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CORE PHYSICS II - MECHANICS - 21UPH02

UNIT I

PROJECTILE: Definition of Range, Time of flight and Angle of projection –Range up and down an inclined plane maximum range – two directions of projections for a given velocity and range.

IMPACT: Laws of impact – coefficient of restitution – the impact of a smooth sphere on a fixed smooth plane – Direct impact between two smooth spheres – Loss of kinetic energy indirect impact – velocity change in oblique impact between two smooth spheres-Loss of kinetic energy in an oblique impact.

UNIT II

CENTRE OF GRAVITY: Definition - Centre of gravity of a solid cone, Solid hemisphere, hollow hemisphere and a tetrahedron – Centre of Buoyancy.

FRICTION: Introduction – Static, Dynamic, Rolling and Limiting Friction - Laws of friction – the angle of friction – resultant reaction and cone of fiction – equilibrium of a body on an inclined plane under the action of a force.

UNIT III

SIMPLE HARMONIC MOTION: Composition of two SHM's of same period along a straight line and at the right angles to each other –Lissajou's figures – Experimental methods for obtaining Lissajou's figures – Applications.

RIGID BODY DYNAMICS: Compound pendulum - Centers of oscillation and suspension - determination of g and k - Bifilar pendulum - Parallel and non-parallel threads - Centre of mass - Conservation of linear and angular momentum - Variable mass Rocket propulsion.

UNIT IV

HYDROSTATICS: Concurrent forces - Parallel forces –couple - static equilibrium of rigid body - the centre of pressure of rectangular and triangular lamina - Metacentric height and its determination.

HYDRODYNAMICS: Equation of continuity of flow – Euler's equation of unidirectional flow - Torricelli's theorem - Bernoulli's theorem and its applications – Venturimeter.

UNIT V

LAGRANGIAN DYNAMICS: Mechanics of system of particles – Conservation of energy - Constraints of motion Generalized coordinates and the transformation equation – simple illustration for the transformation equation – Configuration space - the principle of virtual work – D'Alembert's principle - Lagrange's formulation for conservation theorems – Hamiltonian-Hamiltons Equation.

BOOKS FOR STUDY

1. R. Murugeshan, Mechanics and Mathematical Physics, S.Chand& Company

Ltd, 2008, 3rd Edn.

2. M. Narayanamurthi and N. Nagarathinam Dynamics, The National Publishing

Company 2008, 8rd Edn.

Unit –I

PROJECTILE

Range up and down an inclined plane maximum range

Projectile range on an inclined plane:

A Particle is projected with a velocity 'u' at an angle α to the horizontal from a point O on an inclined plane, inclined at an angle β to the horizontal. The direction of projection lies in the vertical plane through OA the line of greatest slope. Of the plane. Let the particle strike inclined plane at A Then OA (=R) is the range of the inclined plane Fig (1.1)

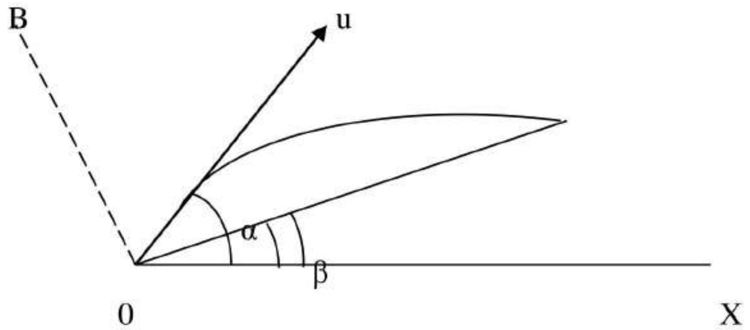


Fig 1-1

Let OX and OA be respectively the horizontal and inclined plane through the point of projection O. OB is a line perpendicular to OA.

Component of initial velocity u along OA = u cos (α- β)

Component of initial Velocity u along OB = u sin (α- β)

The Projectile moves with a vertical retardation g.

Acceleration along OA = - g sin β

Acceleration along OB = - g cos β

Now, Let T be the time taken the particle to go from O to A. when the particle reaches A after time T, the distance moved perpendicular to the plane is zero.

Hence,

$0 = u \sin (\alpha - \beta). T - 1/2\ g \cos b.T^2$ (since $S = ut + 1/2\ at^2$)

$$T = \frac{2u \sin(\alpha-\beta)}{g \cos \beta}$$

When the particle strikes A after time, T, the distance OA (=R) moved is the range on the inclined plane.

Therefore, $R = u \cos(\alpha-\beta).T - 1/2\ g \sin \beta.\ T^2$

$$= u \cos(\alpha-\beta).\frac{2u \sin(\alpha-\beta)}{g \cos \beta} - 1/2\ g - \sin \beta\ \frac{4u^2 \sin^2(\alpha-\beta)}{g^2 \cos^2 \beta}$$

$$= \frac{2u^2 \sin(\alpha-\beta)}{g \cos \beta} [\cos(\alpha-\beta) \cos \beta - \sin(\alpha-\beta) \sin \beta\ ]$$

$$= \frac{2u \sin(\alpha-\beta) \cos \alpha}{g \cos \beta}$$

Range and Time of flight down an inclined plane or Range down an inclined Plane

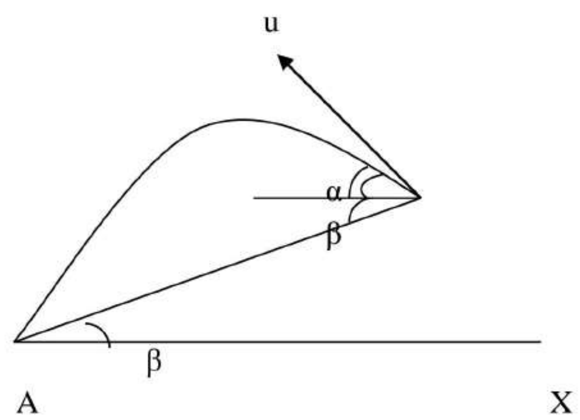


Fig 1-2

The particle is projected own the plane from O at an elevation α Fig. 1-2. Initial velocities along and perpendicular to OA are u cos (α+β) and u sin (α+β). Acceleration along and perpendicular to OA are g sin β and – g cos β. When the particle reaches A after time T_1, the distance moved perpendicular to the inclined plane is zero. Therefore,

$$0 = u \sin(\alpha-\beta).\ T_1 - 1/2\ .g \cos \beta.\ T_1^2 \text{ or } T_1 = \frac{2u \sin(\alpha+\beta)}{g \cos \beta}$$

$$\text{Range} = OA = R_1 = u \cos(\alpha+\beta).\ T_1 + \tfrac{1}{2}\ g \sin \beta\ T_1^2$$

$$= \frac{2u^2 \sin(\alpha+\beta)}{g \cos^2 \beta}$$

Note: 1. Result down the plane can be obtained by putting $-\beta$ for β in the results of the previous article.

Note: 2. In some problems, the elevation relative to the inclined plane may be given. In such cases we must calculate the elevation relative to the horizontal.

Problem: A particle is projected with a velocity of 32 ms^{-1} at an angle of 60^o to the horizontal. Find the range on a plane inclined at 30^o to the horizontal when projected (i) up the plane and (ii) down the plane.

(i) when the particle is projected up the plane, range on inclined plane is given by

$$R = \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

Here , $u = 32\ ms^{-1}$; $\alpha = 60^o$; $\beta = 30^o$; $g = 9.8\ ms^{-1}$

$$\text{Therefore , } R = \frac{2 \times 32 \times 32 \times \sin 30^o \times \cos 60^o}{9.8 \cos^2 30^o}$$

$$= 69.66\text{m.}$$

(ii) When the particle is projected down the plane at an angle a with the horizontal, the range down the plane is given by

$$R = \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta} = 139.3\text{m}$$

$$= = \frac{2 \times 32 \times 32 \times \sin 90^o \times \cos 60^o}{9.8 \cos^2 30^o}$$

Range up an inclined plane or Maximum Range :

To find the direction of projection for the maximum range on the inclined plane.

$$R = \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha - \beta) - \sin\beta]$$

The value or R depends on α, for given values of u and β. Hence R is maximum when $\sin(2\alpha - \beta) = 1$; i.e., when $2\alpha - \beta = 90^o$ or $\alpha = (45 + \alpha - \beta/2)^o$:

$\therefore$ The maximum range we on the inclined plane

$$R_m = \frac{u^2}{g\cos^2\beta}[1 - \sin\beta]$$

$$R_m = \frac{u^2[1-\sin\beta]}{g[1-\sin^2\beta]}$$

$$= \frac{u^2[1-\sin\beta]}{g[1+\sin\beta][1-\sin\beta]}$$

$$= \frac{u^2}{g[1+\sin\beta]}$$

Note : $\alpha = 45^o + \beta/2$. then $\alpha - \beta = 45 - \beta/2$ and $90^o - \alpha = 45^o - \beta/2$

This shows that the direction giving the maximum range bisects the angle between the vertical and the inclined plane.

Now, $$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta]$$

The range R and the values of u and β are given.

Hence sin (2α-β) is constant. There are two values (2α - β), each less than 180^o which satisfy the above equation. Let the corresponding values of α be α_1 and α_2. Then

$2\alpha_1 - \beta = 180 - (2\alpha_2 - \beta)$ or $\alpha_1 - \beta/2 = 90 - (\alpha_2 - \beta/2)$

$\alpha_1 - (45 + \beta/2) = (45 + \beta/2) - \alpha_2$

(45 + β/2) is the angle of projection giving the maximum range. Therefore it follows that the direction giving maximum range bisects the angle between the two angles of projection that can given particular range (Fig 1-3)

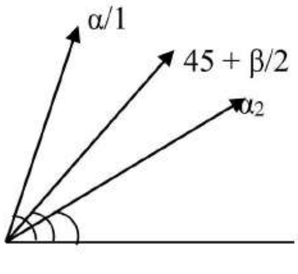


Fig 1-3

Problem: Prove that for a given velocity of projection the ratio between the maximum ranges up and down an inclined plane inclined at an angle

B to the horizon is $\frac{1-\sin\beta}{1+\sin\beta}$

We have already up the in proved that,

Maximum range up the inclined plane = $R_m = \frac{u^2}{g[1+\sin\beta]}$

Range down inclined plane = $R_1 = \frac{2u^2 \sin(\alpha+\beta)\cos\alpha}{g \cos^2\beta}$

$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta) + \sin\beta]$

This range is maximum when $\sin(2\alpha+\beta) = 1$.

Maximum Rangbe down inclined plane = $R_{m1} = \frac{u^2[1+\sin\beta]}{g\cos^2\beta}$

$= \frac{u^2}{g[1-\sin\beta]}$

Hence $\frac{R_m}{R_{m1}} = \frac{u^2}{g[1+\sin\beta]} \times \frac{g[1-\sin\beta]}{u^2}$

$= \frac{1-\sin\beta}{1+\sin\beta}$

Impulse:

The impulse 1 of a constant force F acting for a time t is defined as F x t.

$1 = F \times t$

By Newton's second law, $F = ma$.

If u and v are the initial and final velocities of the paticle,

$a = (v - u)/t$

$\therefore \; I = Ft \text{ mat} = m\left(\frac{v-u}{t}\right)t = m(v-u)$

Thus the impulse of a force is equal to the change in momentum produced.

Impulsive Force: Definition. An impulsive force is an infinitely great force acting for a very short interval of time, such that their product is finite.

The force and the time cannot be measured because one is too great and the better is too small. Nevertheless, their product, which is definite, is capable of measurement. This we have seen, id the impulse of the impulsive force and is equal to the change in momentum produced. Hence an impulsive force is always measured by the change in momentum produced. In practice, the conditions of an impulsive force are never realized. Some approximate examples of impulsive force are: (1) the blow of a hammer on a pie and (2) the force exerted by the bat on a cricket ball.

Laws of Impact:

1. ***Newton's law of impact – coefficient of restitution.***

   When two bodies imping directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact and is in the opposite direction. This constant ratio depends only on the material of the bodies and not on their masses of velocities. It is called the coefficient of restitution and is denoted by the letter e. If u_1, u_2 be velocities of two bodies before the impact and v_1, v_2 the velocities after impact.

   $$\frac{v_1 - v_2}{u_1 - u_2} = -e \text{ or } v_1 - v_2 = -e\,(u_1 - u_2)$$

   Where $(u_1 - u_2)$ and $(v_1 - v_2)$ are their relative velocities, before after the impact, e lies between 0 and 1. If $e = 0$, the bodies are called perfectly Plastic bodies. If $e = 1$, the bodies are called perfectly elastic bodies. For two glass balls, $e = 0.94$; For two lead balls, $e = 0.2$.

Definition of coefficient of restitution

The ratio, with a negative sign, of the relative velocity of two bodies after impact to their relative velocity before impact is called the coefficient of restitution.

2. ***Motion of two smooth bodies perpendicular to the line of impact.***

When two smooth bodies impinge, there is no tangential action between them. Hence there is no change of velocity for either body along the tangent. In other words, there is no change in the velocity of a body in a direction perpendicular to the normal due to impact.

Direct impact between two spheres:

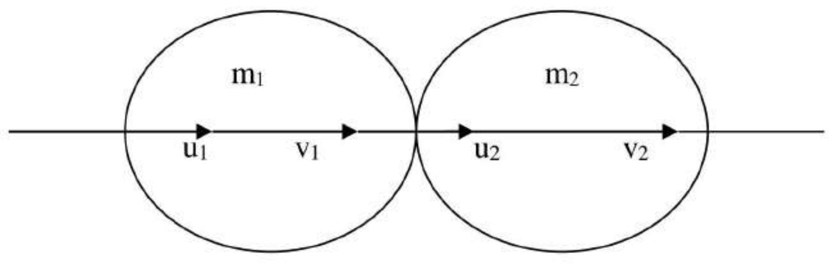


Fig 1-4

A smooth sphere of mass m_1 moving with a velocity u_1 impinges on another smooth mass m_2 moving in the same direction with velocity u_2. If e is the coefficient of restitution between them, find the velocities of the spheres after impact.

Sphere after impact:

Since the spheres are smooth, there is no impulsive force on either along the common tangent. Hence in this direction their velocities after impact are the same as their original velocities i.e., zeroes. Let v_1 and v_2 be the velocities of the two spheres along the common normal after impact

By the principle of conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \text{ \_\_\_\_\_\_\_\_\_\_ (1)}$$

By Newton's experimental law.

$$v_1 - v_2 = e(u_1 - u_2) \text{ \_\_\_\_\_\_\_\_\_\_ (2)}$$

Multiplying (2) by m_2 and adding to (1)

$$v_1 (m_1 + m_2) = m_2 u_2 (1+e) + u_1(m_1 - em_2)$$

$$\therefore\ v_1 = \frac{m_2 u_2 (1+e) + u_1 (m_1 - em_2)}{m_1 + m_2} \text{ \_\_\_\_\_\_\_\_\_\_ (3)}$$

Multiplying (2) by m_2 and subtracting to (1)

$$V_2 (m_1 + m_2) = m_1 u_1 (1+e) + u_2 (m_1 - em_1)$$

$$\therefore\ v_2 = \frac{m_1 u_1 (1+e) + u_2 (m_1 - em_1)}{m_1 + m_2} \text{ \_\_\_\_\_\_\_\_\_\_ (4)}$$

Equations (3) and (4) give the velocities of the two spheres after impact.

Cor.1. This impulse of the blow on the sphere of mass m_1 = change of momentum produced in it = $m_1(v_1 - u_1) = \frac{m_1 m_2 (1+e)(u_2 - u_1)}{m_1 + m_2}$ this is equal and opposite to the impulse on the sphere of mass m_2.

Cor.2. If $e = 1$ and $m_1 = m_2$ then, $v_1 = u_2$ and $v_2 = u_1$. Thus, if two equal perfectly, elastic spheres impinge directly, they interchange their velocities.

Loss of K.E. due to direct impact of two smooth spheres

Let m_1,m_2 be the masses, u_1 and u_2 v_1 and v_2 their velocities before and after and e the coefficient of restitution. Then, by the principle of conservation of linear momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \longrightarrow (1)$$

By Newton's experimental law,

$$v_1 - v_2 = -e (u_1 - u_2) \longrightarrow (2)$$

Square both equations , multiply square of the second by $m_1 m_2$ and add the results. Then,

$$\left.\begin{aligned}(m_1^2 + m_1 m_2) v_1^2 + \\ (m_2^2 + m_1 m_2) v_2^2\end{aligned}\right\} = (m_1u_1 + m_2u_2)^2 + e^2 m_1 m_2 (u_1 - u_2)^2$$

$\therefore$ $m_1(m_1 + m_2) v_1^2 + m_2(m_1 + m_2) v_2^2 =$
$(m_1u_1 + m_2u_2)^2 + m_1 m_2 (u_1 - u_2)^2 + e^2 m_1 m_2 (u_1 - u_2)^2 - m_1 m_2 (u_1 - u_2)^2$

$\therefore (m_1 + m_2)(m_1 v_1^2 + m_2 v_2^2) = (m_1 + m_2) m_1u_1^2 + m_2u_2^2 - m_1 m_2 (u_1 - u_2)^2 (1-e^2)$

$$\therefore \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 - \tfrac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2 (1-e^2)$$

Now, $\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 =$ K.E after impact.

$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 =$ K.E before impact.

$\therefore$ The loss in K.E $= \tfrac{1}{2} \dfrac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2 (1-e^2)$

Note: When e =1, the loss of K.E is zero. In general e<1 so that $(1-e^2)$ is positive. $(u_1 - u_2)^2$ is always positive. Hence, there is always a loss of K.E due to impact. The K.E lost during impact is covered into (i) sound, (ii) heat or (iii) vibration or rotation of the colliding bodies.

When e = 0, the loss in K.E $= \tfrac{1}{2} \dfrac{m_1 m_2 (u_1 - u_2)^2}{m_1 + m_2}$

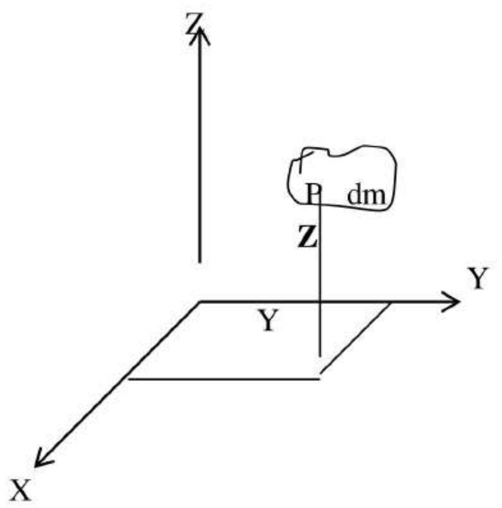
i.e., there is maximum loss of K.E on impact of plastic bodies.

Unit –II

CENTRE OF GRAVITY

Definition: The centre of gravity of a body is the point at which the resultant of the weights of all the particles of the body acts,whatever may be the orientation of the body.The total weight of the body may be supposed to act at its centre of gravity.

Suppose the particles A,B,C,……. of a body have masses $m_1,m_2,m_3,\ldots$ Let their coordinates in a rectangular caetesian coordinate system be$(x_1,y_1,z_1),(x_2,y_2,z_2),\ldots(x_n,y_n,z_n)$.



Then, the coordinates of the centre of gravity G of the body are

$$\bar{x} = \frac{\Sigma\, m_n y_n}{\Sigma\, m_n};$$

$$\bar{y} = \frac{\Sigma\, m_n y_n}{\Sigma\, m_n}; \qquad \bar{z} = \frac{m_n z_n}{\Sigma\, m_n}$$

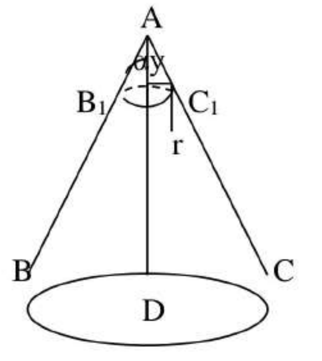
Suppose an element P of the body has a mass dm (Fig. 3.1) and is co ordinate are x,y,z. Then,

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm; \quad \bar{y} = \frac{1}{M}\int y\, dm; \quad \bar{z} = \frac{1}{M}\int z\, dm$$

Here,the integrals extend over all elements of the body,and $M=\int dm$=Total mass of the body.

Centre of Gravity of a rigid solid cone:

Let ABC represent a solid cone of height h and semi-vertical angle α.The cone may be considered to be made up of a large number of circular discs parallel to the base.The centre of gravity ofb each disc lies at its centre.Therefore,the C.G., of the cone should lie along the axis AD of the cone.



Consider a disc B1C1 of thickness dy at a distance y below the vertex A. If r is the radius of the disc, than

$$r = y \tan \alpha$$

Volume of the cone = $\pi r^2 h$ where h=dy

Volume of the disc = Area x thickness = $\pi y^2 \tan^2 \alpha\, dy$

Mass of the disc= $dm = \pi y^2 \rho \tan^2 \alpha\, dy$.

Where ρ = density of the cone.

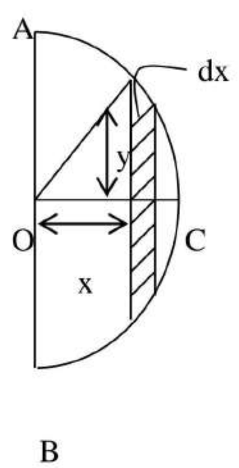
The distance of the C.G., of the cone from the vertex is given by

$$\bar{y} = \frac{\int y\, dm}{\int dm} = \frac{\int_0^h \pi y^3 \rho \tan^2 \alpha\, dy}{\int_0^h \pi y^2 \rho \tan^2 \alpha\, dy} = \frac{\int_0^h y^3\, dy}{\int_0^h y^2\, dy} = 3/4\ h.$$

Therefore, the C.G., of the cone is along its at a distance of 3/4h from the vertex.

Centre of Gravity of a solid hemisphere:

L et ABC represent a solid hemisphere of radius r, centre 0 and density ρ.Consider an elementary slice of the hemisphere with radius y and thickness dx,at a distance x from 0.



Volume of the slice = $\pi y^2 dx = \pi (r^2-x^2)\, dx$.

Mass of the slice = $dm = \rho\pi (r^2- x^2)\, dx$.

The distance of the C.G of the hemisphere from O is given by

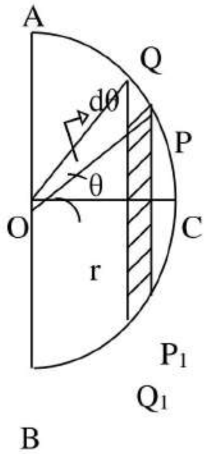
$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^r x\, \rho\pi(r^2-x^2)\, dx}{\int_0^r \rho\pi(r^2-x^2)\, dx} = \frac{\int_0^r (r^2 x - x^3)dx}{\int_0^r (r^2 - x^2)dx}$$

$\bar{x} = 3/8$ r.

Hence, the C.G., of the solid hemisphere is on its axis at a distance 3/8 r. from the centre.

Centre of gravity of a hollow hemisphere

Let ACB be a section of a hemisphere of radius r, centre O and surface density ρ (Fig. 3.5). Imagine the surface of the hemisphere to be divided into slices like $PQQ_1 P_1$ by planes parallel to AB. If ∠POC = θ and ∠ POQ = dθ, then



Radius of the ring = r sin θ

Width of the ring = r dθ

Area of the ring = $2\pi r \sin\theta . r d\theta$

Therefore, mass of the ring = $dm = 2\pi r^2 \rho \sin\theta\, d\theta$.

The C.G., of this ring is at the centre of the ring at a distance r cos θ from O.

The distance of the C.G., of the hollow hemisphere from O is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^{\pi/2} (r \cos\theta) 2\pi^2 \rho \sin\theta\, d\theta}{\int_0^{\pi/2} 2\pi r^2 \rho \sin\theta . d\theta} = \frac{\int_0^{\pi/2} \sin\theta \cos\theta\, d\theta}{\int_0^{\pi/2} \sin\theta . d\theta}$$

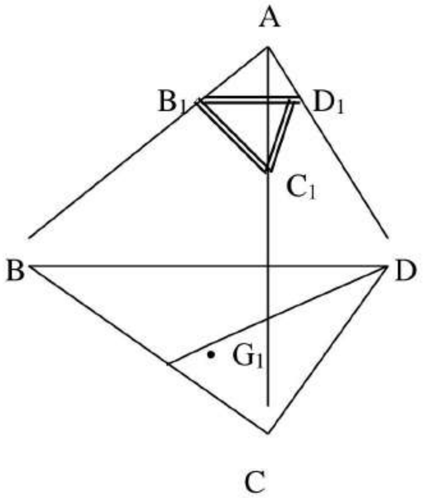
$\bar{x} = r/2$

The C.G., of ahollow hemisphere is on its axis at a distance r/2 from the centre. i.e., the ravity is at the mid point of the radius OC.

Centre of gravity of a solid tetrahedron

Let ABCD be the tetrahedron and G_1 the centre of gravity of the base BCD. Let h be the altitude of the tetrahedron and ρ its density. Suppose the tetrahedron is divided into thin slices by planes parallel to the base BCD. Consider one such slice $B_1C_1D_1$ of thickness dx at a depth x below A. let s be the area of the triangular base BCD. Then we have,

$$\frac{B_1C_1}{BC} = \frac{x}{h}$$



If a_1 and a are the altitudes of triangles $B_1C_1D_1$ and BCD respectively.

$$\frac{a_1}{a} = \frac{x}{h}$$

Now, area of $\triangle B_1C_1D_1 = \frac{1}{2} B_1C_1 \times a_1$

Area of $\triangle$ BCD $= \frac{1}{2}$ BC x a = S

Hence, $$\frac{\text{Area of } \triangle B_1C_1D_1}{S} = \frac{B_1C_1}{BC} \times \frac{a_1}{a} = \frac{x^2}{h^2}$$

Therefore, Area of $\triangle B_1C_1D_1 = Sx^2/h^2$

Volume of the slice $B_1C_1D_1 = Sx^2dx/h^2$

Mass of the slice = dm = $\rho\, Sx^2dx/h^2$

The distance of the centre of gravity of the tetrahedron from A is given by

$$\overline{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^h x\rho\, Sx^2dx/h^2}{\int_0^h \rho\, Sx^2dx/h^2} = \frac{\int_0^h x^3dx}{\int_0^h x^2dx} = \frac{3}{4} h$$

Hence, the C.G., of a uniform tetrahedron lies at a point G on the line AH such that AG : GH = 3:1.

Friction:

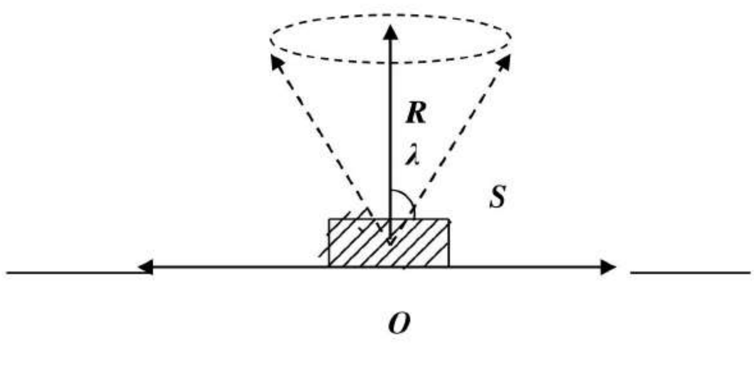
Laws of static Friction

1. The direction of the frictional force is always opposite to the direction in which one body tends to slide over another.
2. The magnitude of the force of friction when there is equilibrium between two bodies is just sufficient to prevent the motion of one body over the other. The frictional force attains a maximum value when one body is just on the point of sliding over the other. The maximum value of the force of friction is called limiting friction.
3. The magnitude of the force of limiting friction bears a constant ratio to the normal between the two bodies. This ratio is called coefficient of friction and is denoted by μ. If F is the limiting friction and R the normal reaction between the two bodies, then $\mu = F/R$. μ depends only on the nature of surface in contact.
4. The limiting friction is independent of the extent and shape of the surfaces in contact, provided the normal reaction is unaltered.
5. When a body is in motion, the direction of friction is still opposite to the direction of motion of the body and is independent of the velocity. But the ratio of the force of friction to the normal reaction is slightly less than that when the body is just on the point of motion.

Angle of Friction:

Let F be the force of limiting friction and R, the normal reaction. Let S be the resultant of these two forces. Then the angle which this resultant force makes with the normal reaction is called the angle of friction. It is denoted by λ.

Then, $\tan \lambda = \frac{F}{R} = \frac{\mu R}{R} = \mu$ (Since $\mu = F/R$)



Cone of Friction:

Consider a cone with the point of contact of two bodies as the vertex, the normal reaction as axis and semi- vertical angle λ. Then the resultant reaction (S) may lie any where within or on the surface of the cone. This imaginary cone is called the cone of friction.

Equilibrium of a body on a Rough inclined Plane Acted upon by an External Force

Proposition:

A body of weight w is in equilibrium on a rough inclined plane of angle $\alpha > \lambda$ under the action of an external force inclined upwards at an angle θ with the plane. Find the value of P for limiting equilibrium.

Case 1: Let the body be just on the point of sliding down the plane. Let P be the magnitude of the external force, applied at an angle θ with the plane. The forces acting on the body are : (i) the weight of the body (w) acting vertically acting on the body are: (i) the weight of the body(w) acting vertically down, (ii)The normal reaction (R)acting perpendicular to the plane (iii) The force of limiting friction(μR) acting up the plane and (iv) The external force (effort) P making an angle θ with the line of greatest slope of bthe inclined plane.Resolving the forced parallel and perpendicular to the plane.

$$P \cos \theta + \mu R = w \sin \alpha \quad \text{-----------------------------1}$$

$$P \sin \theta + \mu R = w \cos \alpha \quad \text{-----------------------------2}$$

Multiplying(2) by μ and substracting from(1)

$$P(\cos \theta - \mu \sin \theta) = w(\sin \alpha - \mu \cos \alpha).$$

$$P = w\,\frac{(\sin \alpha - \mu \cos \alpha)}{(\cos \theta - \mu \sin \theta)}$$

But $\mu = \tan \lambda$, where λ is the angle of friction.

Hence $P = w\,\frac{(\sin \alpha - \tan \lambda \cos \alpha)}{(\cos \theta - \tan \lambda \sin \theta)} = w\,\frac{(\sin \alpha \cos \lambda - \sin \lambda \cos \alpha)}{(\cos \theta \cos \lambda - \sin \lambda \sin \theta)}$

$$P = \frac{w \sin(\alpha - \lambda)}{\cos(\theta + \lambda)} \quad \text{-----------------------------3}$$

Case 2: Let the body be just on the point of sliding up the plane. Let P_1 be the magnitude of the external force.In this case,the force of limiting friction(μR) acts down the plane.Resolving the forces parallel and perpendicular to the plane,

$$P_1 \cos \theta = w \sin\alpha + \mu R \quad \text{-----------------------------4}$$

$$P_1 \sin \theta + R = w \cos\alpha \quad \text{-----------------------------5}$$

Simplifying, we get, $P_1 = \frac{w \sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$ ----------------------------6

Corollary 1: P_1 is a maximum when $\cos(\theta - \lambda)$ is maximum i.e., when $\cos(\theta - \lambda) = 1$ i.e., when $\theta = \lambda$.Hence force required to move the body up the plane will be least when it is applied in a direction making with inclined plane an angle equal to the angle of friction.

Corollary 2: Let a body be at rest on a rough inclined plane whose inclination to the horizontal $\alpha > \lambda$.Let it be acted upon by an external force applied parallel to the plane.Here $\theta = 0$.From (3) and (6)

$$P = \frac{w \sin(\alpha - \lambda)}{\cos \lambda} \quad \text{--------------------------------7}$$

and

$$P_1 = \frac{w \sin(\alpha + \lambda)}{\cos \lambda} \quad \text{--------------------------------8}$$

Unit –III

SIMPLE HORMONIC MOTION

Introduction:

A simple type of motion which occurs frequently in nature is the motion of a particle with its acceleration always directed towards a fixed point in the path and varying as the distance of the particle from the fixed point. Such a motion is known as simple harmonic motion. The oscillation of a mass suspended by an pendulum, the vertical oscillations of mass suspended by an elastic string for a small oscillations, the oscillations of charge in an ideal LC circuit, the Helmholtz resonator, the oscillations of liquid in U-tube are few examples of simple harmonic motion or harmonic oscillations.

Definition of Simple Harmonic Motion:

If a particle be constrained to move in such a way that its acceleration is always directed towards a fixed point in the line of motion and is directly proportional to the distance of the particle is said to be simple harmonic.

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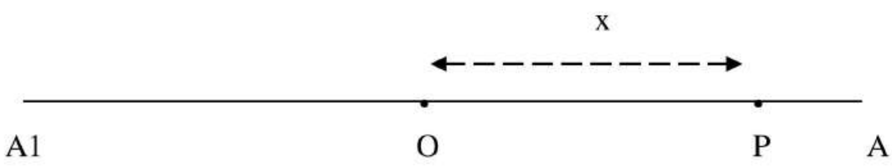


Fig 2.1

Let O be the midpoint of straight lint AOA_1. Let a particle P move along this straight line such that its acceleration at any instant of time t is directed towards O, and is proportional to the distance of the particle from O. Let OP = x. Then the acceleration of P at the instant is d^2x/dt^2 and the displacement is x. When the motion of P is simple harmonic

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Where ω^2 = acceleration per unit displacement which is a constant

$$= \frac{\text{Force constant}}{\text{Mass}} = \frac{k}{m}$$

The negative sign implies that the acceleration is in a direction opposite to the direction along which x increases.

Composition of two simple harmonic motions in a straight line of the same period but different amplitudes and phase:

Let the displacement for the motions of the particle be

$$x_1 = a \sin \omega t \longrightarrow \quad (1)$$
$$x_2 = b \sin (\omega t + \varepsilon) \longrightarrow \quad (2)$$

Where a and b are the amplitude and ε the phase difference.

If x be the resultant displacement

$$x = a \sin \omega t + b \sin (\omega t + \varepsilon)$$
$$= a \sin \omega t + b \sin \omega t \cos \varepsilon + b \cos \omega t \sin \varepsilon$$
$$= (a + b \cos \varepsilon) \sin \omega t + b \sin \varepsilon \cos \omega t$$

$$\text{Let } a + b \cos \varepsilon = A \cos \delta \longrightarrow \quad (3)$$
$$b \sin \varepsilon = A \sin \delta \longrightarrow \quad (4)$$

then $x = A \sin \omega t \cos \delta + A \cos \omega t \sin \delta$

$$= A \sin (\omega t + \delta) \longrightarrow \quad (5)$$

The resultant motion is also simple harmonic of the same period of different amplitude and phase.

Squaring equation (3) and (4) and adding

$$A^2 = a^2 + b^2 + 2ab \cos \varepsilon$$
$$\text{Or } A = \sqrt{}\,(a^2 + b^2 + 2ab) \cos \varepsilon \longrightarrow \quad (6)$$

Dividing equation (4) by (3)

$$\tan \delta = \frac{b \sin \varepsilon}{a + b \cos \varepsilon}$$

Equation (6) and (7) gives the amplitude and phase of the resultant motion.

Corollary 1. If $\varepsilon = 0$ $\quad A = n + b$ and $\delta = 0$
Corollary 2. If $\varepsilon = \pi$ $\quad A = a - b$ and $\delta = 0$
Corollary 3. If $\varepsilon = \frac{1}{2}\pi$ $\quad A = \sqrt{}\,(a^2 + b^2)$ and $\delta \tan^{-1} b/a$

Composition of two simple harmonic motions in a straight line of the same period but different amplitudes and phase along two perpendicular directions:

Taking the two perpendicular directions as X and Y axis, the displacement of the particles may be written as

$$X = a \sin \omega t \quad \text{……..} (1)$$
$$\text{And} \quad Y = b \sin (\omega t + \varepsilon) \quad \text{……..} (2)$$

Where a and b are the amplitudes and ε the phase difference.

From the equation (2)

$$\frac{y}{b} = \sin \omega t \cos \varepsilon + \cos \omega t \sin \varepsilon \qquad ......(3)$$

But from equation (1)

$$\text{Sin } \omega t = x/a$$

Therefore $\cos \omega t = \sqrt{1 - x^2/a^2}$

Substituting the values of sin ωt and cos ωt in equation (3) we have

$$\frac{x}{b} = \frac{x}{a} \cos \varepsilon + \sqrt{1 - \frac{x^2}{a^2}} \sin \varepsilon$$

$$\frac{y}{b} = \frac{x}{a} \cos \varepsilon + \sqrt{1 - \frac{x^2}{a^2}} \sin \varepsilon \qquad .... (4)$$

Squaring both sides of equation (4)

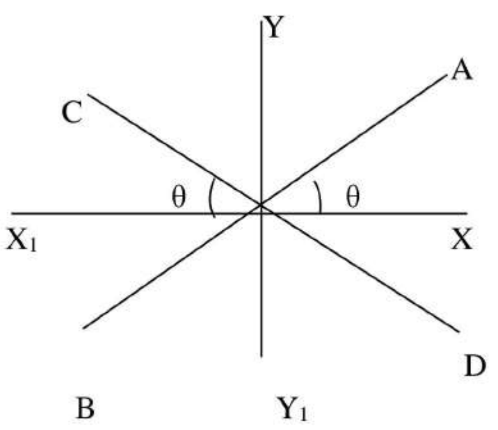
$$\frac{y^2}{b^2} - \frac{2xy}{ab} \cos \varepsilon + \frac{x^2}{a^2} \cos^2 \varepsilon = \sin^2 \varepsilon - \frac{x^2}{a^2} \sin \varepsilon$$

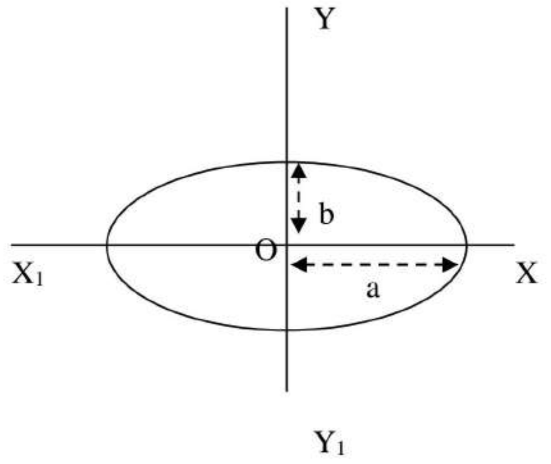
Re – arranging $\frac{y^2}{b^2} - \frac{2xy}{ab} \cos \varepsilon + \frac{x^2}{a^2} = \sin^2 \varepsilon$

This is a general equation for an ellipses whose major and minor axes are inclined x and y co ordinate axes.

Corollary 1. If $\varepsilon = 0$, $\frac{y}{b} = \frac{x}{a}$ or $y = b/a\ (x)$

The resultant motion is along the straight line AB inclined at an angle $\theta = \tan^{-1} b/a$ with the X – axis. The resultant motion is therefore rectilinear. (Fig. 2.4)





Corollary 2. If ε = л

$$y = -\,{}^{b}/_{a}\ x$$

This equation is represented by the straight line CD inclined at an angle $\theta = \tan^{-1} -{}^{b}/_{a}$

The resultant motion is rectilinear with a negative slope

Corollary 3.

If ε = ½ л

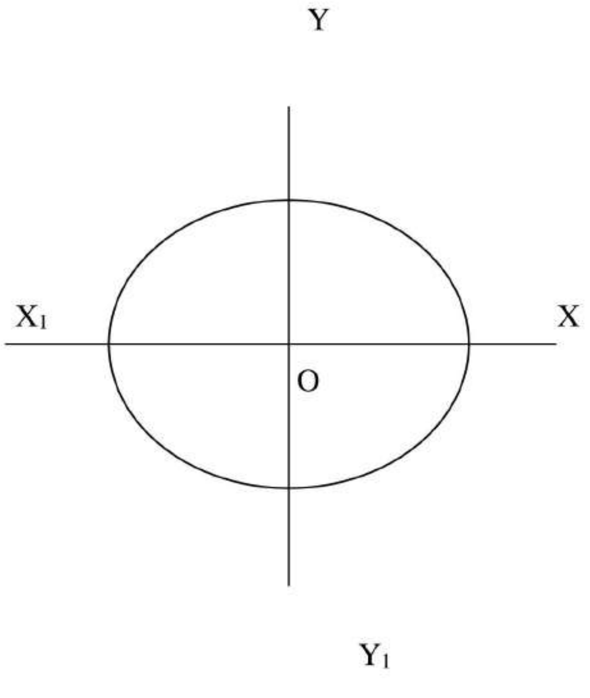
$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

This equation represents an ellipse whose major and minor axes coincide with the X and Y axes. The resultant vibration is elliptical (fig 2.3)

Corollary 4. If ε = ½ л and a = b.

$$x^2 + y^2 = a^2$$

This equation represents a circle and so the resultant motion is circular.



Lissajous Figure:

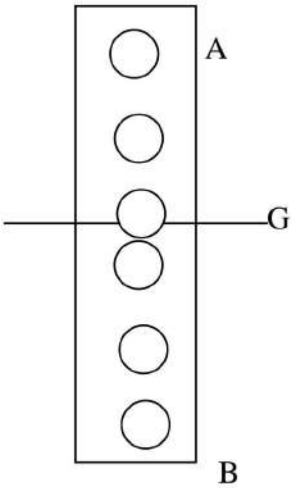
When a particle is influenced simultaneously two SHM at right angles to each other, the resultant motion of a particle takes a curve. These curves are called as Lissajous figures. The shape of the curve depends on the time period and amplitude of the two constituent vibrations.

'g' using compound pendulum :

A compound pendulum consists of a heavy uniform metal bar about a meter long. It has a number of holes drilled of regular intervals on either side of the center of mass G.

The horizontal knife edges are passed through the hole near the end A. The period of oscillation is determined and the distance of the knife edge from the end A is measured. The experiment is repeated and the bar is made to oscillate about the knife edge placed. Successively in the different hole from A to B.In each case the period of oscillation and the distance of position of knife edge from the same and are noted .

A graph is plotted between the period [on Y-axis] and distance from A [on X- axis].



Two curves as shown in figure are obtained . A horizontal line PQRS is drawn cutting both the curves at the point P,Q,R and S. P,Q,R and S are then the four points on the bar collinear with of mass having the same period

PR = QS = L, the length of the equivalent simple pendulum. Therefore, if T be its tme period given by the of anyone of the points P,Q,R,S.

We have $T = 2\pi \sqrt{l/g}$

$T^2 = 4\pi^2 l / g$

$G = 4\pi^2 l / T^2$

Knowing l and T we can able to calculate the value of g at the given place

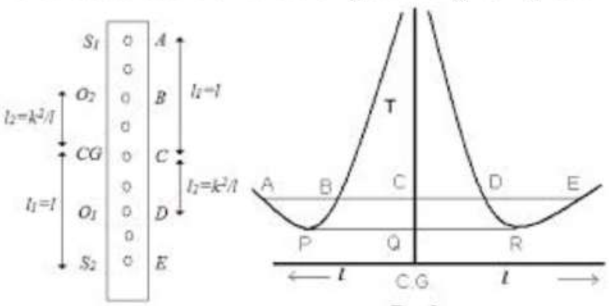


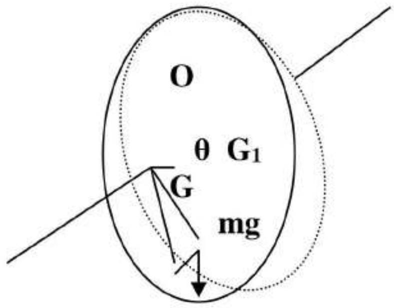
Fig 1 Fig 2

The compound pendulum:

A compound pendulum consists of a rigid body capable of rotation about a fixed horizontal axis under gravity. Let the axis of rotation pass through the point **O** in a vertical section of the body taken through the centre of gravity **G** of the body. In the equilibrium position **OG** will be vertical. **OG=h.** If **θ** is the small angular displacement of the body form the equilibrium position in time **t** and **M** the mass of the body , the couple tending to restore the body to it's equilibrium position is **Mgh sin θ**. The couple will produce an angular acceleration $d^2\theta/dt^2$. If *I* be the moment of inertai of the body about the axis of rotataion, the product of moment of inertia and the angular acceleration is also equal to the couple acting.Therefore

$$I\frac{d^2\theta}{dt^2} = -Mgh\sin\theta \qquad \text{-------------------1}$$

The significance of the negative sign is that the angular acceleration and the angular displacement are oppositely directed.



When **θ** is small $\sin\theta = \theta$

$$\text{Therefore } I\frac{d^2\theta}{dt^2} = -Mgh\,\theta$$

$$\text{or } \frac{d^2\theta}{dt^2} = -\frac{Mgh}{I}.\theta \qquad \text{-------------------2}$$

If k be the radius of gyration about the axis of rotation then $I = Mk^2$

$$\text{Therefore } \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}.\theta \qquad \text{------------------3}$$

This represent a simple harmonic oscillation of period

$$T = \frac{2\pi}{\sqrt{gh/k^2}} = 2\pi\sqrt{k^2/gh} \qquad \text{-----------------4}$$

If **K** be the radius of gyration about an axis through **G,** parallel to the axis of rotation, then by parallel axis theorem we have

$$k^2 = Mh^2 \text{ or } k^2 = k^2 + h^2 \qquad \text{-----------------5}$$

$$\text{therefore } T = 2\pi\sqrt{\frac{k^2+h^2}{hg}} \qquad \text{-----------------6}$$

$$\text{hence } T = 2\pi\sqrt{\frac{k^2+h^2}{hg}}$$

Centre of suspension and centre of oscillation:

The point *0* where the axis of rotation meets the vertical plane through the centre of gravity *G* of the rigid body is called the centre of suspension.

A simple pendulum which has the same period as the given component pendulum is called the simple pendulum $L=k^2/h$ or $\frac{K^2+h^2}{h}$

If OG is produced to a point C such that OC=L the length of the equivalent simple pendulum,the point c is called he centre of oscillation.The centr of oscillation is obviously a point at which the mass of the body may be considered to be concentrated without any change in the periodic time.

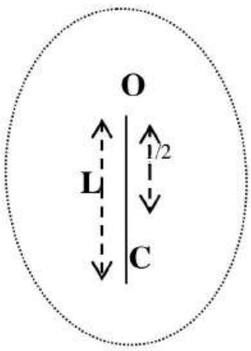
If the body is suspended about a parallel axis through C, we have CG=L-h.The length of the equivalent simple pendulum will be

$$L_1 = \frac{K^2+(L-h^2)}{L-h}$$

But $$L=\frac{K^2+h^2}{h}$$

we have $$K^2 = Lh-h^2$$

or $$L_1 = \frac{Lh-h^2+L^2-2Lh+h^2}{L-h} = \frac{L(L-h)}{L-h} = L$$



Centre of Percussion:

Minimum periods of a compound pendulum: From the expression $\mathbf{T=2\pi\sqrt{(k^2+h^2/hg)}}$ we find that the value of the period T depends on the length of the equivalent simple pendulum namely $\frac{K^2+h^2}{h}$

If T is minimum, $\frac{dT}{dh}=0$

i.e., $\frac{d}{dh}(K^2/h+h)=0$ i.e., $1-\frac{K^2}{h^2}=0$

or $K^2=h^2$ or $K=\pm h$ A compound pendulum will have its period a minimum when the depth of the centre of gravity of the pendulum below the centre of suspension is equal in magnitude to the radius of gyration about an axis through the centre of gravity parallel to the axis of rotation.

The Bifilar pendulum:

A Bifilar pendulum consists of a rigid body AB suspended by two eual non-parallel threads CA and DB attached at eual distances on either side of the centre of gravity G of the body.Let AB=2a,CD=2b.Let the body be given a small angular displacement about a vertical axis through G and released.Let A_1B_1 be the position of AB at any instant of time t,the angular displacement ∠AGA_1 being a small angle θ. The strings A_1C and B_1D will now be included to the vertical at the same angel as when the body was in its equilibrium position.Let CE be drawn perpendicular to AB and let ∠ACE = ϕ,which wil also be small. AE=GA-GE=a-b

$$CE^2=CA^2-AE^2=I^2-(a-b)^2$$

∠ A_1CE is also equal to ϕ .

If AC=BD=1 ;

$$\sin\phi = \frac{a-b}{l} \text{ and } \cos\phi = \frac{\sqrt{I^2-(a-b)^2}}{l}$$

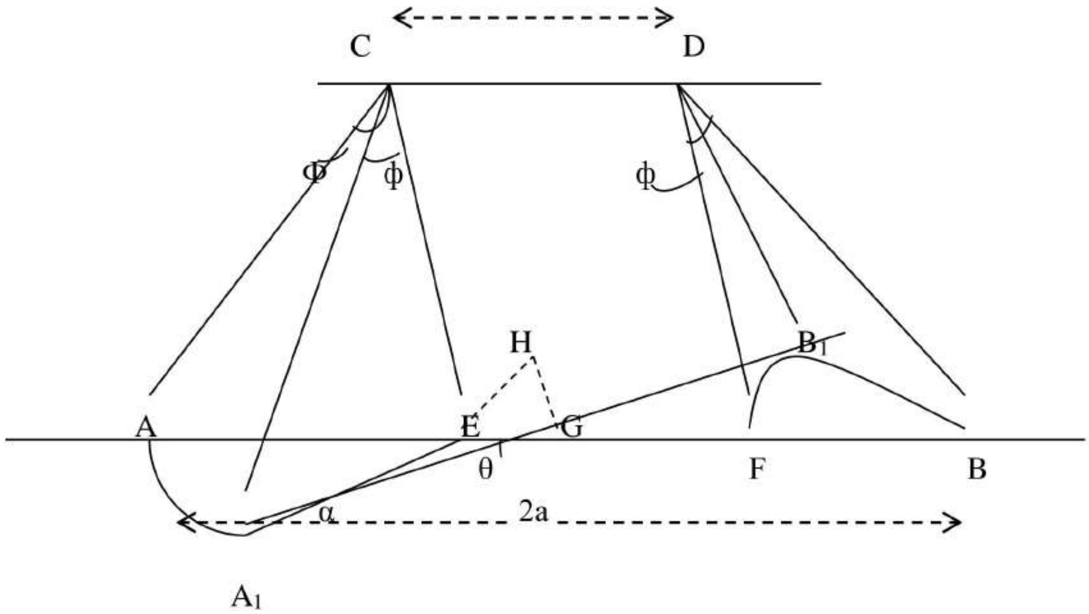
The tensions of the strings A_1C and B_1D are each inclined at angles equal to ϕ with the vertical.The sum of the vertical component of the tensions at A_1 and B_1 is equal to 2T cos ϕ.These balance the weight Mg of the rigid body.

Therefore 2T cos ϕ=Mg. --------------------------1

The components of the tensions analog A1E and B1F in a horizontal plane are each equal to T sin ϕ.These are unlike parallel and equal,and therefore constitute a couples the moment of which tends to restore the displaced body to the equilibrium position. The moment of the restoring couple.

=T sin ϕ X 2GH --------------------------2

Where H is the foot of the perpendicular drawn from G on the component T sin ϕ acting along A_1E.



Let ∠ GA_1E= α ,then in the triangle GA_1E

$$\frac{b}{\sin\alpha} = \frac{a-b}{\sin\theta} \text{ or } \sin\alpha = \frac{b\sin\theta}{a-b}$$

In the right angled triangle GHA_1

$$GH = \alpha \sin \alpha = a \frac{b \sin \theta}{a-b} \qquad \text{----------------------------3}$$

Substituting the value of GH in equation (2) the moment of the restoring couple.

$$T \sin \phi \times \frac{2ab \sin \theta}{a-b} \qquad \text{----------------------------4}$$

but $T = \frac{Mg}{2\cos \phi}$

Substituting this value for T in equation (4).

Moment of the restoring couple

$$= \frac{Mg}{2\cos \phi} \times \sin \phi \times \frac{2ab}{a-b} \sin \theta$$

Substituting the values of cos ϕ and sin ϕ.

Moment of the restoring couple

$$= \frac{Mgl}{\sqrt{[I^2-(a-b)^2]}} \times \frac{a-b}{l} \times \frac{ab}{a-b} \sin \theta$$

$$= \frac{Mgab}{\sqrt{[I^2-(a-b)^2]}} \sin \theta =$$

$$= \frac{Mgab}{\sqrt{[I^2-(a-b)^2]}} .\theta \text{ .since } \theta \text{ is small.}$$

The angular acceleration of the body is $\frac{d^2 \theta}{dt^2}$ .If I be the moment of inertia of the body about the vertical axis through G, the deflecting couple = $I \frac{d^2 \theta}{dt^2}$ .Hence for equilibrium

$$I \frac{d^2 \theta}{dt^2} = -\frac{Mgab}{\sqrt{[I^2-(a-b)^2]}} . \theta$$

$$\text{therefore } \frac{d^2 \theta}{dt^2} = -\frac{Mgab}{I\sqrt{[I^2-(a-b)^2]}} . \theta$$

now $\frac{Mgab}{I\sqrt{[I^2-(a-b)^2]}} . \theta$ is a constant.

Hence the angularb acceleration of the body is directly proportional to the angular displacement.The oscillations are therefore simple harmonic and periodic time is given by

$$T = \frac{2 \pi}{\sqrt{[Mgab/I\sqrt{l^2-(a-b)^2}]}}$$

$$= 2 \pi \sqrt{I\sqrt{l^2-(a-b)^2}/ Mgab}$$

If a and b are nearly equal,a-b can be neglected.In this case,

$$t = 2 \pi \sqrt{Il/Mgab}$$

Special case:

If the strings are parallel,a=b and the periodic time reduces to

$$t = 2 \pi \sqrt{Il/Mga^2}$$

The Bifilar pendulum(parallel threads):

A Bifilar pendulum consists of a heavy uniform body suspended by two parallel strings of equal length and symmetrically arranged so that the body executes small oscillations in a horizontal plane under gravity.

Let the strings supporting the body be parallel.Let l be the length of each string and 2a the distance between them.suppose the suspended system is given a small angular displacement θ about a central vertical axis so that the supporting threads are inclined at a small angle to with the vertical.

Now are $AA_1 = l\,\phi = a\theta$

Hence $\phi = \frac{a}{l}\,\theta$ ------------------------------1

If T be the tension in each string,then for the vertical equilibrium of the rod,

$2T\cos\phi = Mg.$

When ϕ is small $\cos\phi = 1$

Hence $2T = Mg$

$T = 1/2\,Mg$ ------------------------------2

The component of T at right angles to the axis of the body which tends to restore the body to equilibrium position are each equal to $T\sin\phi$

Moment of the restoring couple

$= T\sin\phi \times 2a = T\,\phi.2a$,since the angle ϕ is small.

$= 1/2 Mg.\frac{a}{l}\,\theta.2a = \frac{Mga^2}{l}.\theta$

If I be the moment of inertia of the body about a vertical axis through its centre of gravity,then the equation of the body is given by

$$I\frac{d^2\theta}{dt^2} = -\frac{Mga^2}{I.l}.\theta \text{ or } \frac{d^2\theta}{dt^2} = -\frac{Mga^2}{I.l}.\theta$$

The angular acceleration of the body is proportional to the angular displacement.The oscillations are therefore simple harmonic.The period of oscillation of the bifilar pendulum is given by

$$T = 2\pi\sqrt{(Il/Mga^2)}$$

Putting $I = Mk^2$

$$T = 2\pi\sqrt{(Mk^2l/Mga^2)} = 2\pi.k/a.\sqrt{(l/g)}$$

If the quantities T,k,a and l are known,g may be determined.

UNIT – IV

HYDRO STATICS AND HYDRO DYNAMICS

Definition:

The Centre of pressure of a plane surface in contact with a fluid is the point on the surface through which the line of action of the resultant of the thrusts on the various elements of the area passes.

Centre of pressure of a rectangular lamina immersed vertically in a liquid with one edge in the surface of the liquid:

Let ABCD be a plane rectangular lamina immersed vertically in a liquid of density ρ with one edge AB in the surface XY of the liquid.Let AB=a and AD=b.Divide the rectangle into a number of narrow strips parallel to AB. Consider one strip of width dx at a depth x below the surface of the liquid.

The thrust acting on the strip

$=(x\rho g)x(adx)=x\rho ga\ dx$

Moment of this thrust about AB

$=(x\rho ga\ dx)x\ x = x^2\rho ga\ dx$

Sum of the moments of the thrusts on all the strips= $\int_0^b x^2\rho ga\ dx$

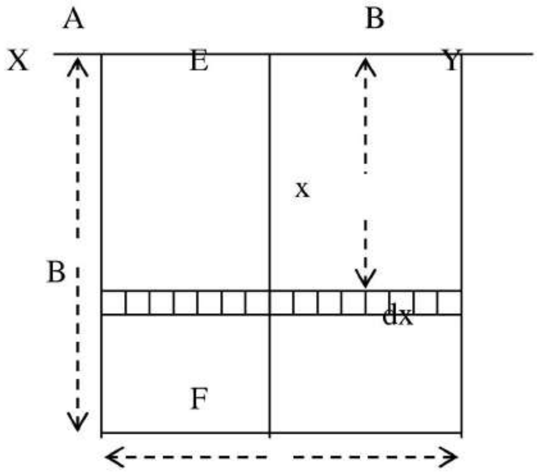
Resultant of the thrusts on all the strips $=\int_0^b x\rho ga\ dx$

Moment of the resultant thrust about AB=$H\int_0^b x\rho ga\ dx$

Where H=depth of the centre of pressure below AB.

$$H\int_0^b x\rho ga\ dx= \int_0^b x\rho ga\ dx$$

Or $H\rho ga \frac{b^2}{2} = \rho ga \frac{b^3}{3}$ or $H=\frac{2}{3} b$



The thrust on every elementary strip acts through its midpoint.Hence the centre of pressure will lie on EF where E and F are the midpoints of AB and DC.

Centre of pressure of a triangular lamina immersed vertically in a liquid with its vertex in the surface of the liquid and its base horizontal.

Let ABC be a triangularlamina immersed vertically in a liquid with its vertex A in the surface XY of the liquid and with its base BC horizontal.BC=a. Let the depth of the base of the lamina be b from the free surface of the liquid.Divide the triangle into a number of elementary strips of width dx parallel to the base BC.Consider one such strip B_1C_1 of width dx at a depth x below the surface XY.

Area of the strip $B_1C_1 = B_1C_1 dx = (ax/b)dx$

Thrust on the strip $B_1C_1 = (x\rho g)x(ax/b)dx$

Moment of this thrust about XY=$[aX^3 \rho g\, dx/b]$

Total moment due to all the strips= $\int_0^b \frac{a\rho g}{b} x^3\, dx$

Resultant Of The Thrusts on all the strips= $\int_0^b \frac{a\rho g}{b} x^2\, dx$

Moment of the resultant thrust about XY = $H \int_0^b \frac{a\rho g}{b} x^2\, dx$

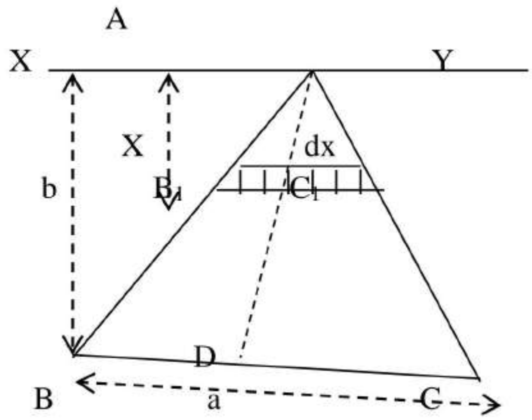
Here H=the depth of the centre of pressure below XY.

Since the two moments are equal,

$$\int_0^b \frac{a\rho g}{b} x^3\, dx = H \int_0^b \frac{a\rho g}{b} x^2\, dx$$

or $$\frac{a\rho g}{b}(b^4/4) = H\, \frac{a\rho g}{b}(b^3/3)$$

or $$H = \frac{3}{4} b$$



The centre of pressure lies on the line joining the mid points of the strips.i.e., lies on the median AD at adepth 3b/4 below the surface XY.

Centre of pressure of a triangular lamina immersed vertically in a liquid with one side in the surface when there is no external pressure:

Let ABC a triangular lamina immersed in a liquid with its base BC=a in the surface XY of the liquid.Let AD be a median of the triangle.Let b be the depth of the vertex A below the surface XY.Divide the triangle into a number of elementary strips of width dx parallel to the base BC.Consider one strip B_1C_1 at a depth x beow BC.

Area of the strip $B_1C_1 = B_1C_1 dx = \frac{a(b-x)}{b} dx$

$$\left(\text{since } \frac{B_1C_1}{a} = \frac{b-x}{b}\right)$$

Thrust on the strip $B_1C_1 = x\rho g \frac{a(b-x)}{b} dx$

Moment of this thrust about XY$= x^2\rho g \frac{a(b-x)}{b} dx$

Total moment due to all the strips$= \int_0^b x^2\rho g \frac{a(b-x)}{b} dx$

Resultant of the thrusts on all the strips$= \int_0^b x\rho g \frac{a(b-x)}{b} dx$

Let H be the depth of the centre of pressure below XY

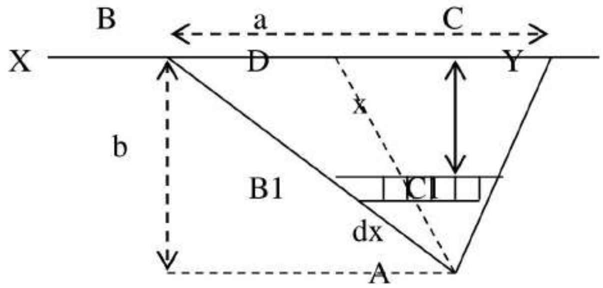
Moment of the resultant thrust about XY$= H\int_0^b x\rho g \frac{a(b-x)}{b} dx$

therefore $\int_0^b x^2\rho g \frac{a(b-x)}{b} dx = H\int_0^b x\rho g \frac{a(b-x)}{b} dx$

or $\quad H\int_0^b x(b-x)dx = \int_0^b x^2(b-x)dx$

or $\quad H[b^3/2-b^3/3]= [b^4/3-b^4/4]$

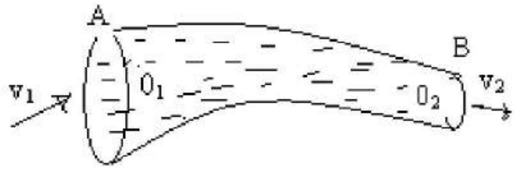
or $\quad H=b/2$



The centre of pressure is on the median AD at a depth b/2 below XY.

Hydrodynamics:

The equation of continuity is an expression of the law of conservation of mass in fluid mechanics.fig represents a tube of varying cross-section through which a non-viscous incompressible fluid of density ρ flows.Let α_1 and α_2 be the cross-sectional areas of the tube at the points A and B.Let the velocity of the fluid atA and B be v_1 and v_2 respectively.Since the fluid is incompressible,in the steady state,mass of fluid entering the tube per second through the section A=mass of fluid leaving the tube per second through the section B.



Mass of fluid entering the tube per second across the section A=$a_1 v_1 \rho$

Mass of fluid leaving the tube per second across the section B=$a_2 v_2 \rho$

Therefore $a_1 v_1 \rho = a_2 v_2 \rho$ or $a_1 v_1 = a_2 v_2$

Thua the product av is constant along any given flow tube.It follows that the speed of flow through a tube is inversely proportional to the cross-sectional area of the tube.It means that wherethe area of cross-sectional area of the tube is large,the velocity is small and vice versa.

EXAMPLE: water flowing with a velocity of 3 m/s in a 4cm diameter pipe enters a narrow pipe having a diameter of only 2cm. Calculate the velocity in the narrow pipe.
Here, $a_1=л(0.02)^2$; v_1= 3m/s; $a_2=л(0.01)^2$; v_2=?
$v_2= a_1 v_1/ a_2=(0.02)^2x3/\{ л(0.01)^2\}$=12m/s.

Bernoulli's theorem

Statement The total energy of an incompressible non-viscous fluid flowing from one point to another,without any friction remains consant throughtout the motion.

Explanation:

According to the theorem , the sum of knetic, potential and pressure energies of any element of an incompressible fluid in streamline flow remains constant. Suppose the height of an fluid of density ρ above
Ground level is h. Let it be moving with a velocity v. Let it have pressure p. then, its total energy per unit volume is

$$E=\rho v^2 /2+\rho gh + p.$$

Bernoulli's theorem states that E is a constant.

If at two points in the fluid the velocity are v_1, v_2 the heights are h_1, h_2 and the pressures are p_1, p_2 , then ,

$$P v_1^2 /2+ \rho gh1+p1=\rho v_2^2/2+\rho gh+p_2$$

The K.E per unit weights is called velocity head and is equal to $v^2/2g$.
The P.E per unit weight is called the gravitational head and is equal to h.p/ρg.Bernoulli eqation can br written as

$$v^2/(2g)+h+p/(\rho g)=constant$$

i.e., velocity head+gravitational head+pressure head=constant

In the case of liquid flowing along a horizontal pipe,the gravitational head h is a constant.

Therefore $v^2/(2g)+ p/(\rho g)=$ constant or $v^2/2+ p/\rho=$constant.

Or $p+\rho v^2/2=$constant.

Or static pressure+dynamic pressure=constant.

This expression shows that greater velocity corresponds to a decrease in pressure and vice versa.This principle may be used to determine fluid speeds by means of pressure measurements.

Example.Venturimeter,pilot tube,etc

Proof.consider a fluid in stream line motion along a nonuniform tube.Let A and B be two transverse sections of the tube at heights h_1 and h_2 from a reference plane(the surface of the earth).Let a_1 and a_2 be the areas of cross section an A and B.Let v_1 and v_2 be the velocities of the fluid at A and B.Let p_1 be the pressure at A due to the driving pressure head.Let p_2 be the pressure at B.Since $a_1>a_2, v_2>v_1$. Hence the fluid accelerated as it flows from A and B.

Workdone per second on the liquid entering at A is

W_1=Force at AxDistance moved by the liquid in 1 second.

$= p_1 a_1 x v_1= p_1 a_1 v_1$

Workdone per second by the liquid leaving the tube at B is

$W_1= p_2 a_2 v_2$

Therefore Network done by the fluid in passing from A to B

$W= W_1- W_2 = p_1 a_1 v_1- p_2 a_2 v_2$

But $a2\ v2 = a_1 v_1$

Therefore $W= (p_1- p_2) a_1 v_1$

The workdone by the fluid is used in changing its potential and Kinetic energies.

Decrease in P.E. $= (a_1 v_1\rho)g(h_1-h_2)$

Increase in K.E $= ½(a_1 v_1\rho) (v_2^2- v_1^2)$

Hence,the total gain in the energy of the system when the liquid flows from A and B

$=1/2 (a_1 v_1\rho) (v_2^2- v_1^2)- (a_1 v_1\rho g) (h_1-h_2)$

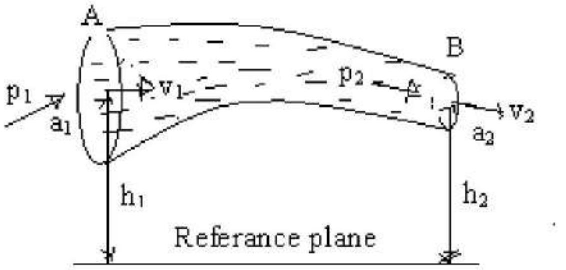
Therefore $(p_1- p_2) a_1 v_1= 1/2 (a_1 v_1\rho) (v_2^2- v_1^2)- (a_1 v_1\rho g) (h_1-h_2)$

Or $p_1+1/2\ \rho v_1^2+ \rho g\ h_1= p_2 +1/2\ \rho v_2^2+ \rho g\ h_2$

Or $p+ 1/2\ \rho v^2+hpg =$ constant

Or $p/\rho +1/2\ v^2+ hg =$ constant

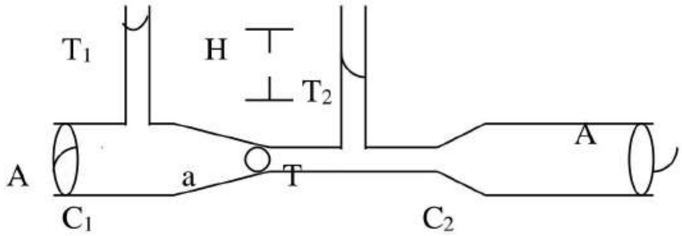
Or $p/\rho g+v^2/2g +h=$ constant.



p/(ρg) is called the pressure head. $v^2/(2g)$ is called the velocity head and h is called the gravitational head.Hence, Bernoulli theorem may be stated as follows.When an incompressible non-viscous fluid flows in stream line motion,the sum of the pressure head,velocity head and gravitational head remains constant throughout its motion.

Venturimetre:

It is a device based on Bernoukki's principle.It is used for measuring rate of flow of liquids in pipes.It consists of two wide conical tubes C_1 and C_2 with a constriction T between them.T is called throat.Let the are of cross-section of C_1 and C_2 be A.Let a be the area of cross-section of the throat.



When the flow is steady ,let V be the volume of water flowing per second through the venturimeter.
Then,$V=Av_1=av_2$
Hence Velocity of water in T is greater than the velocity in C_1 and C_2.Consequently,the pressure in T is smaller than the pressure in C_1 and C_2.This difference in pressure H is measured by the difference of the water levels in the vertical glass tubes T_1 and T_2 connected C_1 and T respectively.Let p_1 and p_2 be the pressure in the wider limb and throat respectively.
According to Bernoulli's equation for the horizontal flow,

$$\frac{Pp_1}{\rho g}+\frac{v_1^2}{2g}=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}$$

or $$\frac{p_1 - p_2}{\rho g} = \frac{v_2^2 - v_1^2}{2g}$$

The difference in pressure in C_1 and T= $p_1 - p_2 = H\rho g$.

Hence $$\frac{H\rho g}{Pg} = 1/2g\left[\frac{v^2}{a^2} - \frac{v^2}{A^2}\right] = v^2/ g\left[\frac{A^2 - a^2}{a^2A^2}\right]$$

Therefore $V=Aa \sqrt{2gH}./(A^2- a^2)$

Pilot tude:

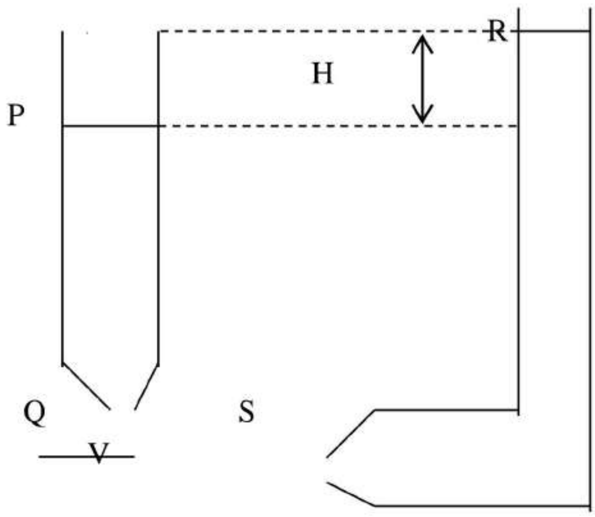
It is an instrument used to measure the rate of flow of water through a pipe-line.It is based on Bernoulli's Principle.It consists of two vertical tubes PQ and RS with small apertures at their lower ends.The plane of aperture of the tube PQ is parallel to the direction of flow of water.The plane of aperture pf the tube RS faces the flow of water perpendicularly.The rise of the water column in the tube RS therefore,measures the pressure at S.

Let p_1 and p_2 be the pressures of water at Q and S respectively.Let v be the velocity of water at Q.Since the water is stopped in the plane of the aperture S of the tube RS,its velocity at S becomes Zero.Hence the pressure increases to p_2 at S.Let H be difference of level in the two tubes.Applying Bernoulli's theorem to the ends Q and S.

Therefore $\frac{1}{2}v^2+\frac{p_1}{\rho}=v^2=\frac{2}{\rho}(p_2-p_1)=\frac{2H\rho g}{\rho}$

Therefore $v=\sqrt{2gH}$

Where a=area of cross-section of the pipe.



UNIT – V

LAGRANGIAN DYNAMICS

Conservation theorem for linear momentum:

The net linear momentum of a system of n-particles is

$$P=\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i$$

From Newton's second law, $F^{ext} = \frac{dP}{dt}$

i.e., the rate of change of linear momentum of a system of particles is equal to the net external force acting on the system.

If $F^{ext}=0$, $\frac{dP}{dt}=0$. Integrating ,P=constant.

This gives the therome for conservation of kinear momentum of the system according to which *"If the sum of external forces acting on the system of particles is zero,the total linear momentum of the system is constant or conserved"*

Conservation theorem for angular momentum:

The angular momentum of its particle of the system about any point 0,from definition is given by

$$J_i = r_i x p_i \qquad \text{-----------1}$$

Where ri is the radius vector of ith particle from the point 0 and pi its linear momentum.

For a system of n particles,we have

$$J=\sum_i J_i = \sum_i r_i x p_i$$

$$\frac{dJ}{dt} = \sum_I r_i x \frac{dp_i}{dt} = \sum_i r_i x F_i \qquad \text{----------------2}$$

here, $F_i = \frac{d}{dt} p_i$ = net force acting on i^{th} particle

Internal forces occur in equal and opposite pairs.Hence the net internal force acting on the system of particles is zero.Thus,

$$\frac{dJ}{dt} = \sum_i r_i F_i^{ext} = \tau^{ext}$$

Here, $\tau^{ext} = \sum_i r_i x F_i^{ext}$ is the torque arising due to external forces only.

If, $\tau^{ext}=0$, $\frac{dJ}{dt}=0$ or J=constant

Thus,if external torque acting on a system of particles is zero,the angular momentum of the system remains constant.This is the conservation theorem for angular momentum of a system of particles.

Conservation of energy.If the workdone by a force is independent of path,the force is saud to be conservative.

If the forces acting on the system of particles are conservative,the otal energy of the system of particles which is the sum of the total kinetic energy and the total potentiel energy of the system is conserved.

This the energy conservation theorem.

On the other hand if the forces are non-conservative,the total energy of universe (mechanical energy+chemical energy+sound energy+light energy+heat energy etc.)remains constant.

Constraints:

Constraints are restrictions imposed on the position or motion of a system,because of geometrical conditions

Examples.

(1) The beads of an abacus are constrained to one dimensional motion by the supporting wires.

(2)Gas molecules within a container are constrained by the walls of the vessel to move only inside the container.

(3) The motion of rigid bodies is always such that the distance between any two particles remains unchanged.

(4) A particle placed on the surface of a solid sphere is restricted by the constraint so that it can only move on the surface or in the region exterior to the sphere.

Holonomic and non-holonomic constraints

If the constraints can be expressed as equations connecting the coordinates of the particles (and possibly time) in the form.

$$f(r_1,r_2,r_3.....r_n,t)=0 \text{ ------------------1}$$

then the constraints are said to be holomonic.

Examples.

(1)The constraints involved in the motion of rigid bodies in which the distance between any two particular points is always fixed,are holomonic since the conditions of constraints are expressed as.

$$(r_i-r_j)^2-c_{ij}^2=0$$

(2) The constraints involved when a particle is rejected to move along a curve or surface are holomonic.Here the equation defining the curve or surface is the equation of constraint.

If the constraints cannot be expressed in the form of Eq.(1),they are called non-holomonic constraints.

Examples.

(1)The constraints involved in the motion of the particle placed on the surface of a solid sphere are non-holomonic.The conditions constraints in this are expressed as

$$r^2 - a^2 \geq 0,$$

where a is the radius of sphere.This is an inequality and hence not in the form of Eq.(1).

(2)The walls of the gas vessel constitute a nonholomonic constraint.

(3)An object rolling on a rough surface without slipping is also an example of non-holomonic constraint.

Scleronomic and Rheonomic constraints:

If the constraints are independent of time,they are called Scleronomic. If the constraints are explicity dependent on time ,they are called rheonomic.

The constraint in the case of rigid body motion is Scleronomous.A bead sliding on a moving wire is an example of rheonomic constraint.

In the solution of mechanical problems,the constraints introduce two type of dofficulties:

(1) The co-ordinates r_i are connected by the equations of constraints.therefore,they are not independent.
(2) The forces of constraint are not a priori known.In fact,they cannot be estimated till a complete solution of the problem is obtained.

The first problem can be soved by introducing generalized coordinates,whereas the second is practically an insurmountable problem.We therefore reformulate the problem such that the forces of constraint disappear.

Generalized co-ordinates:

A system consisting of N particles,fee from constraints,has 3N independent coordinates or degrees of freedom.If the sum of the degrees of freedom of all the particles is k,then the system may be regarded as a collection of free particles is subjected to (3N-k) independent constraints. So only k coordinates are needed to describe the motion of the system.These new coordinates $q_1,q_2,q_3....q_k$ are called the generalized coordinates of Lagrange.Generalised coordinates may be lengths or angles or any other set of independent quantities which define the position of the system.

Definition:

The generalized coordinates of a material system are the independent parameters $q_1,q_2,q_3....q_k$ which completely specify the configuration of the system,i.e., the position of all its particles with respect to the frame of reference.

Example.

Consider the simple pendulum of mass m_1 with fixed length r_1.The single coordinate θ_1 will determine uniquely the position of m_1 since the simple pendulum is a system of one degree of freedom.Since the only variable involved is θ_1,it can be chosen as a generalized coordinate.Thus

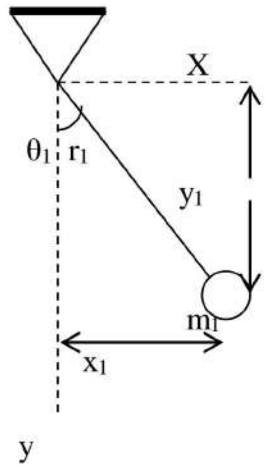
$q = \theta_1$. The two coordinates x_1 and y_1 could also be used to locate m1 but would require the inclusion of the equation of the constraint $x_1^2+y_1^2=r_1^2$

Since x_1 and y_1 are not independent,they are not generalized coordinates.

Generalised velocities:

The generalized velocities of a system are the total time derivatives of the generalized coordinates of the system.

Thus $q_i = \frac{dq_i}{dt}$ (i=1,2,3….k)



Transformation equations:

The rectangular Cartesian coordinates can be expressed as the functions of generalized coordinates.Let x_i,y_i and z_i be the Cartesian coordinates can be expressed as functions of generalized coordinates $q_1,q_2,q_3....q_k$ i.e.,

$$\left.\begin{aligned} x_i &= x_i(q_1,q_2....q_k,^t) \\ y_i &= y_i(q_1,q_2....q_k,^t) \\ z_i &= z_i(q_1,q_2....q_k,^t) \end{aligned}\right\} \text{-----------------1}$$

let ri be the position vector of ith particle,i.e.,r_i=i x_i +j y_i +k z_i

Then $r_i= r_i(q_1,q_2....q_k,^t)$, ---------------------2

Eq.2 is the vector form of eg.1

The equations like (1) and(2) are called transformation equations.The functions and thir derivatives in the above two equations are supposed to be continuous.The equations also contain the constraints explicitly.

Principle of virtual work:

Consider a system described by n generalized coordinates q_j=(j=1,2,3,…,n).Suppose the system undergoes a certain displacement in the configuration space in such a way that it does not take any time and that it is consistent with the constraints on the system.Such displacements are called virtual because they do not represent actual displacement of the system .Since there is no actual motion of the system,the work done by the forces of constraint in such a virtual displacement is zero.

Let the virtual displacement of the ith particle of the given system be δr_i.If the given system is in equilibrium,the resultant force acting on the ith particle of the system must be zero,i.e., $F_i=0$.It is ,then,obvious that virual work Fi. $\delta r_i= 0$ for the ith particle and hence itb is also zero for all the particles of the system.

Thus $$dW=\sum_i F_i . \Delta r_i=0 \qquad \text{------------------}1$$

The resultant force F_i acting on the ith particle is

$$F_i= F_i^a+ f_i \qquad \text{------------------}2$$

Here, Fi^a is the applied force and fi is the force of constraint.

Eq.1 then becomes

$$\sum_i F_i^{a.} \delta r_i+ \sum f_i .\delta r_i =0 \qquad \text{------------------}3$$

We now consider systems for which the virtual workdone by the forces of constraints is zero,i.e.,

$$\sum_i f_i .\delta r_i =0 \qquad \text{---------------------}4$$

Then eq 3 becomes

$$\sum_i F_i^{a.} \delta r_i=0 \qquad \text{-----------------------}5$$

This equation is termed as principle of virtual work.

D'alembert's principle:

Most of the systems we come across in mechanics are not in static equilibrium.Hence the principle must be modified to include dynamic systems as well.According to Newton's second law of motion,

$$F_i=\dot{p}_i \text{ or } F_i- \dot{p}_i = 0 \qquad \text{-----------------------}6$$

According to the above equation ,a moving system of particles can be considered to be in equilibrium under the force (Fi - pi), i.e., the actual applied force Fi plus an additional force –pi which is known as reversed effectiveb force on ith particle. Let us again assume that the forces of constraint do no work. Then, we can generalize Eq. (5) by the use of Eq. (6) to the form.

$$\sum_i(F_i - p_i) . \delta r_i = 0 \text{ -- } 7$$

Eq (7) is the mathematical statement of D' Alembert principle.

It is to be noted here that we have restricted ourselves to the systems where the virtual work done by the forces of constraints disappears. With this in mind we cabn drop the superscript a in equation (2) i.e., D'Alembert's principle may be written as

$$\sum_i(F_i - p_i) . \delta r_i = 0 \text{ -- } 8$$

Generalised Momentum

The generalized momentum conjugate to the generalized co-ordinate qk is defined as the quantity $\frac{\partial L}{\partial \dot{q}_k}$. It is represented by p_k.

i.e., $p_k = \frac{\partial L}{\partial \dot{q}_k}$

Lagrangian formulation of conservation theorems.

(a) Conservation theorem for generalized momentum

According to Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

If the generalized co – ordinate qk is cyclic, then $\frac{\partial L}{\partial q_k}$

Thus, the Lagrange equation of motion reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

or $\frac{dL}{d\dot{q}_k}$ = constant

i.e., p_k = constant

Thus, whenever a coordinate qk does not appear explicitly in the Lagrangian function L, corresponding linear momentum pk is a constant of the motion. Hence we can state as a general conservation theorem that the generalized momentum conjugate to a cyclic coordinate is conserved.

Conservation theorem of energy.

Consider a conservative system. The Lagrangian L does not depend upon time explicitly.

Then $L = L(q_1, q_2, \ldots\ldots q_k, \ldots.., \dot{q}_1, \dot{q}_2 \ldots.., \dot{q}_k, \ldots.)$

Hence, the total time derivative of $L \equiv L(q_k, \dot{q}_k)$ is given by

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k \quad \ldots\ldots\ldots (1)$$

Lagrange's Eq.is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$

or $$\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) = \frac{\partial L}{\partial q_k}$$

We can rewrite Eq (1) as

$$\frac{dL}{} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial q_k} q_k\right) + \sum_k \frac{\partial L}{\partial q_k} q_k = \sum_k \frac{d}{dt} q_k \frac{\partial L}{\partial q_k}\left(\frac{}{}\right)$$

$$\text{or}\left(\frac{d}{dt} L - \Sigma q_k \frac{\partial L}{\partial q_k}\right) = 0$$

therefore, $L - \Sigma q_k \frac{\partial L}{\partial q_k}$ = constant …………………(2)

Now $\frac{\partial L}{\partial q_k} = \frac{\partial}{\partial q_k}(T - V) = \frac{\partial T}{\partial q_k} = \frac{\partial}{\partial q_k}(\Sigma \frac{1}{2} m_k q^2_k)$ (Since , $T = \Sigma \frac{1}{2} m_k q^2_k$)

Therefore, $\frac{\partial L}{\partial q_k} = mk\ q_k$

Eq (2) becomes $L - = \Sigma\ m_k q^2_k$ = constant

Or L -2T = Constant or T-V = Constant

Or - (T+V) = constant

Therefore, T + V = E = total energy = constant.

Thus the energy conservation theorem states that, *if the Lagrangian function does not contain the time explicitly, the total energy of the conservative system is conserved.*

Conservation theorem for Linear Momentum.

Consider a conservative system so that the potential energy V is dependent on position only and the kinetic energy is independent of position, i.e.,

$\frac{\partial V}{\partial q_k} = 0$ and $\frac{\partial T}{\partial q_k} = 0$ ….. (1)

Then, Lagrange equation of motion for such a coordinate is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

i.e., $\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) + \frac{\partial V}{\partial q_k} = 0$ (Using Eq (1))

$$p_k = \frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) - \frac{\partial V}{\partial q_k} = Q_k \qquad \ldots\ldots\ldots\ldots..(2)$$

Now, if we show that Qk represents the component of total force along the direction of translation of the system and p_k is the component of total linear momentum along the same direction, then Eq. (2) will represent the equation of motion for linear momentum.

Generalised force is given by

$$Q_k = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_k}$$

If n is the unit vector along the direction of translation, then

$$\delta r_i = n\, \delta q_k$$

or $$\frac{\delta r_i}{\delta q_k} = n$$

Thus $$Qk = \sum_i F_i \cdot n = n.F \qquad \ldots\ldots\ldots(3)$$

which represents the componentof total force along the direction of n.

The Kinetic energy, $T = \frac{1}{2} \sum_i m_i\; r^2_i$

So the generalized momentum is

$$pk = \frac{\partial L}{\partial q_k} = \frac{\partial T}{\partial q_k} = \Sigma\, m_i\, r_i \cdot \frac{\partial ri}{\partial q_k}$$

$$= \Sigma\, m_i\, v_i \cdot \frac{\partial r_i}{\partial q_k}$$

$$= \sum_i m_i\, v_i \cdot n = n \cdot \sum_i m_i\, v_i$$

Which shows that p_k represents the component of total linear momentum along the direction of translation.

Now we can say that the equation $p_k = Q_k$.

Is the equation of motion for total linear momentum of the system.

If $Q_k = 0$, $p_k = 0$ i.e., = constant.

This gives the conservation theorem for linear momentum. It states that if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.