linear combination $\phi(x)$ is

$$\boxed{\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)}$$ is called a Simple function.

If the Set E_i are Measurable.

however, we know that a function ϕ is Simple. $\Leftrightarrow$ It is Measurable.

And assumes only a finite number of values.

If ϕ is a Simple function and $\{a_1, a_2, \dots a_n\}$ is set of non-zero value of ϕ.

$$\boxed{\phi = \sum_{i=1}^{n} a_i \varphi_{A_i}(x)}$$ (*)

where $A_i = \{x : \phi(x) = a_i\}$

This representation for ϕ is call

Canonical representation.

If ϕ Vanishes outside a set of finite Measure, we define a integral of ϕ by

$$\boxed{\int \phi(x)\, dx = \sum_{i=1}^{n} a_i m A_i}$$ (*)

where ϕ has the canonical representation,

$$\phi = \sum_{i=1}^{n} a_i \varphi_{A_i}$$

If E is any Measurable set, we define

$$\boxed{\int_E \phi = \int \phi\, \varphi_E}$$ (*)

mE

Lemma 1:-

Let $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}(x)$ with $E_i \cap E_j = \phi$ for $i \neq j$. Suppose each set E_i is a Measurable set of finite measure. Then $\int \phi = \sum_{i=1}^{n} a_i mE_i$

U.Q 5M

2 M

Proof:-

Let $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}$, where

$E_i = \{x : \phi(x) = a_i\}$

Let $A_a = \{x : \phi(x) = a\} = \bigcup_{a_i = a} E_i$

$$mA_a = m\left(\bigcup_{a_i=a} E_i\right)$$

$$mA_a = \sum_{a_i=a} mE_i$$

$$a\,mA_a = a \sum_{a_i=a} mE_i$$

$$a\,mA_a = \sum_{a_i=a} a_i mE_i$$

By def $\int \phi = \sum_{i=1}^{n} a\,mA_a$

$$\boxed{\therefore \int \phi = \sum_{i=1}^{n} a_i mE_i}$$

Hence The Proof.

Theorem :-

Let ϕ and ψ be simple functi[ons] which vanishes outside a set of finite measure then $\int(a\phi + b\psi) = a\int\phi + b\int\psi$ and if $\phi \geq \psi$ almost everywhere, then $\int\phi \geq \int\psi$.

Proof :-

Let $\{A_i\}$ and $\{B_j\}$ be the sets covering a canonical representation of ϕ and ψ

Let A_0 and B_0 be the sets where ϕ and ψ vanishes.

Then the set E_k obtained by taking the intersection of A_i and B_j from a finite disjoint collection of Measurable sets

we may write

$$\phi = \sum_{k=1}^{N} a_k \chi_{E_k} \rightarrow (1)$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k} \rightarrow (2)$$

$\int a\phi + b\psi = a\int\phi + b$

then, $$a\phi + b\psi = a\sum_{k=1}^{N} a_k \chi_{E_k} + b\sum_{k=1}^{N} b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} a a_k \chi_{E_k} + \sum_{k=1}^{N} b b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} (a a_k + b b_k)\chi_{E_k}$$

By Previous lemma,

$$\boxed{\int \phi = \sum a_i \, mE_i}$$

$$\int a\phi + b\psi = \sum_{k=1}^{N} (a a_k + b b_k)\, \chi E_k$$

$$= \sum_{k=1}^{N} a a_k \chi E_k + \sum_{k=1}^{N} b b_k \chi E_k$$

$$= a \sum_{k=1}^{N} a_k \chi E_k \ (1) + b \sum_{k=1}^{N} b_k \chi E_k \ (2)$$

$$\boxed{\int a\phi + b\psi = a\int \phi + b\int \psi}$$

W.K.T $$\int \phi - \int \psi = \int (\phi - \psi)$$

Also given that $\phi \geq \psi$ almost everywhere

$$\Rightarrow \therefore \phi - \psi \geq 0.$$

Since the integral of a simple function is ≥ 0. almost everywhere is non-negative.

By the definition of integral

$$\int(\phi - \psi) \geq 0$$

$$\int \phi - \int \psi \geq 0$$

$$\Rightarrow \boxed{\int \phi \geq \int \psi.}$$

Hence the Proof.

inf ≤ f
defined by ↑

Theorem 2:-

Let f be a definite and bound defined by
on a measurable set E with mE wi
finite, in order that $\inf_{f \le \psi} \int \psi(x)\,dx = \sup_{f \ge \phi} \int$
$\forall$ Simple functions ϕ and ψ it is
necessary and Sufficient that f is measurab
$\phi \le f \le$

Proof:-

Let f be bounded by M
and suppose that f is Measurable.
then the set $E_k = \left\{x : \frac{kM}{n} \ge f(x) \ge \frac{(k-}{}\right.$
ψ
$-n \ge k \ge n$ are measurable an
disjoint union in E.

$$E = \bigcup_{k=-n}^{n} E_k$$

Now,
$$mE = m\left(\bigcup_{k=-n}^{n} E_k\right)$$

$$mE = \sum_{k=-n}^{n} mE_k \rightarrow (1)$$

$\therefore$ The simple function is defined by

$$\psi_n(x) = \sum_{k=-n}^{n} \frac{km}{n} \chi_{E_k}(x)$$ and

$$\phi_n(x) = \sum_{k=-n}^{n} \frac{(k-1)m}{n} \chi_{E_k}(x)$$

It satisfies $\phi_n(x) \le f(x) \le \psi_n(x)$

$\inf \le f(x) \le \sup$.

Thus, $\displaystyle \underset{(X)}{\mathrm{Inf}} \int_E \psi(x)\, dx \le \int_E \psi_n(x)\, dx$.

$$\le \int_E \sum_{k=-n}^{n} \frac{km}{n} \psi\, E_k(x)\, dx$$

$$\le \sum_{k=-n}^{n} \int_E \frac{km}{n} \psi\, E_k(x)\, dx.$$

$$\le \sum_{k=-n}^{n} \frac{kM}{n} \int_E \psi\, E_k(x)\, dx.$$

$\left[\because \int_E \phi(x)\, dx = mE \right.$

$$\boxed{\mathrm{Inf} \int_E \psi(x)\, dx \le \sum_{k=-n}^{n} \frac{kM}{n}\, mE_k} \rightarrow (2)$$

$\displaystyle \underset{(X)}{\mathrm{Sup}} \int_E \phi(x)\, dx \ge \int_E \phi_n(x)\, dx$.

$$\ge \int_E \sum_{k=-n}^{n} \frac{(k-1)m}{n} \psi\, E_k(x)\, dx.$$

$$\ge \sum_{k=-n}^{n} \frac{(k-1)m}{n} \int_E \psi\, E_k(x)\, dx.$$

$$\ge \sum_{k=-n}^{n} \frac{(k-1)m}{n}\, mE_k$$

$$\boxed{-\mathrm{Sup} \int_E \phi(x)\, dx \le -\sum_{k=-n}^{n} \frac{(k-1)m}{n}\, mE_k} \rightarrow (3)$$

Compare (2) & (3)

$$\mathrm{Inf} \int_E \psi(x)\, dx - \mathrm{Sup} \int_E \phi(x)\, dx \le \sum_{k=-n}^{n} \frac{km}{n}\, mE_k - \sum_{k=-n}^{n} \frac{(k-1)m}{n}\, mE_k$$

$$\le \sum_{k=-n}^{n} \left(\frac{km}{n} - \frac{(k-1)m}{n} \right) mE_k$$

$$\leq \sum_{k=-n}^{n} \left(\frac{km}{n} - \frac{km}{n} + \frac{m}{n} \right) mE_k$$

$$\leq \sum_{k=-n}^{n} \frac{m}{n}\, mE_k$$

$$\leq \frac{m}{n} \sum_{k=-n}^{n} mE_k \qquad \text{by (1)}$$

$$\leq \frac{m}{n}\, mE \qquad (mE \neq 0)$$

Since n is arbitrary

we have,

$$\inf \int_E \psi(x)\,dx - \sup \int_E \phi(x)\,dx = 0$$

$$\boxed{\inf \int_E \psi(x)\,dx = \sup \int_E \phi(x)\,dx}$$

Conversely,

Given, $\inf \int_E \psi(x)\,dx = \sup \int_E \phi(x)\,dx$

we have to prove that f is measurable.

Given any 'n' their exist a simple function $\phi_n(x)$ and $\psi_n(x)$ such that.

$$\phi_n(x) \leq f(x) \leq \psi_n(x).$$

$$\Rightarrow \int \phi_n(x)\,dx \leq \int \psi_n(x)\,dx$$

$$\Rightarrow \int \phi_n(x)\,dx - \int \psi_n(x)\,dx \leq 0.$$

$$\Rightarrow \int \phi_n(x)\,dx - \int \psi_n(x)\,dx \leq 1/n$$

($\because$ Since n is arbit

Let $\psi^* = \inf \psi_n$ and

$\phi^* = \sup \phi_n$

$\therefore$ ϕ^* and ψ^* are Measurable. The set

$\Delta = \{x : \phi^*(x) < \psi^*(x)\}$ is the union of

the sets $\Delta_\nu = \{x : \phi^*(x) < \psi^*(x) - \frac{1}{\nu}\}$

ie) $\Delta = \cup \Delta_\nu$

If $\Delta'_\nu = \{x : \phi_n(x) < \psi_n(x) - \frac{1}{\nu}\}$

Also $\Delta_\nu \subset \{x : \phi_n(x) < \psi_n(x) - \frac{1}{\nu}\}$

ie) $\Delta_\nu \subset \Delta'_\nu$

ie) $m\,\Delta_\nu \leq m\,\Delta'_\nu \leq m\{x : \phi_n(x) < \psi_n(x) - \frac{1}{\nu}\}$

we have

$$\int \psi_n(x)\,dx - \int \phi_n(x)\,dx \leq 1/n$$

$$\int [\psi_n(x) - \phi_n(x)]\,dx < 1/n$$

$\phi_n(x) < \psi_n(x) - \frac{1}{\nu}$

$\therefore \frac{1}{\nu} < \psi_n(x) - \phi_n(x)$

if $\psi_n(x) - \phi_n(x) > 1/\nu$

then $\int \frac{1}{\nu}\,dx < \int [\psi_n(x) - \phi_n(x)]\,dx < 1/n$

$$\int \frac{1}{\nu}\,dx < 1/n$$

$$\frac{1}{\nu}\,m\,\Delta_\nu \leq 1/n \Rightarrow m\,\Delta_\nu \leq \frac{\nu}{n}$$

$$m\,\Delta'_\nu < \nu/n$$

$m\,\Delta_\nu \leq m\,\Delta'_\nu$

$m\,\Delta'_\nu \to 0$ as $n \to \infty$

$m\,\Delta_\nu \leq m\,\Delta'_\nu \to 0$

$$\boxed{\;\cdot\; \quad = 0}$$

$$m\Delta = m \cup \Delta_\gamma$$

$$m\Delta = \Sigma\, m\,\Delta_\gamma$$

$$\boxed{m\Delta = 0}$$

$$m\{x : \phi^*(x) < \psi^*(x)\} = 0$$

ie) $m\{x : \phi^*(x) \pm \psi^*(x)\} = 0.$

Thus $\phi^* = \psi^*$ except on the set of measure zero.

$\Rightarrow$ $\phi^* = \psi^* = f$ except on the set measure zero.

Then by the theorem.

if f is measurable and $f = g$ almost everywhere then g is measurable.

$\therefore$ f is Measurable.

Hence the Proof.

<u>Definition</u> :- Lebesgue integral

2m U.Q If f is bounded measurable function defined on measurable set E with <u>mE is finite</u>.

we define the <u>Lebesgue integral</u> of f over E by $\boxed{\int_E f(x)\,dx = \inf \int_E \psi(x)\,dx}$ for all simple function $\psi \geq f$

Note :-

we can write $\int_E f$, if $E = [a, b]$

If f is a bounded measurable function that vanishes outside a set E of finite measure we write $\int f$ for $\int_E f$.

$$\boxed{\int_E f = \int f \varphi_E .}$$

Theorem 8 :-

Let f be bounded function defined on $[a,b]$ in f is Riemman integral on $[a,b]$ then it is measurable. and

$$R \int_a^b f(x)\, dx = \int_a^b f(x)\, dx .$$

Proof :-

Since Every step function is also a simple function.

we have,

$$R \underline{\int_a^b} f(x)\, dx \leq \sup_{\phi \leq f} \int_a^b \phi(x)\, dx \leq \inf_{\psi \geq f} \int_a^b \psi(x)\, dx$$
$$\leq R \overline{\int_a^b} f(x)\, dx .$$

Since f is Riemann-integrable.

the inequality are equal.

ie) $$\sup_{\phi \leq f} \int_a^b \phi(x)\, dx = \inf_{\psi \geq f} \int_a^b \psi(x)\, dx$$

By Previous theorem, (Converse part)

$\therefore f$ is measurable.

$$\boxed{\therefore R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx.}$$

Hence the Proof.

Theorem 4 :-

If f and g are bounded measurable function defined on a set E of finite measure then

i) $\int_E (af+bg) = a\int_E f + b\int_E g$

ii) If $f=g$ almost everywhere then $\int_E f = \int_E g$

iii) If $f \leq g$ almost everywhere then $\int_E f \leq \int_E g$ and hence $\left|\int f\right| = \int |f|$

iv) If $A \leq f(x) \leq B$, then $AmE \leq \int_E f \leq BmE$

v) If A and B are disjoint measurable sets of finite measure then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof :-

i) Let φ be a simple function, Then $a\varphi$ is also a simple function and Conversely. if $a \neq 0$, hence for $a > 0$

$$\int_E af = \inf_{\varphi \geq f} \int_E a\varphi$$

$$= a \inf_{\varphi \geq f} \int_E \varphi$$

$$\boxed{\int_E af = a \int_E f}$$

if for $a < 0$

Let $a = -a'$

$$\int_E af = \inf_{\varphi \geq f} \int_E (-a')\varphi$$

$$= -a' \inf_{\varphi \geq f} \int_E \varphi$$

$$= a \inf_{\varphi \geq f} \int_E \varphi$$

$$\boxed{\int_E af = a \int_E f}$$

$\therefore$ Hence in both the cases

$$\int_E af = a \int_E f$$

If φ_1 and φ_2 are the two simple functions Such that $\varphi_1 \geq f$, $\varphi_2 \geq g$

Then $\psi_1 + \psi_2$ is also a simple fun

$\therefore f+g \leq \psi_1 + \psi_2$

hence, $\int_E f+g \leq \int_E (\psi_1 + \psi_2)$

$\leq \int_E \psi_1 + \int_E \psi_2$

$\underline{\int_E f+g < \int_E f + \int_E g} \rightarrow (1)$

If ϕ_1 and ϕ_2 are two simple function

such that $\phi_1 \leq f$, $\phi_2 \leq g$.

then $\phi_1 + \phi_2$ is also a simple function

such that $f+g \geq \phi_1 + \phi_2$

$\int_E f+g \geq \int_E \phi_1 + b_2$

$\geq \int_E \phi_1 + \int_E \phi_2$

$\underline{\int_E f+g > \int_E f + \int_E g} \rightarrow (2)$

From (1) & (2) we have

$\int_E f+g = \int_E f + \int_E g$

$\Rightarrow \boxed{\int_E (af+bg) = a\int_E f + b\int_E g}$

Hence the Proof

ii) To Prove :- If $f=g$ almost everywhere then $\int_E f = \int_E g$.

It is sufficient to show that

$\int_E f - g = 0$

Since, $f = g$ almost everywhere

$$f - g = 0.$$

It follows that

$$\psi \geq f - g$$

$$\Rightarrow \psi \geq 0$$

$$\int_E \psi \geq 0$$

$$\inf_{\psi \geq f-g} \int_E \psi \geq 0$$

$$\int_E f - g \geq 0$$

$$\int_E f - \int_E g \geq 0$$

$$\int_E f \geq \int_E g \rightarrow (3)$$

IIIly

$$\int_E \psi \leq 0.$$

$$\inf_{\psi \leq f-g} \int_E \psi \leq 0.$$

$$\int_E f - g \leq 0$$

$$\int_E f - \int_E g \leq 0$$

$$\int_E f \leq \int_E g \rightarrow (4)$$

From (3) & (4), we get

$$\boxed{\int_E f = \int_E g}$$

Hence the Proof

iii) Since $f \leq g$ almost everywhere

$$f - g \leq 0$$

It follows that

$$\phi \leq f - g$$

$$\Rightarrow \phi \leq 0$$

$$\Rightarrow \int_E \phi \leq 0$$

$$\sup_{\phi \leq f-g} \int_E \phi \leq 0$$

$$\int_E f - g \leq 0$$

$$\int_E f - \int_E g \leq 0$$

$$\boxed{\int_E f \leq \int_E g} \rightarrow (5)$$

Hence the Proof

Also,
we know that

$$f \leq |f|$$

$$\int_E f \leq \int_E |f|$$

$$\left| \int_E f \right| \leq \int_E |f|$$

Also w.k.t $\left|\int f\right| \geq \int |f|$

$$\Rightarrow \boxed{\left| \int f \right| = \int |f|}$$

Hence the Proof.

iv) To Prove :-

$$A \le f(x) \le B \text{ , then,}$$

$$AmE \le \int_E f \le BmE$$

Seperate corollaries.

W.K.T

$$f \le g$$

$$\Rightarrow \int_E f \le \int_E g \quad \text{and} \quad \int_E 1 = mE$$

Given, $A \le f(x) \le B$.

2m

$$\int_E A \le \int_E f(x) \le \int_E B$$

$$A \int_E 1 \le \int_E f(x) \le B \int_E 1$$

$$\boxed{AmE \le \int_E f \le BmE}$$

Hence the Proof

v) To Prove :-

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Seperate corollaries

Now,

$$\psi_{A \cup B} = \psi_A + \psi_B$$

2m

Since $\int_E f = \int f \psi_E$

$$\int_{A \cup B} f = \int f \psi_{A \cup B}$$

$$= \int f (\psi_A + \psi_B)$$

$$= \int f \psi_A + \int f \psi_B .$$

$$\int_{A \cup B} f =$$

Corollary :-

Let f be bounded measurable function on a set of finite Measure E. Suppose A and B are disjoint measurable subsets of E, then $\int_{A \cup B} f = \int_A f + \int_B f$.

Proof:-

Both $f\psi_A$ and $f\psi_B$ are bounded measurable function on E.

Since A and B are disjoint

$\Rightarrow$ $$\psi_{A \cup B} = \psi_A + \psi_B$$

Since $\boxed{\int_E f = \int f \psi_E}$

write the previous Corollary (v).

ie) $$\int_{A \cup B} f = \int f \psi_{A \cup B}.$$

$$= \int f (\psi_A + \psi_B)$$

$$= \int f \psi_A + \int f \psi_B$$

$$\boxed{\int_{A \cup B} f = \int_A f + \int_B f}$$

Hence the Proof.

I.

Corollary :-

Let f be a bounded measurable function on a set of finite measure then $\left|\int_E f\right| \le \int_E |f|$

Proof:-

The function $|f|$ is measurable and bounded

Now, on E

$$-|f| \le f \le |f|$$

$$-\int |f| \le \int f \le \int |f|$$

$$\Rightarrow \boxed{\left|\int_E f\right| \le \int_E |f|}$$

Hence the Proof.

Bounded Convergence Theorem :- 5

Statement :-

Let sequence $\{f_n\}$ be a sequence of Measurable functions defined on the set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \le M$ for all n and all x. If $f(x) = \lim_{n\to\infty} f_n(x)$ for each x in E then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof :-

By the version of Little Wood's 3^{rd} principle

Given $\epsilon > 0$ $\exists$ n and a measurable set

$A \subset E$ with $mA < \frac{\epsilon}{4M} \rightarrow (*)$

such that $n \geq N$ and $x \in E \sim A$

We have

$$| f_n(x) - f(x) | < \epsilon / 2ME$$

Thus.

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right|$$

Think

$$\leq \int_E | f_n - f |$$

Thinking

$$\leq \int_{E-A} | f_n - f | + \int_A | f_n - f |$$

Thoughts.

$$\leq \int_{E-A} \frac{\epsilon}{2ME} + \int_A |f_n| + \int_A |f|$$

$\leq \frac{\epsilon}{2mE} m(E-A) + (M+M) \frac{\epsilon}{4m}$

$$\leq \frac{\epsilon}{2ME} \int_{E-A} 1 + 2M \int_A 1 + \int_A |f|$$

$|f_n| \leq M$

$|f_n - f| \leq |f_n| + |f|$

$\leq M + M \int_A 1$

$\leq 2M (mA)$

$$\leq \frac{\epsilon}{2ME} m(E-A) + 2M(mA)$$

$E - A \subset E$

$m(E-A) \subset mE$ $\leq \frac{\epsilon}{2\cancel{ME}} \cancel{ME} + 2M \frac{\epsilon}{4M}$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\left| \int_E f_n - \int_E f \right| < \epsilon$$

$$\therefore f(x) = \lim_{n \to \infty} \int_E f_n$$

The integral of a non-negative fun

If f is a non-negative me function defined on a measurable set we define $\boxed{\int_E f = \sup_{h \leq f} \int_E h}$

where h is bounded measurable func such that $m\{x : h(x) \neq 0\}$ is finite (or)

mx is finite / h

Theorem 6:-

If f and g are non-negative measurable functions then

i) $\int_E cf = c \int_E f$

ii) If $f \leq g$ almost everywhere then $\int_E f \leq$

iii) $\int_E f+g = \int_E f + \int_E g$.

Proof:-

i) If h is a bounded measurable funct then ch is also a bounded measurable for $c > 0$,

$$\int_E cf = \sup_{h \leq f} \int_E ch$$

$$= c \sup_{h \leq f} \int_E h$$

$$\boxed{\int_E cf = c\int f}$$

Def:- For a non-negative measurable function on E. we define then $\int_E f = \sup\left\{\int_E h \,/\, h \text{ bounded, measurable of finite and } 0 \leq h \leq f \text{ on } E\right\}$ $m[x : h(x) \neq 0]$

ii) <u>To Prove</u>:- $f \le g$ almost everywhere then $\int_E f \le \int_E g$

If $h(x) \le f(x)$ and $k(x) \le g(x)$

$\Rightarrow h(x) \le k(x)$ almost everywhere on E.

$\Rightarrow h(x)$ & $k(x)$ are bounded measurable function

$$\Rightarrow \int_E h \le \int_E k$$

Taking Sup on both sides

we have

$$\sup_{h \le f} \int_E h \le \sup_{k \le g} \int_E k$$

$$\boxed{\int_E f \le \int_E g}$$

iii) <u>To Prove</u>:- $\int_E f+g = \int_E f + \int_E g$.

Let $h(x) \le f(x)$ and $k(x) \le g(x)$

$\Rightarrow h(x) + k(x) \le f(x) + g(x)$

Taking $\int_E$ on both sides

$$\int_E h+k \le \int_E f+g$$

here h & k are bounded measurable functions

then $\int_E h + \int_E k \le \int_E f+g$

Taking Sup on L.H.S side, we get $\sup_{h \le f} \int_E h + \sup_{k \le g} \int_E k \le \int_E f+g$

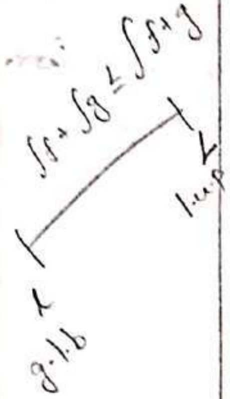
$\cancel{\int_E f + \int_E g \le} \int_E f+g$

$$\therefore \int_E f + \int_E g \le \int_E f+g \rightarrow (1)$$

Let "l" be a bounded measurable function which vanishes outside a set of finite measure and which is not greater than $f+g$

$m^* F = 0$.

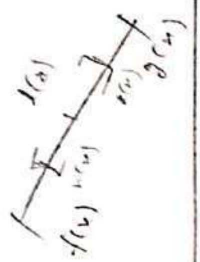
ie) $l \le f+g$.



Then we define the function.

h and k by getting $h(x) = \min(f(x), l(x))$

$\&$ $k(x) = \max(l(x) - h(x))$



we have $h(x) \le f(x)$ $\&$ $k(x) \le g(x)$.

while h and k are bounded by the bounds and vanish

hence $\int_E l = \int_E h+k$

$$\int_E l = \int_E h + \int_E k$$

Taking Sup on R.H.S side, we get

$$\int_E l = \sup_{h \le f} \int_E h + \sup_{k \le g} \int_E k$$

$$\int_E l = \int_E f + \int_E g$$

$$\int_E f+g \le \int_E f + \int_E g \rightarrow (2).$$

From (1) & (2)

$$\boxed{\int f + \int g = \int (f+g)}$$

Fatou's Lemma :- ⑦.

Statement :-

If $\{f_n\}$ be a sequence of non-negative measurable function on E.
and $f_n(x) \to f(x)$ point wise almost everywhere a set
~~If~~ $\{f_n\} \to f$ almost everywhere on a E.

then $\int_E f \leq \underline{\lim} \int_E f_n$

Proof :-

without loss of generality, we may assume that, the converges is everywhere.
Since $\int$ over set of measure zero are zero

Let h be a bounded measurable function which is not greater than f
Such that $h \leq f$ which vanish outside a set E' of finite measure.

Define a function $h_n(x) = \min(h(x), f_n(x))$
$|h_n| \leq h \leq f_n$

Then h_n is bounded by the bound for h i.e.
and vanish outside E'.

Now $h_n \to h$ for each x in E'

By Bounded convergence theorem

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \underline{\lim} \int_E f_n$$

Taking sup over h.

$$\sup_{h \le f} \int_E h \le \underline{\lim} \int_E f_n$$

$$\Rightarrow \boxed{\int_E f \le \underline{\lim} \int_E f_n}$$

Hence the Proof.

Monotone Convergence Theorem :- (2)

Statement :-

Let $\{f_n\}$ be an increasing sequence of non-negative measurable function. and let $f = \lim f_n$ almost everywhere then $\int_E f = \lim \int_E f_n$

Proof :-

Given, $f = \lim f_n$ almost everywhere.

$$\Rightarrow f_n \to f \text{ on } E$$

By Fatou's Lemma,

$$\int_E f \le \underline{\lim} \int_E f_n \to (1)$$

Since, $\{f_n\}$ is an increasing sequence

$$f_n \le f$$

$$\Rightarrow \int_E f_n \le \int f$$

$$\Rightarrow \overline{\lim} \int_E f_n \leq \int_E f \rightarrow (2).$$

From (1) & (2)

$$\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f$$

$|||^{ly}$ $\geq$ condition, so

We have

$$\int_E f = \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n = \int_E f$$

$$\boxed{\therefore \int_E f = \lim \int_E f_n}$$

Hence the Proof.

Corollary :-

Let u_n be an sequence of non-negative measurable function and let $f = \sum_{i=1}^{\infty} u_i$ and then $\int f = \sum_{n=1}^{\infty} \int u_n$

Proof:-

Given, $f = \sum_{i=1}^{\infty} u_i$

Let $S_n = \sum_{i=1}^{n} u_i$

$S_{n+1} = \sum_{i=1}^{n+1} u_i$

$\Rightarrow S_n \subset S_{n+1} \subset S_{n+2} \subset \ldots$

$\Rightarrow S_n$ is an increasing sequence.

By Monotone Convergence theorem

$$\int_E f = \lim \int_E s_n \rightarrow (4)$$

we have $\quad s_n = \sum_{i=1}^{n} u_i$

$$\int s_n = \int \sum_{i=1}^{n} u_i$$

$$= \int u_1 + u_2 + \dots + u_n$$

$$\int s_n = \sum_{i=1}^{n} \int u_i$$

taking limit

$$\lim_{n \to \infty} \int s_n = \sum_{n=1}^{\infty} \int u_n$$

(4)

$$\boxed{\int f = \sum_{n=1}^{\infty} \int u_n}$$

Hence the Proof

Integrable :-

(X) om A non-negative measurable function is called Integrable over a measurable set E if $\boxed{\int_E f < \infty}$

Theorem :- (9)

Let f and g are two non-negative measurable functions. If f is integrable over E and $g(x) \leq f(x)$ on E. Then g is also integrable on E. and $\int_E f-g = \int_E f - \int_E g$

Proof:-

Gn: $\int_E f < \infty$

Also, given that, $g(x) \leq f(x)$

$$\int_E g(x) \leq \int_E f(x) < \infty$$

$$\boxed{\therefore \int_E g < \infty}$$

$\Rightarrow$ g is integrable over E

Given that

f and g are non-negative measurable functions.

We know that

$$\int_E f+g = \int_E f + \int_E g$$

Consider, $f = f - g + g$

$$\Rightarrow \int_E f = \int_E (f-g) + g$$

$$= \int_E (f-g) + \int_E g$$

$$\boxed{\int_E f - \int_E g = \int_E (f-g)}$$

Hence the Proof.

Theorem 10:-

Let f be a non-negative function which is integrable over a set E. Then given $\epsilon > 0$, there is a $\delta > 0$. Such that For every set $A \subset E$ with measure of A $(mA < \delta)$ we have

$$\int_A f < \epsilon$$

Proof:-

i) Suppose f is bounded.

$$\Rightarrow \quad |f(x)| \leq M \qquad \forall x \in E$$

$$\boxed{\text{Take } \delta = \frac{\epsilon}{M}}$$

$$\int_A f \leq \int_A M \qquad \because A \subset E$$

$$\leq M \int_A 1$$

$$\leq M \, mA$$

$$< M \delta$$

$$< M \frac{\epsilon}{M}$$

$$\boxed{\int_A f < \epsilon}$$

Hence The Proof.

ii) Suppose f is not bounded.

Define . $f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{otherwise} \end{cases}$

then f_n is bounded and $f_n \to f$ at each point

By Monotone Convergence theorem

There is an N such that

$$\int_E f_N > \int_E f - \epsilon/2$$

$$\int_E f - \int_E f_N < \epsilon/2$$

$$\int_E f - f_N < \epsilon/2$$

Choose $\boxed{\delta < \frac{\epsilon}{2N}} \rightarrow$ ①

If $mA < \delta$, we have

$$\int_A f = \int_A f + \int_A f_N - \int_A f_N$$

$$= \int_A (f - f_N) + \int_A f_N$$

$$< \frac{\epsilon}{2} + N \int_A 1$$

$$< \frac{\epsilon}{2} + N\, mA$$

$$< \frac{\epsilon}{2} + N\delta$$

$$< \frac{\epsilon}{2} + N \frac{\epsilon}{2N}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\boxed{< \epsilon}$$

Hence the Proof.

General Lebesgue Integral :-

A measurable function f is said to be integral over E if f^+ and f^- are both integrable over E. In this case, we define

2M

integral $\boxed{\int_E f = \int_E f^+ - \int_E f^-}$

Theorem 11:-

Let f and g be integral over E.

i) Then the function cf is integral over E and

$$\int_E cf = c\int_E f.$$

ii) The function $f+g$ is integrable over E and

10M

$$\int_E f+g = \int_E f + \int_E g.$$

iii) If $f \leq g$ almost everywhere then $\int_E f \leq \int_E g$

iv) If A and B are disjoint measurable set

U.Q

Contained in E. then $\int_{A\cup B} f = \int_A f + \int_B f$

Proof:-

Since f is non-negative $(f = f^+ - f^-)$

i) $\int_E cf = \int_E c\,(f^+ - f^-)$

$$= \int_E (cf^+ - cf^-)$$

$$= \int_E cf^+ \ldots$$

$$= c\int_E f^+ - c\int_E f^-$$

$$= c\left[\int_E f^+ - \int_E f^-\right]$$

$$\boxed{\int_E cf = c\int_E f}$$

---

ii) Given that, f and g are integrable.

then f^+, g^+, f^-, g^- are also non-negative

Such that $f = f^+ - f^-$ and $g = g^+ - g^-$

And also $\int_E f = \int_E f^+ - f^- < \infty$ (integrable condition)

$$\int_E g = \int_E g^+ - g^- < \infty$$

So, f^+, f^-, g^+, g^- are all integrable.

$$f+g = (f^+ + g^+) - (f^- + g^-)$$

$$\int_E f+g = \int_E f^+ + g^+ - \int_E f^- + g^-$$

$$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$\boxed{\int_E f+g = \int_E f + \int_E g}$$

iii) Suppose $f \le g$ almost everywhere

we have to P.T

$$\int_E f \le \int_E g$$

Gn: $f \le g$

$$f^+ - f^- \le g^+ - g^-$$

$\Rightarrow \quad g^+ - g^- - f^+ + f^- \ge 0$ (integrable of non. msble fun is non -

$$\Rightarrow \int_E g^+ - \int_E g^- - \int_E f^+ + \int_E f^- \ge 0$$

$$\Rightarrow \int_E g^+ - \int_E g^- \ge \int_E f^+ - \int_E f^-$$

$$\int_E g \ge \int_E f$$

$$\Rightarrow \boxed{\int_E f \le \int_E g}$$

iv) If A and B are disjoint measurable set contained in E.

then $\int_{A \cup B} f = \int f \varphi_{A \cup B}$

$$= \int f (\varphi_A + \varphi_B)$$

$$= \int f \varphi_A + \int f \varphi_B$$

$$\boxed{\int_{A \cup B} f = \int_A f + \int_B f}$$

Hence the theorem.

Theorem :- (17)

Lebesgue Convergence Theorem.

Statement :-

Let g be a integrable over E and let $\{f_n\}$ be a sequence of Measurable function such that $|f_n| \le g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$, then $\int_E f = \lim \int_E f_n$

Proof :-

Given $|f_n| \le g$

$\Rightarrow -g \le f_n \le g$

here we have $f_n \le g$

$\Rightarrow g - f_n \ge 0$

$\Rightarrow \int g - \int f_n \ge 0$

Hence, the given function $(g - f_n)$ is non-negative

By Fatou's Lemma,

$$\int_E g - f_n \le \underline{\lim} \int_E g - f_n$$

Since $|f_n| \le g$ for every n.

we have

$|f| \le g$

and also g is integrable.

So f is integrable

$$\int_E g - \int_E f \le \int_E g - \underline{\lim} \int_E f_n$$

$$\int f \ge \underline{\lim} \int_E f_n$$

$|||^{ly}$ $$\int_E f \geq \overline{\lim} \int_E f_n \rightarrow (A)$$

Consider, $$-g \leq f_n$$

$$\Rightarrow 0 \leq f_n + g$$

$$\Rightarrow f_n + g \geq 0$$

$\therefore$ $f_n + g$ is non -ve .

By Fatou's Lemma

$$\int_E g + f \leq \underline{\lim} \int_E g + f_n$$

$$\Rightarrow \cancel{\int_E g} + \int_E f \leq \underline{\lim} \int_E f_n + \cancel{\int_E g}$$

$$\int_E f \leq \underline{\lim} \int_E f_n \rightarrow (B) .$$

From (A) & (B) , we have

$$\boxed{\int_E f = \lim \int_E f_n}$$

Hence the Proof.

—x—

Unit - 3.

17/7/17

Differentiation of Monotone function:-

Let $\mathcal{I}$ be a collection of interval then we say that $\mathcal{J}$ covers a set E in the sense of vitali, if for each $\epsilon > 0$ and for any x in E, there is an interval $I \in \mathcal{J}$ such that $x \in I$ and $l(I) < \epsilon$

The interval may be open, closed, half open but we do not allow degenerate intervals consisting of only one point

Theorem :-

Vitali Covering Lemma

10 m repeated

Let E be a set of finite outer measure and $\mathcal{J}$ be a collection of intervals that covers E in the sense of vitali then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, I_2, \dots I_N\}$ of intervals in $\mathcal{J}$ such that $m^*[E - \bigcup_{n=1}^{N} I_n] < \epsilon$

Proof:-

It is sufficient to prove the lemma in the case that each interval in $\mathcal{J}$ is closed. For otherwise we replace each interval by its closure $\bar{I} = I$ and observe that the set end points of $\{I_1, I_2, \dots I_N\}$ has measure zero.

Let O be an open set of finite measure containing E.

$$m^* O < \infty$$

To Prove :- $\mathcal{J}$ is a Vitali covering of E.

Since I is a covering of E without loss of generality we may assume that each I of $\mathcal{J}$ contained in O.

ie) $I \subseteq O$ for every $I \in \mathcal{J}$.

we choose a $\{I_n\}$ of disjoint interval of $\mathcal{J}$ By induction as follows,

Let I_1 be any interval in $\mathcal{J}$ and let $k_1 = \sup\{ l(I) : I \in \mathcal{J}$ and I does not meet $I_1\}$

$$k_1 = \sup_{I \in \mathcal{J}} l(I)$$

$$I \cap I_1 = \phi$$

Let I_2 be any interval in $\mathcal{J}$ which is disjoint from I_1 and

$$l(I_2) > \frac{1}{2} k_1$$

and $k_2 = \sup_{I \in \mathcal{J}} \{ l(I) : I \in \mathcal{J}$ and I does not meet $I_2\}$

$$I \cap I_2 = \phi$$

Let I_n be any interval in $\mathcal{I}$

Such that

$I \cap I_1, \; I \cap I_2, \ldots I \cap I_n \ldots = \phi$

$$k_n = \sup l(I)$$
$$= I \cap \mathcal{I}$$
$$= I \cap I_1 = I \cap I_2 = \ldots = \phi$$

Each I is contained in O

we have $I \subset O \rightarrow$ (open covering)

$$I \subset O \Rightarrow l(I) \leq m^* O = mO$$

$$\Rightarrow \sup_{I \in \mathcal{I}} l(I)$$

$$I \cap I_1, \; I \cap I_2, \ldots, I \cap I_{n-1} = \phi$$

$$\sup l(I) \leq mO$$
$$< \infty$$

$$\Rightarrow \boxed{k_n < \infty}$$

unless $E \subset \bigcup_{i=1}^{n} I_i$

we can find I_{n+1} in $\mathcal{I}$ with

$$\boxed{l(I_{n+1}) > \frac{1}{2} k_n} \rightarrow (1)$$

and I_{n+1} is disjoint from $I_1, I_2, \ldots I_n$

Thus we have a sequence $\{I_n\}$ of disjoint intervals of $\mathcal{J}$ and

Since $\cup I_n \subset O$

$$m(\cup I_n) < mO < \infty$$

ie> $\boxed{\sum l(I_n) < mO < \infty} \rightarrow (A)$

To Prove N :-

Hence we can find an integer 'N' such that

$$\boxed{\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}} \rightarrow (B)$$

(A)-(B)

$$\boxed{\text{Let } R = E - \bigcup_{n=1}^{N} I_n} \rightarrow (\#)$$

To Prove the lemma :-

it is sufficient to prove $m^*R < \epsilon$.

Let x be any point of R.

Since $\bigcup_{n=1}^{N} I_n$ is a closed set not containing x

we can find a interval I in $\mathcal{J}$ which contains x and whose length is so small that I does not mean any of the interval $(I_1, I_2, \dots I_N)$.

Now, $\boxed{I \cap I_i = \phi}$ for $i < n$

we must have $l(I) \leq k_n$

where $2l(I_{n+1}) > k_n$ from $\ominus$

$$\boxed{\therefore\ l(I) < 2l(I_{n+1})}$$

Since, the interval I must need atle

one of the interval I_n

Let n be the smallest integer

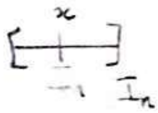
such that I need I_n

we have $n > N$ and

$$l(I) \leq k_{n-1} \leq 2\,l(I_n)$$

Since x is in I and I has a

point in common with I_n it follows.



that the distance x to mid point of I_n

is atmost

Adding $\frac{1}{2} l(I_n)$

$$l(I) + \frac{1}{2} l(I_n) \leq 2\,l(I_n) + \frac{1}{2} l(I_n)$$

$$\leq \frac{5}{2} l(I_n)$$

Thus x belongs to the interval J_n

having the same mid point as I_n and

five times the length

Thus $x \in J_n$

$$R \subset \bigcup_{N+1}^{\infty} J_n$$

Hence,
$$m^* R \leq \sum_{N+1}^{\infty} l(J_n)$$
$$\leq 5 \sum_{N+1}^{\infty} l(I_n)$$
$$\leq 5 \frac{\epsilon}{5} \quad /\because \text{by } (B)$$
$$m^* R < \epsilon$$
$$\boxed{\therefore \; m^* \left[E - \bigcup_{n=1}^{N} I_n \right] < \epsilon} \quad \text{by } (\#).$$

Hence the Proof.

Derivative of f and x :-

$$D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}$$

$$D_+ f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}$$

clearly we have,
$$D^+ f(x) \geq D_+ f(x)$$
$$D^- f(x) \geq D_- f(x)$$

If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \ldots$

we say,

f is differentiable at x

and define $f'(x)$ to be the common value of the derivative at x.

If $D^+ f(x) = D_+ f(x)$

we say, that f has a right hand derivative at x

Theorem :-

If f is continuous on $[a,b]$ and one of its derivative (say D^+) is everywhere non-negative, an open interval (a,b), then f is non-decreasing on $[a,b]$.

ie) $f(x) \leq f(y)$ for $x \leq y$.

2m

Proof:-

Given f is continuous on $[a,b]$ and one of its derivative is non-negative

ie) $D^+ f(x) \geq 0$ (non-negative)

$$\Rightarrow \inf_{\delta > 0} \sup_{0 < h < \delta} \frac{f(x+h) - f(x)}{h} \geq 0$$

$\Rightarrow$ for every $\epsilon > 0$, $\delta > 0$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} \geq 0$$

$\Rightarrow$ For every $\delta > 0$, $\epsilon > 0$

$\exists$ $h \in (a,b)$ (open interval)

$$[f(x+h) - f(x)] \geq 0$$

$$\Rightarrow \quad f(x+h) \geq f(x)$$

$$\Rightarrow \quad f(y) \geq f(x)$$

ie) $y \geq x$.

$$\Rightarrow \quad \boxed{f(y) \geq f(x)}$$

$\therefore$ f is non decreasing on $[a,b]$.

Hence the Proof.

<u>Theorem</u> :-

Let f be an increasing real valued function on the interval $[a,b]$. Then f is differentiable almost everywhere & the derivative of f is measurable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a)$$

U.Q 10m

<u>Proof</u> :-

We know that, f is differentiable on $[a,b]$ then the four derivatives are

$$D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}$$

$$D_+ f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$

$\Rightarrow$ The set of points in $[a,b]$

where the 4 derivatives are not equ

is a set of measure zero.

So, To Prove :- f is differentiable.

we will show that

The set where any two derivatives

are un-equal have measure zero.

Consider the set E

where $D^+ f(x) > D_- f(x)$

ie) $E = \{x \in [a,b] \mid D^+ f(x) > D_- f(x)\}$

we have to prove

$E = \{x \in [a,b] \mid D^+ f(x) > D_- f(x)\}$ is of measure zero.

Let E is a union of sets $E_{u,v}$

$E_{u,v} = \{x \in [a,b] \mid D^+ f(x) > u > v > D_- f(x)\}$

$E = \cup E_{u,v}$

It is sufficient to prove that

$m^*(E_{u,v}) = 0$.

Let $m^*(E_{u,v}) = S$.

Choose $\epsilon > 0$ $\exists$ an open set O

Such that
$$E_{u,v} \subset O \quad \text{and}$$
$$m^* O \leq m^* E_{u,v} + \epsilon$$
$$\Rightarrow \quad m^* O \leq S + \epsilon \quad \rightarrow (*)$$

If $x \in E_{u,v}$ then we can find arbitrary small interval $[x-h, x]$

Now, $x \in E_{u,v} \Rightarrow v > D_- f(x)$
$$\Rightarrow \quad v > \frac{f(x) - f(x-h)}{h}$$
$$\Rightarrow \quad v(h) > f(x) - f(x-h)$$

ie) $f(x) - f(x-h) < vh < \epsilon$

Then the collection of all such intervals $[x-h, x]$ is a vitali covering for $E_{u,v}$

$\therefore$ we can choose a finite collection $\{I_1, I_2, \dots I_n\}$

Such that
$$m^* \left[E_{u,v} - \bigcup_{n=1}^{N} I_n \right] < \epsilon \text{ (vitali covering)}$$

ie) $m^* [E_{u,v}] - m^* [\bigcup_{n=1}^{N} I_n] < \epsilon$
$$S - m^* \left(\bigcup_{n=1}^{N} I_n \right) < \epsilon$$
$$*\left(\bigcup_{n=1}^{N} I_n \right) < \epsilon - S$$

$$m^*\left(\bigcup_{n=1}^{N} I_n\right) > s-\epsilon \rightarrow (\#)$$

$\therefore$ The interior of this interval $\{I_1, I_2, \cdots I_N\}$ covers subset A of $E_{u,v}$ whose outer measure is $> s-\epsilon$ and summing over this interval

Now,

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) < \sum_{n=1}^{N} v h_n$$

$$< v m^* O \rightarrow (1)$$

$$< v(s+\epsilon) \quad [\because \text{ by } (*)]$$

$\rightarrow$ (A)

Let y be any point in A. Then we can choose a small interval of the form $(y, y+k)$ that contained in some I_n and for which $f(y+k) - f(y) > uk$

So $y \in I_n \Rightarrow y \in E_{u,v}$

$$\Rightarrow D^+ f(y) > u$$

$$\Rightarrow \frac{f(y+k) - f(y)}{k} > u$$

$$\Rightarrow f(y+k) - f(y) > u(k)$$

$J = \{[y, y+k] \,/\, y \in A\}$ is a Vitali covering of A is finite measure.

Applying Vitali Covering lemma.

$\exists$ a disjoint collection $\{J_1, J_2, \dots J_m\}$ |

$$m^*\left(A - \bigcup_{n=1}^{m} J_n\right) < \epsilon$$

$$\Rightarrow \text{Take } m^*(A) = m^*\left(\bigcup_{n=1}^{m} J_n\right) + m^*\left(A - \bigcup_{n=1}^{m} J_n\right)$$

$$\Rightarrow m^*(A) = m^*\left(\bigcup_{n=1}^{n} J_n\right) + \epsilon$$

$$m^* A = m^*\left(\bigcup_{n=1}^{N} I_n\right) + \epsilon \qquad / \because J_n \subset I_n$$

Separate step

$$\Rightarrow m^* A < m^*\left(\bigcup_{n=1}^{m} J_n\right) + \epsilon$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{m} J_n\right) > m^* A - \epsilon$$

$$m^* A > m^*\left(\bigcup_{n=1}^{m} I_n\right) - \epsilon$$

$$> S - \epsilon - \epsilon$$

$$m^* A > S - 2\epsilon$$

Union of the intervals $J_1, J_2, \dots J_m$

Contains a subset of A of outer measure

greater than $S - 2\epsilon$

Also.

$$\sum_{i=1}^{m} f(y_i+k_i) - f(y_i) > \sum_{i=1}^{m} u k_i$$

$$> u \sum_{i=1}^{m} k_i$$

$$> u\, m^* A$$

$$> u(S-2\epsilon)$$

$\hookrightarrow$ (I)

Since each J_i is contained in some I_n and the sum over those I for which J_i contained in I_n ($J_i \subset I_n$) we have,

$$\sum_{i=1}^{m} f(y_i+k_i) - f(y_i) < \sum_{n=1}^{N} f(x_n) - f(x_n-h_n)$$

(B) (A)

$$u(S-2\epsilon) < v(S+\epsilon)$$

$$u(S-2\epsilon) < v(S+2\epsilon)$$

$$\Rightarrow \quad uS < vS$$

$$\Rightarrow \quad u < v$$

But $u > v$ and so must be zero.

$$\therefore S = 0$$

$$\therefore m^* E_{u,v} = 0$$

$$m^* E = 0$$

$\therefore$ the set of points in $[a,b]$ where the four derivatives of x are unequal and those of measure zero.

$\Rightarrow$ f is differentiable almost on $[a,b]$

$$\Rightarrow \quad g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

is defined almost everywhere on $[a,b]$

and f is differentiable whenever g is finite

Define

$$\boxed{g_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}} \qquad / \because h = 1/n$$

and for $x \geq b$ and $f(x) = f(b)$

$$\lim_{h \to 0} g_n(x) = \lim_{h \to 0} \frac{f(x + 1/n) - f(x)}{1/n}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{\lim_{h \to 0} g_n(x) = g(x)}$$

$\therefore$ $g_n(x) \to g(x)$ as $n \to \infty$

i.e) $g_n(x) \to g(x)$ almost everywhere

and g is measurable.

$\therefore$ f' is measurable.

Since f is increasing function.

we have, $g_n \geq 0$

<u>By Fatou's Lemma</u> :-

$$\int_a^b g \leq \underline{\lim} \int_a^b g_n$$

$$\int_a^b g \leq \underline{\lim} \int_a^b n\left[f(x+1/n) - f(x)\right] dx$$

$$\leq \underline{\lim} \left[\int_a^b n\left[f(x+1/n)\right] dx - \int_a^b n f(x)\, dx\right]$$

$$\leq \underline{\lim} \left[n \int_a^b f(x+1/n)\, dx - n \int_a^b f(x) dx\right]$$

$$/ \because \text{by } 11^{th} \text{ Property } \int_a^b f(x)\, dx = \int_a^b f(t)\, dt$$

$$\leq \underline{\lim} \left[n \int_{a+1/n}^{b+1/n} f(t) dt - n \int_a^b f(x) dx\right]$$

$$\leq \underline{\lim} \left[n \int_{a+1/n}^{b+1/n} f(x)\, dx - n \int_a^b f(x)\, dx\right]$$

$$\leq \underline{\lim} \left[n \int_{a+1/n}^{b} f(x)\, dx + n \int_b^{b+1/n} f(x)\, dx - n \int_a^{a+1/n} f(x)\, dx - n \int_{a+1/n}^{b} f(x)\, dx\right]$$

$$\int_a^b g \leq \underline{\lim} \left[n \int_b^{b+1/n} f(x)\, dx - n \int_a^{a+1/n} f(x)\, dx \right]$$

$$\left\{ \begin{array}{l} \because x \geq b \quad \text{and} \quad f(x) = f(b) \\ \because x \geq a \quad \text{and} \quad f(x) = f(a) \end{array} \right\}$$

$$\leq \underline{\lim} \left[n f(b) \int_b^{b+1/n} dx - n \int_a^{a+1/n} f(x)\, dx \right]$$

$$\leq \underline{\lim} \left[n f(b) \left[b + \frac{1}{n} - b \right] - n f(a) \int_a^{a+1/n} dx \right]$$

$$\leq \underline{\lim} \left[n f(b) \frac{1}{n} - n f(a) \left[a + \frac{1}{n} - a \right] \right]$$

$$\leq \underline{\lim} \left[n f(b) \frac{1}{n} - n f(a) \frac{1}{n} \right]$$

$$\int_a^b g \leq \underline{\lim} \left[f(b) - f(a) \right]$$

$\therefore$ This shows that g is integrable

and hence, finite almost everywhere.

Thus, f' is integrable almost everywhere

and $g = f'$ almost everywhere

$$\boxed{\therefore \int_a^b f'(x)\, dx \leq f(b) - f(a)}$$

Hence the Proof.

Functions of bounded variation :-

Let f be a real valued function defined on $[a,b]$. Let $a = x_0 < x_1 < x_2 < \dots < x_k = b$ be any subdivision of $[a,b]$. Define p

where $p = \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1}) \right)^+$,

$$n = \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1}) \right)^-$$

Then,

$$f(x_i) - f(x_{i-1}) = \left(f(x_i) - f(x_{i-1}) \right)^+ - \left(f(x_i) - f(x_{i-1}) \right)^-$$

$$\sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1}) \right) = \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1}) \right)^+ - \sum_{i=1}^{k} \left(f(x_i) - f(x_{i-1}) \right)^-$$

$$\left.\begin{array}{l} \cancel{f(x_1)} - f(\overset{a}{x_0}) + \cancel{f(x_2)} - \cancel{f(x_1)} + \dots \\ \dots + f(\underset{b}{x_k}) - \cancel{f(x_{k-1})} \end{array}\right\} = p - n$$

$$f(b) - f(a) = p - n .$$

ie) $f(b) - f(a) + n = p$

$$\boxed{\therefore \; p = n + f(b) - f(a)}$$

here,

$$\left\{ \begin{array}{ll} r^+ = r & \text{if } r \geq 0 \\ \quad = 0 & \text{if } r < 0 \end{array} \right\}$$

$|r| = r^+ + r^-$

$$\left\{ \begin{array}{ll} r^- = 0 & \text{if } r \geq 0 \\ \quad = -r & \text{if } r < 0 \end{array} \right\}$$

Next: $\quad$ modulus. $\qquad\qquad\qquad$ modulus.

$$\left| f(x_i) - f(x_{i-1}) \right| = \left| \left(f(x_i) - f(x_{i-1}) \right)^{+} - \left(f(x_i) - f(x_{i-1}) \right)^{-} \right|$$

$$\sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{i=1}^{k} \left| \left(f(x_i) - f(x_{i-1}) \right)^{+} \right| + \sum_{i=1}^{k} \left| \left(f(x_i) - f(x_{i-1}) \right)^{-} \right|$$

$$t = n + p$$

Define: $P = \sup p$

$N = \sup n$

$T = \sup t$

where Sup is taken overall subdivisions of $[a,b]$ observe that p, n are ≥ 0.

So, $t = n + p$

$\Rightarrow \quad t \geq p \qquad \because n \geq 0.$

Taking sup on both sides.

$\sup t \geq \sup p$

$$\boxed{T \geq P}$$

$|||^{ly}$ $\qquad \because p \geq 0.$

$\Rightarrow t \geq n$

Taking sup on both sides.

$\sup t \geq \sup n$

$$\boxed{T \geq N}$$

We have, $t = p + n$

Taking $\sup t = \sup (p + n)$

$\leq \sup p + \sup n$

$$\boxed{T \leq P + N}$$

we call P, N and T as positive, negative and total variation of f over [a, b].

It is also sometimes denoted by

T_a^b (or) $T_a^b(f)$.

Total T (Total variation)

To denote the interval [a, b] and the function f

Definition :- B.V.

If $T < \infty$ then f is said to be of bounded variation of [a, b] and denoted by $\boxed{f \in B.V}$

Total variation

Lemma :-

If f is of B.V. on [a, b] then

easy

$$T_a^b = P_a^b + N_a^b \text{ and } f(b) - f(a) = P_a^b - N_a^b$$

2M

Proof:-

easy.

For any subdivision of closed interval [a, b]

$$p - n = f(b) - f(a)$$

$$p = n + f(b) - f(a)$$

Taking sup on overall subdivision of [a, b]

$$P_a^b = N_a^b + f(b) - f(a).$$

$$\boxed{f(b) - f(a) = P_a^b - N_a^b} \rightarrow (*)$$

W.K.T

$$t = p + n$$

$$t = p + p - f(b) + f(a)$$

$$t = 2p - f(b) + f(a)$$

Taking Sup on overall subdivision of $[a,b]$

$$T_a^b = 2P_a^b - f(b) + f(a)$$

$$= 2P_a^b - [f(b) - f(a)]$$

$$= 2P_a^b - [P_a^b - N_a^b] \quad \text{by } (*).$$

$$\boxed{T_a^b = P_a^b + N_a^b}$$

Hence The Proof.

Theorem :-

A function f is of Bounded Variation on $[a,b]$ iff f is the difference of two monotone real valued function on $[a,b]$

↑ (or) ↓

Proof :-

Let f be of Bounded variation on $[a,b]$

Set $g(x) = P_a^x$ and $h(x) = N_a^x$

Since the term $[f(x_i) - f(x_{i-1})]^+ \geq 0$

if $x \leq y$, then $P_a^x \leq P_a^y$

from this we get $N_a^x \leq N_a^y$ for $x \leq y$

$$g(x) \leq g(y) \quad \text{if } x \leq y$$

$h(y)$

So,

g and h are Monotone increasing function

Also, $0 \le P_a^x \le T_a^x \le T_a^b < \infty$

IIIly $0 \le N_a^x \le T_a^x \le T_a^b < \infty$

(g & h are non-negative and strictly <)

$\therefore \; 0 \le g(x) < \infty$

$0 \le h(x) < \infty \quad \forall x \in [a, b]$

and so

g and h are real valued function

By Previous Lemma,

$$f(x) - f(a) = P_a^x - N_a^x = g(x) - h(x)$$

Therefore,

$$f(x) = g(x) - h(x) + f(a)$$

$$f(x) = g(x) - [h(x) - f(a)]$$

here $g(x)$ and $h(x) - f(a)$ are monotone increasing real valued function.

So, f is the difference of two monotone real valued function on $[a, b]$

— x —

Conversely.

Case ① if $f = g - h$ with g and h are both increasing. (or) both decreasing.

then any subdivision of $[a, b]$

$$t_a^b = \Sigma \left| f(x_i) - f(x_{i-1}) \right|$$

$$= \Sigma \left| [g(x_i) - h(x_i)] - [g(x_{i-1}) - h(x_{i-1})] \right|$$

$$\leq \Sigma \left| g(x_i) - h(x_i) \right| - \Sigma \left| g(x_{i-1}) - h(x_{i-1}) \right|$$

$$\leq \Sigma \left| g(x_i) - g(x_{i-1}) \right| + \Sigma \left| h(x_i) - h(x_{i-1}) \right|$$

$$t_a^b \leq g(b) - g(a) + h(b) - h(a) < \infty$$

because g & h are increasing functions, $g(x_i) - g(x_{i-1}) \geq 0$

$h(x_i) - h(x_{i-1}) \geq 0$

Taking sup on overall partition we get

$$T_a^b \leq g(b) - g(a) + h(b) - h(a) < \infty .$$

$$\boxed{T < \infty}$$

$\therefore$ f is of B.V on $[a, b]$

$$\boxed{\therefore \ f \in B.V.}$$

Case ② If g is $\uparrow$ (increasing) and h is $\downarrow$ (decreasing), then

$g - h$ is $\uparrow$

i.e) $g - h = f$ is $\uparrow$, for an $\uparrow$ function $f(x_i) - f(x_{i-1}) \geq 0$

So, $\Rightarrow$ $$t_a^b = \Sigma \left| f(x_i) - f(x_{i-1}) \right|$$

$$\leq \Sigma |f(x_i)| - \Sigma |f(x_{i-1})|$$

$$t_a^b \leq f(b) - f(a) < \infty$$

Taking Sup

$$T_a^b \leq f(b) - f(a) < \infty$$

$$\boxed{T < \infty}$$

$\therefore$ f is of B.V.

$$\boxed{f \in B.V.}$$

Hence the Proof.

Corollary :- (*)

If f is of Bounded variation on $[a,b]$ then $f'(x)$ exist for almost all x in $[a,b]$

Proof :-

If $f \in BV$ on $[a,b]$

then $f = g - h$

where g & h are monotonically increasing real valued function.

By theorem 3 :-

$\Rightarrow$ g and h are differentiable almost everywhere

This shows that f is differentiable almost everywhere

ie) $f'(x)$ exists $\forall x$ in $[a,b]$

Hence the Proof

Differentiation of an integral :-

If f is an integrable function of $[a,b]$ we define its indefinite integral to be the function F. defined on $[a,b]$ by

$$\boxed{F(x) = \int_a^x f(t)\,dt.}$$

Theorem :-

If f is integrable on $[a,b]$ then the function F is defined by $F(x) = \int_a^x f(t)\,dt$ is a continuous function of Bounded variation $T<\infty$ on $[a,b]$

Proof :-

Claim ① :-

write $f = f^+ - f^-$

then f^+ and f^- are non negative integrable function

by the theorem ⑩ :- (unit 2)

$$\int_E f < \epsilon$$

Given $\epsilon > 0$, $\exists$ δ_1 and δ_2 are > 0

Such that $\exists mA < \delta_1 \Rightarrow \int_A f^+ < \frac{\epsilon}{2}$ and

$mA < \delta_2 \Rightarrow \int_A f^- < \frac{\epsilon}{2}$

Take $\delta = \min(\delta_1, \delta_2)$

if $m(A) < \delta$ then $\int_A f^+ < \frac{\epsilon}{2}$ and $\int_A f^- < \frac{\epsilon}{2}$

Take x and y in $[a,b]$

Such that $|x-y| < \delta$

then $m[x,y] = |x-y| < \delta$

$$\therefore \int_x^y f^+ < \frac{\epsilon}{2} \quad \text{and} \quad \int_x^y f^- < \frac{\epsilon}{2}$$

$$\therefore \left| \int_x^y f \right| = \left| \int_x^y f^+ - \int_x^y f^- \right|$$

$$\leq \left| \int_x^y f^+ \right| + \left| \int_x^y f^- \right|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\boxed{\left| \int_x^y f \right| < \epsilon} \rightarrow (*)$$

ie) $$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_a^y f(t)\,dt + \int_x^a f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right|$$

$$\boxed{|F(y) - F(x)| < \epsilon} \quad \because \text{by } (*)$$

$\therefore$ this shows that F is <u>continuous</u> on $[a,b]$.

<u>Claim ②</u> :- To Prove F is B.V. on $[a,b]$

Let $a = x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k = b$.

be a subdivision of $[a,b]$

we have

$$t = \sum_{i=1}^{K} \left| F(x_i) - F(x_{i-1}) \right|$$

$$\Rightarrow \sum_{i=1}^{K} \left| \int_a^{x_i} f(t)\,dt - \int_a^{x_{i-1}} f(t)\,dt \right|$$

$$= \sum_{i=1}^{K} \left| \int_a^{x_i} f(t)\,dt + \int_{x_{i-1}}^{a} f(t)\,dt \right|$$

$$= \sum_{i=1}^{K} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right|$$

$$\leq \sum_{i=1}^{K} \int_{x_{i-1}}^{x_i} |f(t)|\,dt.$$

$$\leq \int_{x_0}^{x_1} |f(t)|\,dt + \int_{x_1}^{x_2} |f(t)|\,dt + \dots + \int_{x_{K-1}}^{x_K} |f(t)|\,dt$$

$$t \leq \int_{x_0}^{x_K} |f(t)|\,dt$$

$$t \leq \int_a^b |f(t)|\,dt.$$

Taking Sup on both sides.

$$\text{Sup } t \leq \int_a^b |f(t)|\,dt < \infty.$$

$$\boxed{\therefore T < \infty}$$

$\therefore$ F is a function of BV of $[a, b]$.

Hence the Proof.

+

Theorem :-

$\int_E f < \infty$

If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$ for every $x \in [a,b]$ then $f(t) = 0$ almost everywhere on $[a,b]$.

easy

Proof:- Gn : $\int_a^x f(t)\,dt = 0 \quad \forall\, x \in [a,b]$

If possible, (Taking Conversely)

Let $f(t) \neq 0$ on a set E of positive measure

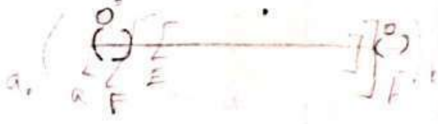
Then $f(t) > 0$, (or) $f(t) < 0$ on a set $E \subset [a,b]$

i) If $f(t) > 0$ on a set E of the +ve measure.

The given $\epsilon > 0$. $\exists$ a closed set $E \subset F$

Such that $mE \leq mF + \epsilon$ (or)

$mF \geq mE - \epsilon$



$\Rightarrow mF \geq 0$

$\Rightarrow$ Let $O = [a,b] - F$

$\Rightarrow [a,b] = O + F \rightarrow (*)$

$F^c = O$

$\Rightarrow O$ is an open set

$\therefore O = \bigcup_n I_n$. Where I_n is an open interval and they are pairwise disjoint

In particular,

$$\int_a^b f(t)\,dt = 0 \quad \text{(or)} \quad \int_a^b f(t)\,dt \neq 0$$

$$\Rightarrow \int_{O+F} f(t)\,dt = 0 \quad \text{(or)} \quad \int_{O+F} f(t)\,dt \neq 0$$

$$\Rightarrow \int_O f(t)\,dt + \int_F f(t)\,dt = 0, \quad \text{(or)} \quad \int_{O+F} f(t)\,dt \neq 0$$

$$\int_O f(t)\,dt + \int_F f(t)\,dt \neq 0$$

$$\Rightarrow \int_O f(t)\,dt = -\int_F f(t)\,dt \neq 0 \qquad \int_O f(t)\,dt = -\int_F f(t)\,dt \neq$$

$$\therefore \int_O f(t)\,dt \neq 0 \text{ in both the cases.}$$

$$\therefore \int_{O = \cup I_n} f(t)\,dt \neq 0$$

$$\Rightarrow \sum \int_{I_n} f(t)\,dt \neq 0.$$

$$\text{ie)} \int_{I_n} f(t)\,dt \neq 0. \qquad I_n = [a_n, b_n]$$

$$\Rightarrow \int_{a_n}^{b_n} f(t)\,dt \neq 0 \qquad \text{where } I_n = [a_n, b_n]$$

$$\Rightarrow \int_{a_n}^{a} f(t)\,dt + \int_a^{b_n} f(t)\,dt \neq 0.$$

$$\Rightarrow \int_{a_n}^{a} f(t)\,dt - \int_{b_n}^{a} f(t)\,dt \neq 0$$

$$\Rightarrow \int_{a_n}^{a} f(t)\,dt \neq 0, \quad \int_{b_n}^{a} f(t)\,dt \neq 0.$$

$$\text{ie)} \int_a^{a_n} f(t)\,dt \neq 0, \quad \int_a^{b_n} f(t)\,dt \neq 0.$$

which is a contradiction to $\int_a f(t)\,dt$

$\therefore$ Our assumption is wrong

$\therefore f(t) \not> 0$

$|||^{ly}$ $f(t) \not< 0$.

$\therefore$ $\boxed{f(t) = 0}$ almost everywhere on $[a,b]$

Hence the Proof.

Theorem :-

If f is bounded & measurable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$ then $F'(x) = f(x)$ for almost all $x \in [a,b]$

Proof :-

Test

Gn: f is bounded & measurable.

Also: $F(x) = \int_a^x f(t)\,dt + F(a)$

1^{st} Condition $\int c f = c \int f$

By the theorem (c) (2 unit) $\int_a^x f(t)\,dt$ is a functions of B.V. then

a constant function $F(a)$ is also a functions of B.V.

$\therefore F(x)$ is a functions of B.V. on $[a,b]$

$F(x) = \int_a^x f(t)\,dt + F(a)$

Also, we know that

Every function of B.V. ... exists ... Corollary (?)

$\therefore F'(x)$ exists for almost all $x \in [a,b]$

Given that f is bounded

$\therefore$ we can find F such that $|f(x)| < M$

for all $x \in [a,b]$

For every integer n define $h = 1/n$

$$\text{and } f_n(x) = \frac{F(x+h) - F(x)}{h} \rightarrow (*)$$

$$= \frac{1}{h}\left\{\int_a^{x+h} f(t)\,dt + \cancel{F(a)} - \left[\int_a^{x} f(t)\,dt + \cancel{F(a)}\right]\right\}$$

$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^{a} f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]$$

$$|f_n(x)| = \left|\frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]\right|$$

$$\leq \frac{1}{h}\int_x^{x+h} |f(t)|\,dt.$$

$$< \frac{1}{h}\int_x^{x+h} M\,dt$$

$$< \frac{1}{h} M \int_x^{x+h} dt$$

$$< \frac{1}{h} M\,[\cancel{x}+h - \cancel{x}]$$

$< \frac{1}{k} M [k]$

$$\boxed{\therefore \left| f_n(x) \right| < M.}$$ |||ly $|f(x)| < M$

Taking lim on $f_n(x)$ in (A)

$$\lim_{h\to 0} f_n(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$\boxed{\lim_{h\to 0} f_n(x) = F'(x).}$$

$\Rightarrow$ $\{f_n(x)\}$ is a sequence of bounded Measurable function covering to $F'(x)$.

By Bounded Convergence theorem :-

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Applying integral

$$\int_a^c F'(x)\,dx = \lim_{h\to 0} \int_a^c \underbrace{\left[\frac{F(x+h) - F(x)}{h}\right]}_{= f_n(x)} dx.$$

$$= \lim_{h\to 0} \frac{1}{h} \int_a^c \left[F(x+h) - F(x)\right] dx.$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_a^c F(x+h)\,dx - \int_a^c F(x)\,dx \right]$$

Let $x+h = t$

$dx = dt$

then $x = a \Rightarrow t = a+h$

$x = c \Rightarrow t = c+h$

$$\therefore \int_a^c F'(x)\,dx = \lim_{h\to 0} \frac{1}{h} \left[\int_{a+h}^{c+h} F(t)\,dt - \int_a^c F(x)\,dx \right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{a+h}^{c+h} F(x)\,dx - \int_a^c F(x)\,dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{\int_{a+h}^{c} F(x)\,dx} + \int_c^{c+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx - \cancel{\int_{a+h}^{c} F(x)\,dx}\right]$$

$$\boxed{\therefore \int_a^c F'(x)\,dx = \lim_{h\to 0} \frac{1}{h}\left[\int_c^{c+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx\right]} \to (1)$$

$\therefore$ F is continuous on $[a, c]$

$\Rightarrow$ F is continuous at a and c.

Given $\epsilon > 0$, $\exists\ \delta > 0$

$\ni$: $|x - c| < \delta$

$\Rightarrow$ $|F(x) - F(c)| < \epsilon$ $\quad \forall\ x \in [c, c+h]$

$\Rightarrow$ $\boxed{F(x) = F(c) - \eta_1 h}$ where $\eta_1 \to 0$ as $h \to 0$.

$|||^{ly}$ $|F(x) - F(a)| < \epsilon$.

$\forall\ x \in [a, a+h]$

$\Rightarrow$ $\boxed{F(x) = F(a) - \eta_2 h}$ where $\eta_2 \to 0$ as $h \to 0$

$$\therefore (1) \Rightarrow \int_a^c F'(x)\,dx = \lim_{h\to 0}\frac{1}{h}\left[\int_c^{c+h} (F(c) - \eta_1 h)\,dx - \int_a^{a+h} (F(a) - \eta_2 h)\,dx\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left\{(F(c) - \eta_1 h)(\cancel{c}+h-\cancel{c}) - \left[(F(a) - \eta_2 h)(\cancel{a}+h-\cancel{a})\right]\right\}$$

$$\left\{(F(c) - \eta_1 h)\cancel{h} - (F(a) - \eta_2 h)\cancel{h}\right\}$$

$$\int_a^c F'(x)\,dx = \lim_{h\to 0}\left\{ F(c) - \underset{0}{\eta_1 h} - F(a) + \underset{0}{\eta_2 h} \right\} \qquad \begin{cases} \eta_1 \to 0 & \text{as } h\to 0 \\ \eta_2 \to 0 & \text{as } h \to 0 \end{cases}$$

$$= \lim_{h\to 0} \left\{ F(c) - F(a) \right\}$$

$$= \lim_{h\to 0}\left\{ \int_a^c f(t)\,dt + F(a) - F(a) \right\} \quad \text{2 steps} \qquad = \int_a^c f(x)\,dx + \cancel{F(a)} - \cancel{F(a)}$$

$$= \lim_{h\to 0} \int_a^c f(t)\,dt$$

$$= \int_a^c f(x)\,dx$$

$\because F(x) = \int_a^x f(t)\,dt + F(a)$

$F(c) = \int_a^c f(t)\,dt + F(a)$

$F(a) = \underbrace{\int_a^a f(t)\,dt}_{0} + F(a)$

$$\boxed{\int_a^c F'(x)\,dx = \int_a^c f(x)\,dx}.$$

$\Rightarrow \boxed{F'(x) = f(x)}$ almost everywhere on $[a,b]$

$\forall\, c \in [a,b]$

Hence the Proof

Theorem :-

$\int f < \infty$

Let f be an integrable function on $[a,b]$. Suppose that $F(x) = F(a) + \int_a^x f(t)\,dt$ then $F'(x) = f(x)$ for almost all x in $[a,b]$

Proof :-

Let f be non-negative integrable function

Let f_n be defined by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ x & \text{if } f(x) > n \end{cases}$$

Then $f_n < f \quad \forall n$

$f - f_n \ge 0 \quad \forall n.$

Define

$$(*)\quad \boxed{G_n(x) = \int_a^x (f - f_n)\, dx} \rightarrow (1)$$

is an increasing function which have a derivative almost everywhere.

from the definition of f_n

f_n is bounded and integrable.

$\therefore$ It is also measurable.

f_n is

By Previous theorem,

we have

$$\begin{cases} F(x) = \int_a^x f_n(t)\, dt + F(a) \rightarrow (2) \\ \text{and} \quad F'(x) = f_n(x) \end{cases}$$

(f_n in place of f by Previous theorem).

$$(2) \Rightarrow \frac{d}{dx} F(x) = \frac{d}{dx}\left[\int_a^x f_n(t)\, dt + F(a)\right]$$

$$F'(x) = \frac{d}{dx}\left[\int_a^x f_n(t)\, dt\right] + 0.$$

$$f_n(x) = \frac{d}{dx}\left[\int_a^x f_n(t)\, dt\right] \rightarrow (3)$$

Also, WKT $F(x) = \int_a^x f(t)\, dt + F(a)$

$$= \int_a^x (f - f_n + f_n)(t)\, dt + F(a)$$

$$= \int_a^x (f - f_n)(t)\, dt + \int_a^x f_n(t)\, dt + F(a)$$

$$\ldots (t)\, dt + \frac{d}{dx}\int_a^x f_n(t)\, dt + 0$$

$$F'(x) = \frac{d}{dx} G_n(x) + f_n(x) \quad / \because \text{by (3)}$$

$$F'(x) - f_n(x) = \frac{d}{dx} G_n(x) \geq 0$$

$$F'(x) - f_n(x) \geq 0$$

$$F'(x) - f(x) = 0 \quad / \because n \text{ is arbitrary}$$

$$\boxed{\therefore F'(x) = f(x)}$$

for almost all x in $[a, b]$

Hence the Proof.

Absolute Continuous function :-

A Real Valued function f is defined on $[a, b]$ is said to be Absolutely Continuous on $[a, b]$ if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\boxed{\sum_{i=1}^{n} | f(x_i') - f(x_i) | < \epsilon}$$

for every finite collection $\{x_i', x_i\}$ are non-overlapping interval.

$$\boxed{\sum_{i=1}^{n} | x_i' - x_i | < \delta}$$

2m

Note :-

1. An absolute continuous function is continuous.

2. The Sum and difference of two Absolute continuous function is also Absolute continuous.

Theorem :-

If f is absolutely continuous on $[a,b]$ then it is of bounded variation on $[a,b]$.

Proof :-

Let f be absolute continuous that corresponding to $\epsilon = 1$

Hence For $\epsilon = 1$, we have

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < 1$$

with $\sum_{i=1}^{n} | x_i' - x_i | < \delta$

Then any subdivision of $[a,b]$ can be split into K sets of intervals, each of bounded length $< \delta$

where K is the largest integer, $K = 1 + \frac{(b-a)}{\delta}$

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < 1 < K = 1 + \frac{b-a}{\delta}$$

i.e) $t < K$

Taking Sup we get

$$\text{Sup } t < K < \infty$$

$$\boxed{T_a^b < \infty}$$

$[T < \infty]$

$\therefore$ f is of Bounded Variation on $[a,b]$

Hence the Proof

<u>Corollary</u> :-

If f is absolutely Continuous, then f has a derivative almost everywhere.

<u>Proof</u>:-

<u>Gn</u>: f is absolutely continuous on $[a,b]$

By Previous theorems

$\Rightarrow$ f is Bounded Variation on $[a,b]$

$\Rightarrow$ $f'(x)$ exists for almost all x on $[a,b]$

ie) f has a derivatives almost everywhere on $[a,b]$

10 M <u>Theorem</u> :-

If f is absolutely Continuous on $[a,b]$ and $f'(x) = 0$ almost everywhere then f is Constant.

(7) <u>Proof</u>:-

U.Q.

<u>To Prove</u> :- $f(x)$ is Constant

We shall P.T. :- $\boxed{f(c) = f(a)} \quad \forall \; c \in [a,b]$

Given f is absolutely Continuous. on $[a,b]$

and $f'(x) = 0$ almost everywhere.

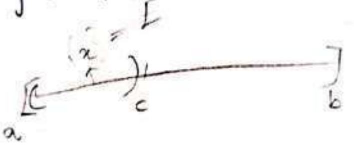
Given $\epsilon > 0$, $\exists \; \delta > 0$ Such that

for every finite collection

$\{(x_k, y_k)\}_{k=1}^{n} \quad (k = 1, 2, \dots n)$

of non-overlapping interval with $\sum_{k=1}^{n} |y_k - x_k| < \delta$

$$\Rightarrow \sum_{k=1}^{n} |f(y_k) - f(x_k)| < \epsilon.$$



Let $c \in [a,b]$.

Let $E = \{x \in (a,c) \;/\; f'(x) = 0\}$

then $E \subset [a,c]$ be the set of measure $(c-a)$; Such that

$mE \leq c - a$

in which $f'(x) = 0$

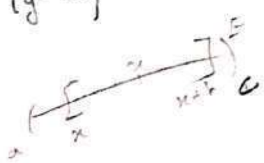
Let ϵ and η be arbitrary +ve numbers

to each $x \in E$. , $f'(x) = 0$

ie) $\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0.$

$$\left| \frac{f(y) - f(x)}{y - x} \right| < \eta$$

$$|f(y) - f(x)| < \eta \, |y - x|$$



ie) $x \in [x, x+h]$

(derivative condition)

$$\Rightarrow \boxed{|f(y) - f(x)| < \eta\, |y - x|}$$

for an interval $[x_k, y_k]$ such that

$$|f(y_k) - f(x_k)| < \eta\, |y_k - x_k|$$

for every $x_k \in E$.

$\therefore \exists \{[x_k, y_k]\}$ is a collection of intervals

for each $h > 0$

There is an interval $x \in (x_k, y_k)$ (I).

and (I) $l(x_k, y_k) = (y_k - x_k) < h$.

$\exists$ a vitali covering of E which is

a finite measure (1)

i.e) By vitali covering Lemma for $\delta > 0$

finite disjoint collection $\{[x_k, y_k]\}_{k=1}^{n}$ of

(I) (II) (III).

interval such that. $m^*[E - \bigcup_{n=1}^{n} I_n] < \epsilon$

$$m\left\{[a, c] - \bigcup_{k=1}^{n} [x_k, y_k]\right\} < \delta.$$

Now, we rearrange x_k

Such that $x_k < x_{k+1}$ $[a, c]$

$$a = x_0 \le x_1 \le y_1 \le x_2 \le y_2 \le \dots \le y_n \le x_{n+1} = c$$

and $\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta$

$$\Rightarrow \boxed{\therefore \sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \epsilon} \rightarrow (1)$$

Since f is

Now, for the interval $\{[x_k, y_k]\}_{k=1}^{n}$

$$\Rightarrow \left| f(y_k) - f(x_k) \right| < \eta \, |y_k - x_k|$$

$$\Rightarrow \sum_{k=0}^{n} \left| f(y_k) - f(x_k) \right| < \eta \sum_{k=0}^{n} |y_k - x_k|$$

$y_k \cdot x_0$

$$\boxed{\therefore \sum_{k=0}^{n} \left| f(y_k) - f(x_k) \right| \leq \eta (c-a)} \rightarrow (2)$$

from (1) + (2)

consider,

$$\left| f(c) - f(a) \right| = \sum_{k=0}^{n} \left| f(x_{k+1}) - f(y_k) \right| + \sum_{k=0}^{n} \left| f(y_k) - f(x_k) \right|$$

$$= \epsilon + \eta (c-a).$$

Since ϵ and η are arbitrary, we have

$$\left| f(c) - f(a) \right| = 0.$$

$$f(c) - f(a) = 0.$$

ie)

$$\boxed{f(c) = f(a)}$$

$\therefore f$ is a constant function.

Hence the Proof

—x—

4/5/2017

Unit - 4

General measure and Integration

σ- Algebra :-

Let X be a non - empty set. Let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is σ- Algebra in X. If it satisfies the following axioms.

i) $X \in \mathcal{B}$

ii) $A^c \in \mathcal{B}$ whenever $A \in \mathcal{B}$

iii) If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of members of $\mathcal{B}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}.$$

Note :-

Any $A \in \mathcal{B}$ is called a measurable set.

Measurable Space :-

Let X be a non - empty set. If $\exists$ a σ- Algebra $\mathcal{B}$ defined on X. Then X is called the measurable space if the members of $\mathcal{B}$ are called measurable sets in X.

Note :-

Measurability of a set is depending on X a choice of σ- Algebra defined on X.

Measure on a Measurable space :-

Let (X, β) be a Measurable space. Let μ be a non-negative set function defined on β. then μ is said to be measure on (X, β) if it satisfies the following axioms.

i) $\boxed{\mu\phi = 0}$

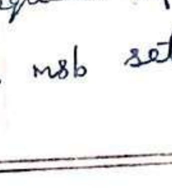
ii) $\boxed{\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu E_i}$ for any sequence E_i of disjoint msb sets.

Measure Space :-

Let (X, β) be a measurable space. Then If a measure μ is defined on β. Then X is said to be measure space.

Theorem 1 :-

If $(A, B) \in \beta$ and $A \subset B$



cycle

then $\mu A \leq \mu B$.

J.Q

Proof:-

Since $B = A \cup (B-A)$ is a disjoint union

Since μ is a measure defined on β

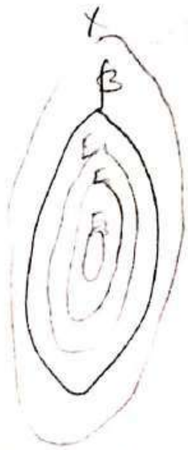
we have,

$$\mu B = \mu A + \mu(B-A)$$

$$\boxed{\therefore \ \mu B \geq \mu A}$$

Hence the Proof.

Theorem 2 :-



If $E_i \in \beta$, $\mu E_i < \infty$

and $E_i \supset E_{i+1}$ $\forall i$ $(E_3 \subset E_2 \subset E_1)$

Then Prove that, $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu E_n$

Proof:-

10 m

First, Prove the theorem (8) in unit-1

By replacing m by μ (for given $E_{i+1} \subset E_i$).

Theorem 3 :-

If $E_i \in \beta$. Then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu E_i$

Proof:-

Let $G_n = E_n - \bigcup_{i=1}^{n-1} E_i$

(or)

$$E_n = G_n + \bigcup_{i=1}^{n-1} E_i$$

Then, $G_n \subset E_n$ and the sets G_n are disjoint.

$$\Rightarrow \quad E_n \supset G_n$$

Hence, $\boxed{\mu E_n \geq \mu G_n} \to (1)$.

also, $\cup E_i = \cup G_i$

$$\left(\because G_n = E_n - \bigcup_{i=1}^{n-1} E_i\right)$$

$$\Rightarrow \mu(\cup E_i) = \mu(\cup G_i) \to (*).$$

Since G_n's are disjoint.

$$\mu\left(\bigcup_{i=1}^{\infty} G_i\right) = \sum_{i=1}^{\infty} \mu G_i \leq \sum_{i=1}^{\infty} \mu E_i$$

$$\boxed{\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu E_i} \text{ by } (1).$$

Hence the Proof.

Theorem 2 :- $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$

Proof:-

A

I.Q

SM

Let $E = \bigcap_{i=1}^{\infty} E_i$

$$E_1 = E \cup \bigcup_{i=1}^{\infty} (E_i \sim E_{i+1}) \quad (*)$$

and R.H.S is a disjoint union in $\mathcal{B}$.

$$\Rightarrow \mu E_1 = \mu E + \sum_{i=1}^{\infty} \mu(E_i \sim E_{i+1}) \to (1)$$

Now, $E_1 = E_2 \cup (E_1 \sim E_2)$

$E_i = E_{i+1} \cup (E_i \sim E_{i+1})$ is a disjoint union

$$\Rightarrow \therefore \mu(E_i) = \mu(E_{i+1}) + \mu(E_i \sim E_{i+1})$$

$$\Rightarrow \mu(E_i \sim E_{i+1}) = \mu(E_i) - \mu(E_{i+1})$$

$$(\mu E_i - \mu E_{i+1}) \to (*)$$

Since,

$$① \Rightarrow \mu E_1 = \mu E + \sum_{i=1}^{\infty} \mu(E_i \sim E_{i+1})$$

$$= \mu E + \sum_{i=1}^{\infty} (\mu E_i - \mu E_{i+1}) \quad / \text{ from } k$$

$$\boxed{\mu E_1 = \mu E + \lim_{n\to\infty} \sum_{i=1}^{n-1} (\mu E_i - \mu E_{i+1})}$$

Illy

$$\mu E_1 = \mu E + \mu E_1 + \lim_{n\to\infty} \mu E_n$$

$$\boxed{\therefore \mu E = \lim_{n\to\infty} \mu E_n}$$

$$\Rightarrow \boxed{\therefore \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n} \quad / \text{ Since } E = \bigcap_{i=1}^{\infty} E_i$$

Hence the Proof.

Finite Measure :-

Let (X, β, μ) be a measurable space, then μ is a Finite Measure if μX is finite. In this case X is called a Finite Measure space.

σ- finite :-

Let $(X, \mathcal{B}, \mu)$ be a measurable space, then μ is a called a σ- finite if there exists a sequence $\{x_n\}$ of measurable set in $\mathcal{B}$. Such that

$$\boxed{X = \bigcup_{n=1}^{\infty} x_n} \quad \text{and} \quad \boxed{\mu(x_n) < \infty} \quad \forall n$$

In this case, X is called a σ- finite measure space.

we can also assume that x_i's are disjoint.

Semi - finite Measure :-

Let $(X, \mathcal{B}, \mu)$ be a measurable. Space, then μ is said to be semi- finite measure, if each measurable set of infinite Measurable contains measurable sets with arbitrary large measurable.

ie) $A \in \mathcal{B}$ with $\mu A = \infty$ $\forall$ large N

we can find $B \subset A$

such that $\mu B < \infty$ and $\mu B > N$

Complete Measure :-

A measurable space $(X, \mathcal{B}, \mu)$ is said to be complete, if it contains all subsets of measure zero.

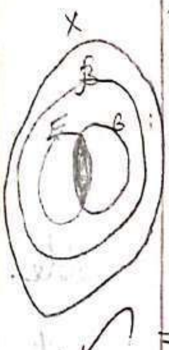
ie) If E belongs to $\mathcal{B}$ and $\mu E = 0$ Then every subset of E also belongs to $\mathcal{B}$

ie) $A \in \mathcal{B}$ with $\mu A = 0$

and c contained in A.

$$\Rightarrow c \in \mathcal{B}$$

Logically Measurable space :-



Let $(X, \mathcal{B}, \mu)$ be a measurable space, a set $E \subset X$ is said to be logically measurable. If $E \cap B \in \mathcal{B}$ $\forall B \in \mathcal{B}$ with $\mu B < \infty$

Countabily Additive :-

By a measure μ on a measurable space $(X, \mathcal{B})$, we mean an extended real valued non-negative set function

$\mu : \mathcal{B} \to [0, \infty]$ for which $\mu\phi = 0$

and which is Countabily Additive in the sense that for any countably disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\boxed{\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu E_k}$$

...urable function :-

Theorem 4 :-

Let (X, β) be a measurable space and f is an extended Real valued function defined on X. Then the following statements are equivalent.

i) for each real number c, the set $\{x \in X / f(x) \le c\}$ is measurable.

ii) for each real number c, the set $\{x \in X / f(x) < c\}$ is measurable.

iii) for each real number c, the set $\{x \in X / f(x) \ge c\}$ is measurable.

iv) for each real number c, the set $\{x \in X / f(x) > c\}$ is measurable.

v) each of these properties implies that for each extended real number c, the set $\{x \in X / f(x) = c\}$ is measurable.

Proof:- Unit - 1 , By theorem (9).

Theorem 5 :-

If c is a Constant and the function f and g are measurable. Then so all the functions $f+c$, fc , fg , $f+g$. more over if sequence $\{f_n\}$ is a sequence of measurable function then $\sup f_n$, $\inf f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ are all measurable.

Proof:-

Unit - 1 , By Theorem (10) & (11)

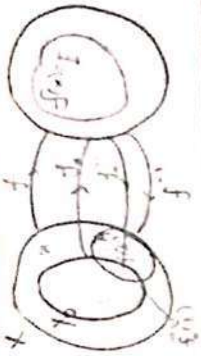
Theorem 6 :-

If μ is a complete measure and f is a measurable function and $f = g$ is almost everywhere. Then g is measurable.

Proof :- Unit - 1, theorem (12)

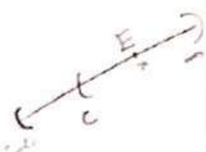
Theorem 7 :-

Let $(X, \mathcal{B}, \mu)$ be a complete measurable space. X_0 be a measurable subset of X in which $\mu(X - X_0) = 0$, then an extended real valued function f on X is measurable $\Leftrightarrow$ Its restriction to X_0 is measurable.



Proof :- Define 'f_0' to be the restriction of 'f' to X_0

Let c be a real number and $E = (c, \infty)$



If f is measurable, then $f^{-1}(E)$ is also measurable.

Hence, So $\underbrace{f^{-1}(E)}_{\text{measurable}} \cap \underbrace{X_0}_{\text{measurable subset of } X} = f_0^{-1}(E)$ is measurable

$\therefore$ f_0 is measurable.

Conversely,

We assume f_0 is measurable

Then $f^{-1}(E) = \underbrace{f_0^{-1}(E)}_{\text{measurable}} \cup \underbrace{A}_{\text{measurable}}$ (where A is a subset of $X - X_0$)

Since $(X, \mathcal{B}, \mu)$ is complete

A is measurable.

hence $f^{-1}(E)$ is measurable.

$\therefore$ f is measurable.

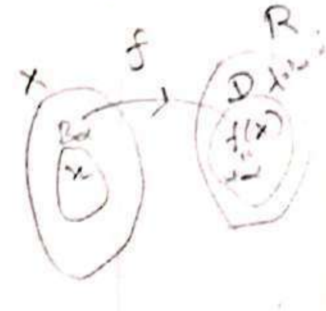
Hence the Proof

Lemma :-

Suppose that, to each α in a dense set $\underline{D}$ of real number, there is assigned a set $B_\alpha \in \mathcal{B}$. Such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a Unique measurable extended real valued function f on X such that

$$f \leq \alpha \quad \text{on } B_\alpha$$

and $f \geq \alpha$ on $X - B_\alpha$



Proof:-

Define a function $f: X \to R$ by $f(x) =$

$$f(x) = \inf \{ \alpha \in D : x \in B_\alpha \} \text{ for every } x \in X.$$

ie) $x \in B_\alpha \Rightarrow f(x) \leq \alpha \rightarrow (*)$

$x \notin B_\alpha \Rightarrow f(x) \geq \alpha.$

We have to show that f is measurable

we take $\lambda \in R$ and choose a sequence $\{\alpha_n\}$ from D with $\underline{\alpha_n < \lambda}$ and $\boxed{\lambda = \lim \alpha_n}$

$f(x) < \alpha < \lambda$

Claim :- $\{x : f(x) < \lambda\} = \bigcup_{n=1}^{\infty} B_{\alpha_n}$

First, Let $x \in \{x : f(x) < \lambda\}$

$\Rightarrow \exists \, \alpha_n$ Such that $\underline{f(x) \leq \alpha_n}$ for some n

by $(*)$

$\Rightarrow \underline{x \in B_{\alpha_n}}$

$\subset \bigcup_{n=1}^{\infty} B_{\alpha_n}$

$$\Rightarrow \{x : f(x) < \lambda\} \subset \bigcup_{n=1}^{\infty} B_{\alpha_n}$$

Next, take

$$x \in \bigcup_{n=1}^{\infty} B_{\alpha_n}$$

$$\Rightarrow x \in B_{\alpha_n}$$

$$\Rightarrow f(x) \leq \alpha_n$$

$$\Rightarrow \bigcup_{n=1}^{\infty} B_{\alpha_n} \subset \{x : f(x) < \lambda\} \rightarrow (2).$$

From (1) & (2)

$$\boxed{\bigcup_{n=1}^{\infty} B_{\alpha_n} = \{x : f(x) < \lambda\}}$$

But B_{α_n} is measurable set.

$\therefore \bigcup B_{\alpha_n}$ is a measurable set.

$\Rightarrow \{x : f(x) < \lambda\}$ is a measurable set.

$\Rightarrow f$ is a measurable function

Hence, $f(x) \leq \alpha$ on B_α.

and $f(x) \geq \alpha$ on $X - B_\alpha$.

Hence the Proof.

Integration :-

If f be a non-negative extended real valued measurable function on the measurable space $(X, \mathcal{B}, \mu)$. then $\int f \, d\mu$ is the supremum of the integral $\int \phi \, d\mu$ and ϕ ranges overall simple function with $0 \leq \phi \leq f$

Fatou's Lemma :-

U.Q Statement :- Let $(X, \mathcal{B}, \mu)$ be a measurable space and $\{f_n\}$ be a sequence of non-negative measurable function on X. for which $\{f_n\}$ converges to f pointwise almost everywhere on X. Assume f is measurable, then $\int_X f \, d\mu \leq \underline{\lim} \int_X f_n \, d\mu$

5M [or].

Let $\{f_n\}$ be a sequence of non-negative measurable function that converge almost everywhere on E to a function f. then $\int_E f \leq \underline{\lim} \int_E f_n$

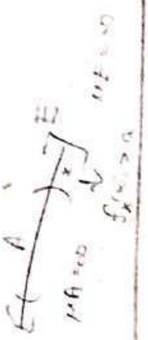
Proof :- Since $\int$ over a set of Measure zero is ~~zero~~ (Measurable)

Assume that $f_n(x) \rightarrow f(x)$ for each $x \in E$

From the definition of $\int f$

... to show that

If ϕ is any non-negative
function with $\phi \le f$ then $\int_E \phi \le$ ...



Case 1 :-

If $\int_E \phi = \infty$.

then there is a measurable set $A \subset E$

with $\mu A = \infty$

such that $\phi > a > 0$ on A .

Set $\boxed{A_n = \{x \in E : f_k(x) > a \;\; \forall\, k \ge n\}}$

Then $\langle A_n \rangle$ is an increasing sequence of measurable

whose union contains A .

$\lim A_n = A$

$\lim \mu A_n = \mu A$

Since $\phi \le \lim f_n$

$\therefore \lim \mu A_n = \infty$

$$\int_E f \, d\mu \ge \int_{A_n} f \, d\mu \ge \int_{A_n} a \, d\mu$$

$f(x) > a$

$\int_{A_n} d\mu = \mu A_n$

Since $\int_E f_n \ge a \, \mu A_n$

we have $\boxed{\lim \int_E f_n = \infty = \int_E \phi}$ $\Rightarrow$ $\boxed{\lim \int_E f_n = \int_E \phi}$

H.P.

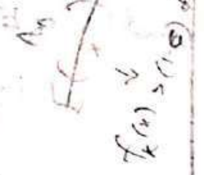
Case 2 :-

If $\int_E \phi < \infty$ then the set $A = \{x \in E : \phi(x) > 0\}$

is a measurable set of finite measure.

Let M be a maximum of ϕ

Let ϵ be a given +ve number and



Set $\boxed{A_n = \{x \in E : f_k(x) > (1-\epsilon)\,\phi(x) \;\; \forall\, k \ge n\}}$

then $\langle A_n \rangle$ is an increasing sequence of sets

whose union contains A.

$\therefore \langle A \setminus A_n \rangle$ is a decreasing sequence of

sets whose intersection is empty.

11/06/16